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ON A GENERALIZED EGNELL INEQUALITY

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Abstract. In this paper we prove an inequality which connects the L^p norm of the gradient of a function u with its $|x|^\nu$-weighted $L^{\frac{p(N+\nu)}{N-p}}$ norm and its L^{p^*}-weak norm. Here $1<p<N$, $-p<\nu\leqslant 0$ and $p^*=\frac{Np}{N-p}$. As a consequence we can provide an alternative proof of the Egnell inequality in $\mathbb{R}^N$.

1. Introduction

The classical Sobolev inequality in $\mathbb{R}^N$ asserts that

$$\left(\int_{\mathbb{R}^N}|\nabla u|^p\,dx\right)^{\frac{1}{p}}\geqslant S(N,p)\left(\int_{\mathbb{R}^N}|u|^{p^*}dx\right)^{\frac{1}{p^*}},\tag{1}$$

with $1<p<N$, $p^*=\frac{Np}{N-p}$ and u is a real-valued function in $L^{p^*}(\mathbb{R}^N)$ such that $|\nabla u|\in L^p(\mathbb{R}^N)$, where ∇u is the distributional gradient of u. The value of the sharp constant $S(N,p)$ in (1) is known to be

$$S(N,p)=\pi^{\frac{1}{2}}2^{\frac{1}{N}}N^{\frac{1}{p}}(N-p)^{\frac{p-1}{p}}(p-1)^{\frac{1}{N}-\frac{p-1}{p}}p^{-\frac{1}{N}}\left[\frac{\Gamma(\frac{N}{p})\Gamma(N-\frac{N}{p})}{\Gamma(N)\Gamma(\frac{N}{2})}\right]^{\frac{1}{N}},$$

where Γ is the standard Euler function. The equality sign holds when u is of the form

$$u(x)=\frac{h}{\left[1+k|x|^{\frac{p}{p-1}}\right]^{\frac{N-p}{p}}},\qquad h,k>0,$$

with h,k positive constants (see [6], [8] and [32]).

When $\mathbb{R}^N$ is replaced by a bounded domain $\Omega\subset\mathbb{R}^N$ and $u\in W_0^{1,p}(\Omega)$, the Sobolev inequality (1) still holds with $S(N,p)$ as best constant, but the constant is never achieved. For this reason, several authors studied the problem of improving the inequality (1) for $u\in W_0^{1,p}(\Omega)$, by adding a right-hand-side *remainder term*. The first

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results in this direction are given in [10] and then in [9], where the authors prove several improvements of (1) by adding the norm of u and of ∇u in suitable L^q spaces. Similar results are still true when we consider $u \in W^{1,p}(\Omega)$ vanishing on a fixed part Γ_0 of the boundary $\partial\Omega$ (see [19], [28], [29]).

Analogous questions can be studied in relation to the celebrated Hardy-Sobolev inequality (see [26] and [25])

$$\int_{\mathbb{R}^N} |\nabla u|^p \, dx \geqslant A(N,p) \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx, \tag{2}$$

with $u \in W^{1,p}\left(\mathbb{R}^N\right)$ and

$$A(N,p) = \left(\frac{N-p}{p}\right)^p. \tag{3}$$

This inequality and its various improvements are used in many contexts, as in the study of stability of solutions to semi-linear elliptic and parabolic equations (see [11], [12], [33]), or in the analysis of the asymptotic behaviour of the heat equation with singular potentials (see [34]). When $\mathbb{R}^N$ is replaced by a bounded domain Ω of $\mathbb{R}^N$ and $u \in W_0^{1,p}(\Omega)$, the Hardy-Sobolev inequality (2) still holds.

The constant $A(N,p)$ in (3) is the best one in both cases but there is no function $u \in W^{1,p}\left(\mathbb{R}^N\right)$ (or $u \in W_0^{1,p}(\Omega)$) for which it is achieved. For this reason several authors have improved inequality (2) by adding at the right-hand side a non-negative correction term (see e.g. [1], [2], [5], [7], [11], [15], [17], [20], [21], [22], [23], [30]).

The Sobolev and Hardy-Sobolev inequalities described so far represent a special case of a more general inequality, known as Egnell inequality (see [28]).
Let us consider $1 < p < N$, $-p < \nu \leqslant 0$ and

$$q = \frac{p(N+\nu)}{N-p}. \tag{4}$$

If we denote by $L^q(\mathbb{R}^N, |x|^\nu)$ the space of measurable functions u such that

$$\|u\|_{q,|x|^\nu} := \left(\int_{\mathbb{R}^N} |u|^q \, |x|^\nu \, dx\right)^{\frac{1}{q}} < \infty,$$

then the Egnell inequality states that

$$\left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx\right)^{\frac{1}{p}} \geqslant C(p,\nu) \left(\int_{\mathbb{R}^N} |u|^q \, |x|^\nu \, dx\right)^{\frac{1}{q}}. \tag{5}$$

The optimal value of $C(p,\nu)$ is obtained in [18] by means of methods similar to the ones used by Talenti in [32] to get the best constant in Sobolev inequality (see also [27]).

To write down the value of $C(p,\nu)$, let us consider for any $u \in W^{1,p}\left(\mathbb{R}^N\right)$, $u \not\equiv 0$ the functional

$$F(u) := \frac{\| |\nabla u| \|_{L^p}}{\|u\|_{q,|x|^\nu}};$$

then the best constant $C(p,\nu)$ is defined as

$$C(p,\nu) := \inf_{\substack{u\in W^{1,p}(\mathbb{R}^N)\\ u\neq 0}} F(u).$$

In [18] Egnell proves that the value of $C(p,\nu)$ is given by

$$C(p,\nu) = \pi^{\frac{N}{2}\frac{p+\nu}{p(N+\nu)}} 2^{\frac{p+\nu}{p(N+\nu)}} (N+\nu)^{\frac{1}{p}} (N-p)^{\frac{p-1}{p}} (p-1)^{-\frac{p-1}{p}+\frac{p+\nu}{p(N+\nu)}} \times (p+\nu)^{-\frac{p+\nu}{p(N+\nu)}} \left[\frac{\Gamma\left(\frac{(N+\nu)(p-1)}{p+\nu}\right)\Gamma\left(\frac{N+\nu}{p+\nu}\right)}{\Gamma\left(\frac{p(N+\nu)}{p+\nu}\right)\Gamma\left(\frac{N}{2}\right)}\right]^{\frac{p+\nu}{p(N+\nu)}}, \tag{6}$$

and the infimum of $F(u)$ is attained when

$$u(x) = \frac{h}{\left[1+k|x|^{\frac{p+\nu}{p-1}}\right]^{\frac{N-p}{p+\nu}}}, \tag{7}$$

with h e k positive constants.

We highlight that the Egnell inequality becomes the Sobolev inequality when $\nu = 0$ and $C(p,0) = S(N,p)$. In terms of embedding between functional spaces, the Egnell inequality represents the continuous embedding of the Sobolev space $W^{1,p}(\mathbb{R}^N)$ in the weighted Lebesgue space $L^q(\mathbb{R}^N,|x|^\nu)$, where q is given in (4). We finally remark that the Egnell inequality can be read as a particular case of a class of interpolation inequalities known as Caffarelli-Kohn-Nirenberg inequalities (see [13] and [14]). Improved Caffarelli-Kohn-Nirenberg inequalities are widely studied in the literature.

In this paper we prove an inequality which connects the $L^p(\mathbb{R}^N)$ norm of the gradient a function u with the weighted $L^q(\mathbb{R}^N,|x|^\nu)$ and L^{p^*}-weak norms of u. Namely, we prove the following inequality:

$$\|u\|_{p^*,\infty}^{rp*} \||\nabla u|\|_{L^p}^{p} \geqslant A(N,p)\|u\|_{q,|x|^\nu}^{q} + B(N,p)\|u\|_{p*,\infty}^{sp^*}, \tag{8}$$

where $A(N,p)$ and q are given by (3) and (4) respectively,

$$r = \frac{p+\nu}{N}, \qquad s = \frac{N+\nu}{N}, \tag{9}$$

and

$$B(N,p) = 2\pi^{\frac{N}{2}}(N-p)^{p-1}(p-1)^{2-p-\frac{(N-p)(p-1)}{p+\nu}} p^{\frac{p(N-p)}{p+\nu}} \left[\frac{\Gamma\left(\frac{(N+\nu)(p-1)}{p+\nu}\right)\Gamma\left(\frac{N+\nu}{p+\nu}\right)}{\Gamma\left(\frac{p(N+\nu)}{p+\nu}\right)\Gamma\left(\frac{N}{2}\right)}\right]. \tag{10}$$

Finally, we show that Egnell inequality can be easily deduced by (8) as corollary.

In order to prove the main result, we firstly apply a symmetrization procedure by replacing u with its rearrangement $u^\#$, which is spherically symmetric and decreases with respect to $|x|$. Then, the Pólya-Szego principle (see [32]) and Hardy-Littlewood inequality (see [26]) ensure that the previous assumption is not restrictive. Finally, we apply classical arguments of one-dimensional Calculus of Variations (see [31]) and techniques similar to the ones used in [4] for the Sobolev inequality, which straightforwardly lead to inequality (8).

2. Definitions and main result

The main result of this paper is a generalized Egnell inequality which links the $L^p(\mathbb{R}^N)$ norm of the gradient a function u with the weighted $L^q(\mathbb{R}^N, |x|^\nu)$ and L^{p*}-weak norm of u, where $1 < p < N$, $-p < \nu \leqslant 0$, $p^* = \frac{Np}{N-p}$ and q is given in (4). The Egnell inequality (5) with its optimal value (6) easily follows as corollary.

We firstly recall the definition of spherically decreasing rearrangement of a function u and some related properties.

DEFINITION 1. Let Ω be a measurable subset of $\mathbb{R}^N$ and $u : \Omega \to \mathbb{R}$ a measurable function in Ω. The distribution function of u is the decreasing map μ from $[0, +\infty[$ into $[0, +\infty[$ defined at any point $t \geqslant 0$ as the measure of a level set of u, $\{x \in \Omega : |u(x)| > t\}$. The decreasing rearrangement u^* of u is the distribution function of μ

$$u^*(s) := \sup\{t \geqslant 0 : \mu(t) > s\}, \qquad s \in (0, |\Omega|).$$

The main property of rearrangements is the fact that the distribution of u^* is μ, in other words u and u^* are equidistribuited.

DEFINITION 2. Let us denote by ω_N the measure of the unit ball of $\mathbb{R}^N$ and by $\Omega^\#$ the ball of $\mathbb{R}^N$ centred at the origin such that $|\Omega| = \left|\Omega^\#\right|$. For every $x \in \Omega^\#$, the spherically decreasing rearrangement of u is defined as the decreasing rearrangement u^* valued in $\omega_N |x|^N$

$$u^\#(x) := u^*\left(\omega_N |x|^N\right), \qquad x \in \Omega^\#.$$

Obviously, $u^\#$ is decreasing and spherically symmetric; moreover u and $u^\#$ are equidistributed, and the level set $\left\{x \in \Omega^\# : \left|u^\#(x)\right| > t\right\}$ is the ball centred at the origin and whose measure is $\mu(t)$. For an exhaustive treatment of rearrangements see, for example, [16] and [24]. Here we just recall the Hardy-Littlewood inequality

$$\int_\Omega |u(x)\, v(x)|\, dx \leqslant \int_{\Omega^\#} u^\#(x)\, v^\#(x)\, dx, \tag{11}$$

with u, v measurable functions (see [26]), and the Pólya-Szego principle (see [32])

$$\int_{\mathbb{R}^N} \left|\nabla u^\#(x)\right|^p dx \leqslant \int_{\mathbb{R}^N} |\nabla u(x)|^p\, dx \tag{12}$$

where $u \in W_0^{1,p}\left(\mathbb{R}^N\right)$, $1 < p < N$.

Now we recall the definitions of the Marcinkiewicz L^p-weak space, of Lorentz space and some of their elementary properties which will be used in the following sections.

DEFINITION 3. Let Ω be an open subset of $\mathbb{R}^N$ and $0<p<\infty$. The Marcinkiewicz L^p-weak space consists of all measurable functions u such that

$$\|u\|_{p,\infty} := \sup_{t>0} \omega_N^{\frac{1}{p}} \left[t^{\frac{N}{p}} u^*(t) \right] \tag{13}$$

DEFINITION 4. Let Ω be an open subset of $\mathbb{R}^N$ and $0 < p, q \leqslant \infty$. The Lorentz space $L(p,q)$ consists of all measurable functions u such that

$$\|u\|_{p,q} := \begin{cases} \left(\int_0^{+\infty} \left[u^*(t) t^{\frac{1}{p}} \right]^q \frac{dt}{t} \right)^{\frac{1}{q}}, & 0 < q < \infty \\ \sup_{t>0} u^*(t) t^{\frac{1}{p}}, & q = \infty. \end{cases} \tag{14}$$

The Lorentz space $L(p,p)$ coincides with the Lebesgue space L^p and

$$\|u\|_{p,p} = \|u\|_p .$$

Moreover, it can be proved that the $L(p,\infty)$ space coincides with the Marcinkiewicz L^p- weak space.

Inclusion relations among $L(p,q)$ spaces, with p varying, are like those for the Lebesgue L^p spaces, in that they depend on the structure of the underlying measure space. The secondary exponent q is not involved. Thus, if $0 < p < r \leqslant \infty$ and $0 < q, s \leqslant \infty$ then

$$L(r,s) \hookrightarrow L(p,q). \tag{15}$$

For what concerns the secondary exponent, we have that if if $0 < p \leqslant \infty$ and $0 < q < s \leqslant \infty$, then

$$L(p,q) \hookrightarrow L(p,s). \tag{16}$$

REMARK 1. Thanks to the Sobolev inequality in the Lorentz space $L(p^*,p)$ (see [3]) and Lorentz spaces properties (15) and (16), we get that for $1 < p < N$ and $-p < \nu \leqslant 0$ then

$$W^{1,p}(\mathbb{R}^N) \hookrightarrow L(p^*,p) \hookrightarrow L\left(p^*, \frac{p(N+\nu)}{N-p}\right) \hookrightarrow L^{\frac{p(N+\nu)}{N-p}}(\mathbb{R}^N, |x|^\nu),$$

hence we can conclude that the Egnell inequality is more refined than the Sobolev one.

The main result of the paper is stated in the following:

THEOREM 1. *Let $u \in W^{1,p}(\mathbb{R}^N)$, $1 < p < N$ and $-p < \nu \leqslant 0$. Then inequality (8) holds:*

$$\|u\|_{p^*,\infty}^{rp*} \, \||\nabla u|\|_{L^p}^p \geqslant A(N,p) \, \|u\|_{q,|x|^\nu}^q + B(N,p) \, \|u\|_{p*,\infty}^{sp^*},$$

where $A(N,p)$ and $B(N,p)$ are given in (3) and (10), while q, r, s are given in (4) and (9) respectively.

3. Proof of Theorem 1

In this section we provide a detailed proof of Theorem 1. As stated in the introduction, Egnell inequality will then be deduced from inequality (8).

Proof. The first step consists in the reduction of the problem to a spherically symmetric one. We replace u with $u^{\#}$ and for the sake of simplicity we keep calling it u. By Hardy-Littlewood inequality (11) and Pólya-Szego principle (12), the left hand side of (5) decreases, while the right side increases. This implies that it is enough to prove Theorem 1 only in the radial case. Moreover without loss of generality we can assume that $u \in C_0^1(\mathbb{R}^N)$, since this assumption can be removed by density.

Let us consider the functional

$$J(u) := \frac{\omega_N}{p}\int_0^{\infty} |u'|^p r^{N-1} dr - \frac{\omega_N}{p}\frac{(N-p)^p}{(p-1)^{p-1}} a^{p+\nu} \int_0^{\infty} u^{\frac{p(N+\nu)}{N-p}} r^{N-1+\nu} dr, \tag{17}$$

and the related Euler equation

$$-\triangle_p u = \left(\frac{N-p}{p-1}\right)^{p-1} a^{p+\nu}(N+\nu)u^{\frac{p(N+\nu)}{N-p}-1} r^{\nu}. \tag{18}$$

It can be proved that the following one-parameter family of functions

$$u_{\varepsilon}(r) := u_{\varepsilon}(|x|) = \frac{\varepsilon^{\frac{N-p}{p}}}{\left[1+(a\varepsilon|x|)^{\frac{p+\nu}{p-1}}\right]^{\frac{N-p}{p+\nu}}}, \quad \varepsilon > 0, \tag{19}$$

satisfy (18). The constant $a > 0$ in (19) is a free constant that will be properly chosen in the following.

If we compute the $L^q(\mathbb{R}^N, |x|^{\nu})$-norm of these extremals, we get that

$$\|u_{\varepsilon}\|^q_{q,|x|^{\nu}} = 2\pi^{\frac{N}{2}} \frac{p-1}{p+\nu} a^{-N-\nu} \frac{\Gamma\left(\frac{(N+\nu)(p-1)}{p+\nu}\right)\Gamma\left(\frac{N+\nu}{p+\nu}\right)}{\Gamma\left(\frac{p(N+\nu)}{p+\nu}\right)\Gamma\left(\frac{N}{2}\right)}, \tag{20}$$

so all the functions of the family (19) have the same $L^q(\mathbb{R}^N\,|x|^{\nu})$-norm, which is independent of ε.

The curve

$$y = \gamma_a(r) = \frac{(p-1)^{\frac{(p-1)(N-p)}{p(p+\nu)}}}{p^{\frac{N-p}{p+\nu}}}(ar)^{-\frac{N-p}{p}}, \quad r > 0, \tag{21}$$

is the envelope of the graphs $y = u_{\varepsilon}(r)$; these cover the region T of the first quadrant which lies below the curve (21).

Let u be a sufficiently smooth, compactly supported radial function, and let us consider its norm in the Marcinkiewicz space of the functions weakly L^{p^*}:

$$\|u\|_{p^*,\infty} = \sup_{r>0}\left[r^{\frac{N-p}{p}} v(r)\right].$$

If we choose

$$a = \frac{(p-1)^{\frac{p-1}{p+\nu}}}{p^{\frac{p}{p+\nu}}}\|u\|_{p^*,\infty}^{-\frac{p}{N-p}}, \tag{22}$$

then we get that it is the minimum value such that $u(r) \leqslant \gamma_a(r)$ for all $r > 0$; the corresponding envelope (21) is given by:

$$\gamma(r) = \|u\|_{p^*,\infty} r^{-\frac{N-p}{p}}. \tag{23}$$

For each $\varepsilon > 0$ the graph of the extremal $y = u_\varepsilon(r)$ defined in (19) touches the envelope $\gamma(r)$ defined in (23) at a point which splits it into two curves, denoted with $C_1(\varepsilon)$ and $C_2(\varepsilon)$. By varying ε, $C_1(\varepsilon)$ and $C_2(\varepsilon)$ define two families of curves that are the trajectories of two different fields of extremals of the functional (17), and are both defined in the same region T. Let us denote by $(1, q_1(r,y))$ the first field and by $(1, q_2(r,y))$ the second one. We explicitly stress that $q_1(r,y)$ represents the slope of the extremal of the first family passing through (r,y); $q_2(r,y)$ has an analogous meaning. The dashed lines in Figure 1 (and Figure 2) represent some arcs of extremals $C_1(\varepsilon)$ (and $C_2(\varepsilon)$), obtained by varying ε.
Moreover, the envelope in (23) touches the graph of u at least in a point $P = (\alpha, \gamma(\alpha))$, which splits the graph of u itself into two arcs Γ_1 and Γ_2, as in Figure 1 and Figure 2. Finally, we denote by C_1, C_2, respectively, the arcs of the families $C_1(\varepsilon)$ and $C_2(\varepsilon)$ passing through such $P = (\alpha, \gamma(\alpha))$. In Figure 1 and 2 the graphs of the envelope $y = y(r)$ and the arcs Γ_1 and Γ_2, C_1 and C_2 are sketched in full lines.
At this stage we apply classical arguments of one-dimensional Calculus of Variations (see [31]). Let us denote with

$$f(r,u,u') := \frac{\omega_N}{p} r^{N-1}\left(|u'|^p - \frac{(N-p)^p}{(p-1)^{p-1}} a^{p+\nu} u^q r^\nu\right),$$

then the functional $J(u)$ defined in (17) can be rewritten as

$$J(u) = \int_0^\alpha f(r,u,u')dr + \int_\alpha^\infty f(r,u,u')dr = J_1(u) + J_2(u).$$

Our target is to show that the one parameter family of extremals $y = u_\varepsilon(r)$ minimizes the functional $J(u)$. We begin by estimating $J_1(u)$ from below and we embed it in the first field $(1, q_1(r,y))$.

Since f is convex with respect to the last variable, then the Weierstass condition

$$\mathscr{E}(r,w,\xi,\xi_1) = f(r,w,\xi) - f(r,w,\xi_1) + (\xi_1 - \xi) f_{v'}(r,w,\xi_1) \geqslant 0,$$

is satisfied. As a consequence

$$J_1(u) \geqslant \int_0^{\alpha} f(r,u,q_1) + \left(u' - q_1\right) f_{u'}(r,u,q_1)dr. \tag{24}$$

Moreover, the differential form

$$\zeta_1 = \left[f(r,u,q_1) - q_1 f_{u'}(r,u,q_1)\right] dr + f_{u'}(r,u,q_1)du, \tag{25}$$

is exact (see [31]), so its integral along any closed path is equal to zero. We compute the integral of ζ_1 along the closed path represented in Figure 1 consisting of the graphs of C_1 and of Γ_1 between the origin and α, and the segment τ of the vertical axis delimited by the intersection points of C_1 and Γ_1 with the vertical axis.

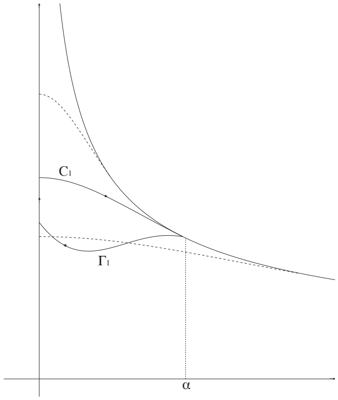

Figure 1

As a consequence, the integral of the right-hand side in (24) equals the line integral of (25) along τ, which is null, plus the integral line along the curve C_1, therefore

$$\int_0^{\alpha} f(r,u,q_1) + \left(u' - q_1\right) f_{u'}(r,u,q_1)dr = \int_0^{\alpha} f(r,u_\varepsilon,u'_\varepsilon) = J_1(u_\varepsilon).$$

This implies that

$$J_1(u) \geqslant J_1(u_\varepsilon). \tag{26}$$

In a similar way we get an estimate from below of $J_2(u)$ and we embed it in the second field $(1, q_2(r,y))$. Let us consider the exact differential form

$$\zeta_2 = \left[f(r,u,q_2) - q_2 f_{u'}(r,u,q_2)\right] dr + f_{u'}(r,u,q_2)du; \tag{27}$$

we integrate ζ_2 between α and β along the path sketched in Figure 2 delimited by C_2, Γ_2, S_β and the segment of the horizontal axis between the intersection points of Γ_2 and S_β with the axis itself.

An asymptotic argument allows us to prove that the line integral of (27) along the

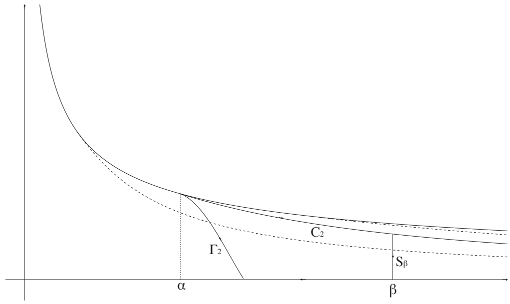

Figure 2

vertical segment S_β in Figure 2 is infinitesimal when β goes to infinity. Therefore

$$J_2(u) \geqslant J_2(u_\varepsilon). \tag{28}$$

From (26) and (28) we get

$$J(u) \geqslant J(u_\varepsilon).$$

Now we compute $J(u_\varepsilon)$. From Egnell inequality and (20) we deduce

$$\begin{aligned} J(u_\varepsilon) &= |||\nabla u_\varepsilon|||^p_{L^p} - \frac{(N-p)^p}{(p-1)^{p-1}} a^{p+\nu} \|u_\varepsilon\|^q_{q,|x|^\nu} = \\ &= C(p,\nu)\,\|u_\varepsilon\|^p_{q,|x|^\nu} - \frac{(N-p)^p}{(p-1)^{p-1}} a^{p+\nu} \|u_\varepsilon\|^q_{q,|x|^\nu} = \\ &= \frac{2}{p}\pi^{\frac{N}{2}} a^{p-N} \frac{(N-p)^{p-1}}{(p-1)^{p-2}} \frac{\Gamma\left(\frac{(N+\nu)(p-1)}{p+\nu}\right)\Gamma\left(\frac{N+\nu}{p+\nu}\right)}{\Gamma\left(\frac{p(N+\nu)}{p+\nu}\right)\Gamma\left(\frac{N}{2}\right)}. \end{aligned}$$

Since $J(u) \geqslant J(u_\varepsilon)$, then

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u|^p\,dx \geqslant &\frac{(N-p)^p}{(p-1)^{p-1}} a^{p+\nu} \|u\|^q_{q,|x|^\nu} + \\ &+ 2\pi^{\frac{N}{2}} a^{p-N} \frac{(N-p)^{p-1}}{(p-1)^{p-2}} \frac{\Gamma\left(\frac{(N+\nu)(p-1)}{p+\nu}\right)\Gamma\left(\frac{N+\nu}{p+\nu}\right)}{\Gamma\left(\frac{p(N+\nu)}{p+\nu}\right)\Gamma\left(\frac{N}{2}\right)}. \end{aligned} \tag{29}$$

Taking into account the value of a in (22) and using a density argument we get the result (8) with $A(N,p)$ and $B(N,p)$ given by (3) and (10) respectively.

4. Conclusions

The inequality (8) can be read as a generalization of the Egnell inequality (5), which can be deduced from it by a minimization argument as follows.

Proof. We start from inequality (29) with $a > 0$ free constant and we rewrite the right-hand side in a more concise way setting $x = a^{p+\nu}$; the inequality becomes:

$$\int_{\mathbb{R}^N} |\nabla u|^p \, dx \geqslant Kx + Hx^{-\frac{N-p}{p+\nu}} = \phi(x),$$

where

$$K = \frac{(N-p)^p}{(p-1)^{p-1}} \, \|u\|_{q,|x|^\nu}^q$$

and

$$H = 2\pi^{\frac{N}{2}} \frac{(N-p)^{p-1}}{(p-1)^{p-2}} \frac{\Gamma\left(\frac{(N+\nu)(p-1)}{p+\nu}\right)\Gamma\left(\frac{N+\nu}{p+\nu}\right)}{\Gamma\left(\frac{p(N+\nu)}{p+\nu}\right)\Gamma\left(\frac{N}{2}\right)}.$$

Since $\phi(x)$ reaches its minimum when

$$x = \left[\frac{H(N-p)}{K(p+\nu)}\right]^{\frac{p+\nu}{N+\nu}},$$

then we obtain the (5) and the optimal value of the Egnell constant (6).

REFERENCES

[1] ADIMURTHI, N. CHAUDHURI AND M. RAMASWAMY, *An improved Hardy-Sobolev inequality and its application*, Proc. Amer. Math. Soc., **130**, (2002), 485–505.

[2] ADIMURTHI, S. FILIPPAS AND A. TERTIKAS, *On the best constant of Hardy-Sobolev inequalities*, Nonlinear Anal., **70**, (2009), 2826–2833.

[3] A. ALVINO, *Sulla diseguaglianza di Sobolev in spazi di Lorentz*, Boll. Un. Mat. Ital. A, **14**, 5 (1977), 148–156.

[4] A. ALVINO, *On a Sobolev-type inequality*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl., **20**, 4 (2009), 379–386.

[5] A. ALVINO, R. VOLPICELLI AND B. VOLZONE, *On Hardy inequalities with a remainder term*, Ric. Mat., **59**, (2010), 265–280.

[6] T. AUBIN, *Problème isopérimétriques et espaces de Sobolev*, J. Diff. Eq., **11**, (1976), 573–598.

[7] G. BARBATIS, S. FILIPPAS AND A. TERTIKAS, *A unified approach to improved L^p Hardy inequalities with best constant*, Trans. Amer. Math. Soc., **356**, 6 (2004), 2169–2196.

[8] G. A. BLISS, *An integral inequality*, J. London Math. Soc., **5**, (1930), 40–46.

[9] H. BREZIS AND E.H. LIEB, *Sobolev inequalities with remainder terms*, J. Funct. Analysis, **62**, (1985), 73–86.

[10] H. BREZIS AND L. NIREMBERG, *Positive solutions of nonlinear equations involving critical Sobolev exponents*, Comm. Pure Appl. Math., **36**, (1983), 437–477.

[11] H. BREZIS AND J.L. VAZQUEZ, *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ. Complut. Madrid, **2**, (10) (1997), 443–469.

[12] X. CABRÉ AND Y. MARTEL, *Weak eigenfunctions for the linearization of extremal elliptic problems*, J. Funct. Anal., **156**, (1998), 30–56.

[13] L. CAFFARELLI, R. KOHN AND L. NIRENBERG, *First order interpolation inequality with weights*, Compositio Math., **53**, (1984), 259–275.

[14] F. CATRINA AND Z. Q. WANG, *On the Caffarelli-Kohn-Niremberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions*, Comm. Pure Appl. Math., **54**, (2001), 229–258.

[15] N. CHAUDHURI AND M. RAMASWAMY, *Existence of positive solutions of some semilinear elliptic equations with singular coefficients*, Proc. Roy. Soc. Edinburgh Sect. A 131, **6**, (2001), 1275–1295.

[16] K. M. CHONG AND N. M. RICE, *Equimeasurable rearrangements of functions*, Queen's Papers in Pure and Applied Mathematics, **28**, Queen's University, Kingston, Ontario, 1971.
[17] S. CUOMO AND A. PERROTTA, *On best constant in Hardy inequalities with a remainder term*, Nonlinear Anal., **74**, (2011), 5784–5792.
[18] H. EGNELL, *Elliptic boundary value problems with singular coefficients and critical nonlinearities*, Indiana Univ. Math. J., **38**, (1989), 235–251.
[19] H. EGNELL, F. PACELLA AND M. TRICARICO, *Some remarks on Sobolev Inequalities*, Nonlinear Anal., **13**, (1989), 671–681.
[20] S. FILIPPAS, V. G. MAZ'JA AND A. TERTIKAS, *Sharp Hardy-Sobolev inequalities*, C. R. Math. Acad. Sci. Paris, **339**, (2004), 483–486.
[21] S. FILIPPAS AND A. TERTIKAS, *Optimizing improved Hardy inequalities*, J. Funct. Anal., **192**, (2002), 186–233, J. Funct. Anal., **255**, (2008), 2095 (Corrigendum).
[22] F. GAZZOLA, H. C. GRANAU AND E. MITIDIERI, *Hardy inequalities with optimal constants and remainder terms*, Trans. Amer. Math. Soc., **356**, 6 (2004), 2149–2168.
[23] N. GHOUSSOUB AND A. MORADIFAM, *On the best possible remaining term in the Hardy inequality*, Proc. Natl. Acad. Sci. USA, **105**, (2008), 13746–13751.
[24] B. KAWHOL, *Rearrangements and convexity of level sets in P.D.E.*, Lecture Notes in Mathematics, **1150**, Springer, Berlin, 1985.
[25] G. H. HARDY, *Notes on some points in the integral calculus*, Messenger Math., **48**, (1919), 107–112.
[26] G. H. HARDY, J. E. LITTLEWOOD AND G. POLYA, *Inequalities*, Cambridge University Press, Cambridge, 1964.
[27] T. HORIUCHI, *Best constant in weighted Sobolev inequality*, Proc. Japan Acad. Ser. A Math. Sci., **72**, (1996), 208–211.
[28] V. G. MAZ'JA, *Sobolev Spaces*, Springer-Verlag, Berlin, 1986.
[29] A. PERROTTA AND B. VOLZONE, *A note on a Sobolev Inequality with a remainder term for functions vanishing on part of the boundary*, Rend. Acc. Sci. fis. Mat. Napoli, **LXXIV**, (2007), 35–50.
[30] V. RADULESKU, D. SMETS AND M. WILLEM, *Hardy-Sobolev inequalities with a remainder term*, Topol. Methods Nonlinear Anal., **20**, 1 (2002), 145–149.
[31] H. SAGAN, *Introduction to the Calculus of Variations*, Dover Pubblications, Inc., New York, 1969.
[32] G. TALENTI, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl., **110**, 4 (1976), 353–372.
[33] J.L. VAZQUEZ, *Domain of existence and blow-up for the exponential reaction diffusion equation*, Indiana Univ. Math. J., **48**, (1999), 677–709.
[34] J.L. VAZQUEZ AND E. ZUAZUA, *The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse square potential*, J. Funct. Anal., **173** (2000), 103–153.

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