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NORM INEQUALITIES OF DAVIDSON–POWER TYPE

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Abstract. Let A,B, and X be $n\times n$ complex matrices such that A and B are positive semidefinite. It is shown, among other inequalities, that

$$\|AX+XB\| \leqslant \frac{1}{2}\max(\|A\|,\|XBX^*\|)+\frac{1}{2}\max(\|X^*AX\|,\|B\|)+\left\|A^{1/2}XB^{1/2}\right\|.$$

This norm inequality extends an inequality of Kittaneh, which improves an earlier inequality of Davidson and Power.

1. Introduction

Let $\mathbb{M}_n(\mathbb{C})$ denote the algebra of all $n\times n$ complex matrices. For $A\in\mathbb{M}_n(\mathbb{C})$, let $s_1(A),s_2(A),\ldots,s_n(A)$ denote the singular values of A (i.e., the eigenvalues of $|A|=(A^*A)^{1/2}$) arranged in decreasing order and repeated according to multiplicity.

Let $|||.|||$ denote any unitarily invariant norm on $\mathbb{M}_n(\mathbb{C})$ (and its extension on $\mathbb{M}_{2n}(\mathbb{C})$). Every unitarily invariant norm satisfies the invariance property $|||UAV|||=|||A|||$ for all $A\in\mathbb{M}_n(\mathbb{C})$ and for all unitary matrices $U,V\in\mathbb{M}_n(\mathbb{C})$. It is known that unitarily invariant norms are increasing symmetric gauge functions of singular values (see, e.g., [1] or [7]). Among the most important examples of unitarily invariant norms are the Schatten p-norms $\|.\|_p$, defined by $\|A\|_p=\left(\sum_{j=1}^{n}s_j^p(A)\right)^{1/p}$ for $1\leqslant p\leqslant\infty$, which include the spectral (operator) norm $\|.\|$ corresponding to the case $p=\infty$. Thus, $\|A\|=s_1(A)$.

Kittaneh [9] proved that if $A,B\in\mathbb{M}_n(\mathbb{C})$ are positive semidefinite, then

$$\|A+B\| \leqslant \max(\|A\|,\|B\|)+\left\|A^{1/2}B^{1/2}\right\|. \tag{1}$$

A weaker version of the inequality (1), where $\left\|A^{1/2}B^{1/2}\right\|$ is replaced by $\|AB\|^{1/2}$ has been given in Davidson and Power [6], and an equivalent formulation of the inequality (1) has been recently given in [5].

An improvement of the inequality (1) has been given in [10] so that

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$$||A+B|| \leqslant \frac{1}{2}\left(\|A\|+\|B\|+\sqrt{(\|A\|-\|B\|)^2+4\left\|A^{1/2}B^{1/2}\right\|^2}\right), \tag{2}$$

which also improves the triangle inequality for norms of positive semidefinite matrices.

Moreover, Kittaneh [9] proved that if $A,B \in \mathbb{M}_n(\mathbb{C})$ are positive semidefinite, then

$$\|A+B\|_p \leqslant \left(\|A\|_p^p+\|B\|_p^p\right)^{1/p}+2^{1/p}\left\|A^{1/2}B^{1/2}\right\|_p \tag{3}$$

for $1 \leqslant p \leqslant \infty$.

In this paper, we give a considerable generalization of the inequality (1). Our new result involves arbitrary unitarily invariant norms and concave increasing functions. Generalizations of the inequalities (2) and (3) are also given. Finally, we present a relevant singular value inequality involving increasing convex functions.

2. Main results

In our analysis, we need the following lemmas. The first lemma is a consequence of the spectral theorem (see, e.g., [1, p. 5]). For the second lemma, we refer to [2]. The third lemma can be found in [4]. Throughout this paper, all functions are assumed to be continuous.

LEMMA 1. *Let* $A \in \mathbb{M}_n(\mathbb{C})$ *and let* f *be a nonnegative increasing function on* $[0,\infty)$. *Then*

$$s_j\left(f\left(|A|\right)\right)=f\left(s_j\left(A\right)\right)$$

for $j=1,2,\ldots,n$.

LEMMA 2. *Let* $A,B \in \mathbb{M}_n(\mathbb{C})$. *Then*

$$s_j(AB^*) \leqslant \frac{1}{2}s_j(A^*A+B^*B)$$

for $j=1,2,\ldots,n$.

LEMMA 3. *Let* $A_1,\ldots,A_m \in \mathbb{M}_n(\mathbb{C})$ *be normal and let* f *be a nonnegative concave function on* $[0,\infty)$. *Then*

$$|||f(|A_1+\ldots+A_m|)||| \leqslant |||f(|A_1|)+\ldots+f(|A_m|)|||$$

for every unitarily invariant norm.

The following is our first main result.

THEOREM 1. *Let* $A,B,X \in \mathbb{M}_n(\mathbb{C})$ *such that* A *and* B *are positive semidefinite and let* f *be a nonnegative concave function on* $[0,\infty)$. *Then*

$$|||f(|(AX+XB)\oplus 0|)||| \tag{4}$$

$$\leqslant \left|\left|\left| f\left(\frac{1}{2}A\right) \oplus f\left(\frac{1}{2}XBX^*\right) \right|\right|\right| + \left|\left|\left| f\left(\frac{1}{2}X^*AX\right) \oplus f\left(\frac{1}{2}B\right) \right|\right|\right|$$
$$+ \left|\left|\left| f\left(\left|A^{1/2}XB^{1/2}\right|\right) \oplus f\left(\left|A^{1/2}XB^{1/2}\right|\right) \right|\right|\right|$$

for every unitarily invariant norm.

Proof. Let $S = \begin{bmatrix} A^{1/2} & XB^{1/2} \\ 0 & 0 \end{bmatrix}$ and $T^* = \begin{bmatrix} A^{1/2}X & 0 \\ B^{1/2} & 0 \end{bmatrix}$. Then, for $j = 1,2,...,2n$, we have

$$\begin{aligned}
&s_j(f(|(AX+XB)\oplus 0|)) \\
&= s_j(f(|ST^*|)) = f(s_j(ST^*)) \text{ (by Lemma 1)} \\
&\leqslant f\left(\frac{1}{2}s_j(S^*S+T^*T)\right) \text{ (by Lemma 2)} \\
&= f\left(s_j\left(\tfrac{1}{2}\begin{bmatrix} A & A^{1/2}XB^{1/2} \\ B^{1/2}X^*A^{1/2} & B^{1/2}|X|^2B^{1/2} \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} A^{1/2}|X^*|^2A^{1/2} & A^{1/2}XB^{1/2} \\ B^{1/2}X^*A^{1/2} & B \end{bmatrix}\right)\right) \\
&= f\left(s_j\left(K+T+Y\right)\right) = s_j(f\left(|K+T+Y|\right)),
\end{aligned}$$

where $K = \begin{bmatrix} \frac{1}{2}A & 0 \\ 0 & \frac{1}{2}B^{1/2}|X|^2B^{1/2} \end{bmatrix}$, $T = \begin{bmatrix} \frac{1}{2}A^{1/2}|X^*|^2A^{1/2} & 0 \\ 0 & \frac{1}{2}B \end{bmatrix}$, and
$Y = \begin{bmatrix} 0 & A^{1/2}XB^{1/2} \\ B^{1/2}X^*A^{1/2} & 0 \end{bmatrix}$.

Since unitarily invariant norms are increasing functions of singular values, it follows that

$$\begin{aligned}
&|||f(|(AX+XB)\oplus 0|)||| \\
&\leqslant |||f(|K+T+Y|)||| \\
&\leqslant |||f\left(|K|\right)+f\left(|T|\right)+f\left(|Y|\right)|||\text{(by Lemma 3)} \\
&\leqslant |||f(|K|)|||+|||f(|T|)|||+|||f(|Y|)||| \\
&= \left|\left|\left| f\left(\frac{1}{2}A \oplus \frac{1}{2}B^{1/2}|X|^2B^{1/2}\right) \right|\right|\right| + \left|\left|\left| f\left(\frac{1}{2}A^{1/2}|X^*|^2A^{1/2} \oplus \frac{1}{2}B\right) \right|\right|\right| \\
&\quad + \left|\left|\left| f\left(\left|\begin{bmatrix} 0 & A^{1/2}XB^{1/2} \\ B^{1/2}X^*A^{1/2} & 0 \end{bmatrix}\right|\right) \right|\right|\right| \\
&= \left|\left|\left| f\left(\frac{1}{2}A\right) \oplus f\left(\frac{1}{2}B^{1/2}|X|^2B^{1/2}\right) \right|\right|\right| + \left|\left|\left| f\left(\frac{1}{2}A^{1/2}|X^*|^2A^{1/2}\right) \oplus f\left(\frac{1}{2}B\right) \right|\right|\right| \\
&\quad + \left|\left|\left| f\left(\begin{bmatrix} |B^{1/2}X^*A^{1/2}| & 0 \\ 0 & |A^{1/2}XB^{1/2}| \end{bmatrix}\right) \right|\right|\right| \\
&= \left|\left|\left| f\left(\frac{1}{2}A\right) \oplus f\left(\frac{1}{2}B^{1/2}|X|^2B^{1/2}\right) \right|\right|\right| + \left|\left|\left| f\left(\frac{1}{2}A^{1/2}|X^*|^2A^{1/2}\right) \oplus f\left(\frac{1}{2}B\right) \right|\right|\right| \\
&\quad + \left|\left|\left| f\left(\left|B^{1/2}X^*A^{1/2}\right|\right) \oplus f\left(\left|A^{1/2}XB^{1/2}\right|\right) \right|\right|\right|
\end{aligned}$$

$$= \left|\left|\left| f\left(\frac{1}{2}A\right) \oplus f\left(\frac{1}{2}XBX^*\right) \right|\right|\right| + \left|\left|\left| f\left(\frac{1}{2}X^*AX\right) \oplus f\left(\frac{1}{2}B\right) \right|\right|\right|$$
$$+ \left|\left|\left| f\left(\left|A^{1/2}XB^{1/2}\right|\right) \oplus f\left(\left|A^{1/2}XB^{1/2}\right|\right) \right|\right|\right|,$$

as required. □

Now, specializing the inequality (4) for the usual operator norm and the Schatten $p-$norms, we have the following two corollaries.

COROLLARY 1. *Let $A,B,X \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive semidefinite. Then*

$$\|AX + XB\| \leqslant \frac{1}{2}\max(\|A\|, \|XBX^*\|) + \frac{1}{2}\max(\|X^*AX\|, \|B\|) + \left\|A^{1/2}XB^{1/2}\right\|.$$

Proof. The result follows from Theorem 1 by letting $f(t) = t$ and by considering the spectral norm. □

COROLLARY 2. *Let $A,B,X \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive semidefinite, $1 \leqslant p \leqslant \infty$. Then*

$$\|AX + XB\|_p \leqslant \left(\left\|\frac{1}{2}A\right\|_p^p + \left\|\frac{1}{2}XBX^*\right\|_p^p\right)^{1/p} + \left(\left\|\frac{1}{2}B\right\|_p^p + \left\|\frac{1}{2}X^*AX\right\|_p^p\right)^{1/p}$$
$$+2^{1/p}\left\|A^{1/2}XB^{1/2}\right\|_p.$$

Proof. The result follows from Theorem 1 by letting $f(t) = t$ and by considering the Schatten p-norms. □

The above corollaries represent generalizations of the inequalities (1) and (3), which can be retained by letting $X = I$. Specializing Theorem 1 for certain types of functions, enables us to get the following two corollaries.

COROLLARY 3. *Let $A,B,X \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive semidefinite. Then*

$$|||\log(|(AX + XB)| + I)|||$$
$$\leqslant \left|\left|\left| \log\left(\frac{1}{2}A + I\right) \oplus \log\left(\frac{1}{2}XBX^* + I\right) \right|\right|\right| + \left|\left|\left| \log\left(\frac{1}{2}X^*AX + I\right) \oplus \log\left(\frac{1}{2}B + I\right) \right|\right|\right|$$
$$+ \left|\left|\left| \log\left(\left|A^{1/2}XB^{1/2}\right| + I\right) \oplus \log\left(\left|A^{1/2}XB^{1/2}\right| + I\right) \right|\right|\right|$$

for every unitarily invariant norm.

Proof. The result follows from Theorem 1 by letting $f(t) = \log(t + 1)$. □

COROLLARY 4. *Let $A,B,X\in\mathbb{M}_n(\mathbb{C})$ such that A and B are positive semidefinite. Then, for $r\in(0,1]$, we have*

$$\begin{aligned}&|||\ |(AX+XB)|^r|||\\ \leqslant\ &\left|\left|\left|\left(\frac{1}{2}A\right)^r\oplus\left(\frac{1}{2}XBX^*\right)^r\right|\right|\right|+\left|\left|\left|\left(\frac{1}{2}X^*AX\right)^r\oplus\left(\frac{1}{2}B\right)^r\right|\right|\right|\\ &+\left|\left|\left|\left|A^{1/2}XB^{1/2}\right|^r\oplus\left|A^{1/2}XB^{1/2}\right|^r\right|\right|\right|\end{aligned}$$

for every unitarily invariant norm.

Proof. The result follows from Theorem 1 by letting $f(t)=t^r$, $r\in(0,1]$. □

A generalization of the inequality (2) is given in the following theorem, which is based on a folklore lemma (see, e.g., [8]).

LEMMA 4. *Let $A,B,C,D\in\mathbb{M}_n(\mathbb{C})$. Then*

$$\left\|\begin{bmatrix}A & B\\ C & D\end{bmatrix}\right\|\leqslant\left\|\begin{bmatrix}\|A\| & \|B\|\\ \|C\| & \|D\|\end{bmatrix}\right\|.$$

THEOREM 2. *Let $A,B,X\in\mathbb{M}_n(\mathbb{C})$ such that A and B are positive semidefinite. Then*

$$\|AX+XB\|\leqslant\frac{1}{4}\left[\begin{array}{c}\left\|A+A^{1/2}|X^*|^2A^{1/2}\right\|+\left\|B+B^{1/2}|X|^2B^{1/2}\right\|\\ +\sqrt{\left(\left\|A+A^{1/2}|X^*|^2A^{1/2}\right\|-\left\|B+B^{1/2}|X|^2B^{1/2}\right\|\right)^2+16\left\|A^{1/2}XB^{1/2}\right\|}\end{array}\right].$$

Proof. Let $S=\begin{bmatrix}A^{1/2} & XB^{1/2}\\ 0 & 0\end{bmatrix}$ and $T^*=\begin{bmatrix}A^{1/2}X & 0\\ B^{1/2} & 0\end{bmatrix}$. Then

$$\begin{aligned}\|AX+XB\|&=\|ST^*\|\leqslant\frac{1}{2}\|S^*S+T^*T\|\\ &=\frac{1}{2}\left\|\begin{bmatrix}A & A^{1/2}XB^{1/2}\\ B^{1/2}X^*A^{1/2} & B^{1/2}|X|^2B^{1/2}\end{bmatrix}+\begin{bmatrix}A^{1/2}|X^*|^2A^{1/2} & A^{1/2}XB^{1/2}\\ B^{1/2}X^*A^{1/2} & B\end{bmatrix}\right\|\\ &=\frac{1}{2}\left\|\begin{bmatrix}A+A^{1/2}|X^*|^2A^{1/2} & 2A^{1/2}XB^{1/2}\\ 2B^{1/2}X^*A^{1/2} & B+B^{1/2}|X|^2B^{1/2}\end{bmatrix}\right\|\\ &\leqslant\frac{1}{2}\left\|\begin{bmatrix}\left\|A+A^{1/2}|X^*|^2A^{1/2}\right\| & 2\left\|A^{1/2}XB^{1/2}\right\|\\ 2\left\|B^{1/2}X^*A^{1/2}\right\| & \left\|B+B^{1/2}|X|^2B^{1/2}\right\|\end{bmatrix}\right\|\\ &\qquad\text{(by Lemma 4)}\\ &=\frac{1}{4}\left[\begin{array}{c}\left\|A+A^{1/2}|X^*|^2A^{1/2}\right\|+\left\|B+B^{1/2}|X|^2B^{1/2}\right\|\\ +\sqrt{\left(\left\|A+A^{1/2}|X^*|^2A^{1/2}\right\|-\left\|B+B^{1/2}|X|^2B^{1/2}\right\|\right)^2+16\left\|A^{1/2}XB^{1/2}\right\|}\end{array}\right].\end{aligned}$$

Using an argument similar to that used in the proof of Theorem 1, we have the following norm inequality. $\square$

THEOREM 3. *Let $A,B,X \in \mathbb{M}_n(\mathbb{C})$ such that A and B are positive semidefinite and let f be a nonnegative concave function on $[0,\infty)$. Then*

$$|||f(|(AX-XB)\oplus 0|)||| \leqslant \left|\left|\left|f(\frac{1}{2}A)\oplus f(\frac{1}{2}XBX^*)\right|\right|\right| + \left|\left|\left|f(\frac{1}{2}X^*AX)\oplus f(\frac{1}{2}B)\right|\right|\right|$$

for every unitarily invariant norm.

Proof. Let $S=\begin{bmatrix} A^{1/2} & XB^{1/2} \\ 0 & 0 \end{bmatrix}$ and $R^*=\begin{bmatrix} A^{1/2}X & 0 \\ -B^{1/2} & 0 \end{bmatrix}$. Then, for $j=1,2,...,2n$, we have

$$\begin{aligned}
& s_j(f(|(AX-XB)\oplus 0|)) \\
&= s_j(f(|SR^*|)) = f(s_j(SR^*)) \text{ (by Lemma 1)} \\
&\leqslant f\left(\frac{1}{2}s_j(S^*S+R^*R)\right) \text{ (by Lemma 2)} \\
&= f\left(s_j\left(\tfrac{1}{2}\begin{bmatrix} A & A^{1/2}XB^{1/2} \\ B^{1/2}X^*A^{1/2} & B^{1/2}|X|^2B^{1/2} \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} A^{1/2}|X^*|^2A^{1/2} & -A^{1/2}XB^{1/2} \\ -B^{1/2}X^*A^{1/2} & B \end{bmatrix}\right)\right) \\
&= f\left(s_j\left(\begin{bmatrix} \tfrac{1}{2}A & 0 \\ 0 & \tfrac{1}{2}B^{1/2}|X|^2B^{1/2} \end{bmatrix} + \begin{bmatrix} \tfrac{1}{2}A^{1/2}|X^*|^2A^{1/2} & 0 \\ 0 & \tfrac{1}{2}B \end{bmatrix}\right)\right) \\
&= s_j\left(f\left(\begin{bmatrix} \tfrac{1}{2}A & 0 \\ 0 & \tfrac{1}{2}B^{1/2}|X|^2B^{1/2} \end{bmatrix} + \begin{bmatrix} \tfrac{1}{2}A^{1/2}|X^*|^2A^{1/2} & 0 \\ 0 & \tfrac{1}{2}B \end{bmatrix}\right)\right).
\end{aligned}$$

Since unitarily invariant norms are increasing functions of singular values, it follows that

$$\begin{aligned}
& |||f(|(AX-XB)\oplus 0|)||| \\
&\leqslant \left|\left|\left|f\left(\begin{bmatrix} \tfrac{1}{2}A & 0 \\ 0 & \tfrac{1}{2}B^{1/2}|X|^2B^{1/2} \end{bmatrix} + \begin{bmatrix} \tfrac{1}{2}A^{1/2}|X^*|^2A^{1/2} & 0 \\ 0 & \tfrac{1}{2}B \end{bmatrix}\right)\right|\right|\right| \\
&\leqslant \left|\left|\left|f\left(\begin{bmatrix} \tfrac{1}{2}A & 0 \\ 0 & \tfrac{1}{2}B^{1/2}|X|^2B^{1/2} \end{bmatrix}\right) + f\left(\begin{bmatrix} \tfrac{1}{2}A^{1/2}|X^*|^2A^{1/2} & 0 \\ 0 & \tfrac{1}{2}B \end{bmatrix}\right)\right|\right|\right| \quad \text{(by Lemma 3)} \\
&\leqslant \left|\left|\left|f\left(\begin{bmatrix} \tfrac{1}{2}A & 0 \\ 0 & \tfrac{1}{2}B^{1/2}|X|^2B^{1/2} \end{bmatrix}\right)\right|\right|\right| + \left|\left|\left|f\left(\begin{bmatrix} \tfrac{1}{2}A^{1/2}|X^*|^2A^{1/2} & 0 \\ 0 & \tfrac{1}{2}B \end{bmatrix}\right)\right|\right|\right| \\
&= \left|\left|\left|f\left(\frac{1}{2}A\right)\oplus f\left(\frac{1}{2}B^{1/2}|X|^2B^{1/2}\right)\right|\right|\right| + \left|\left|\left|f\left(\frac{1}{2}A^{1/2}|X^*|^2A^{1/2}\right)\oplus f\left(\frac{1}{2}B\right)\right|\right|\right| \\
&= \left|\left|\left|f\left(\frac{1}{2}A\right)\oplus f\left(\frac{1}{2}XBX^*\right)\right|\right|\right| + \left|\left|\left|f\left(\frac{1}{2}X^*AX\right)\oplus f\left(\frac{1}{2}B\right)\right|\right|\right|,
\end{aligned}$$

as required. $\square$

Letting $f(t)=t$ and $X=I$ in Theorem 3, we obtain an earlier norm inequality of Bhatia and Kittaneh [2], which says that

$$|||(A-B)\oplus 0||| \leqslant |||A\oplus B|||. \tag{5}$$

For extensions and generalizations of the inequality (5), we refer to [11] and [12].

An application of Theorem 3 for special types of functions can be seen in the following two corollaries.

COROLLARY 5. *Let $A,B,X\in\mathbb{M}_n(\mathbb{C})$ such that A and B are positive semidefinite. Then*

$$\begin{aligned}&|||\ \log\left(|(AX-XB)|+I\right)|||\\ \leqslant &\left|\left|\left|\log\left(\frac{1}{2}A+I\right)\oplus\log\left(\frac{1}{2}XBX^*+I\right)\right|\right|\right|+\left|\left|\left|\log\left(\frac{1}{2}X^*AX+I\right)\oplus\log\left(\frac{1}{2}B+I\right)\right|\right|\right|\end{aligned}$$

for every unitarily invariant norm.

Proof. The result follows from Theorem 3 by letting $f(t)=\log(t+1)$. $\square$

COROLLARY 6. *Let $A,B,X\in\mathbb{M}_n(\mathbb{C})$ such that A and B are positive semidefinite. Then, for $r\in(0,1]$, we have*

$$|||\ |(AX-XB)|^r|||\leqslant\left|\left|\left|\left(\frac{1}{2}A\right)^r\oplus\left(\frac{1}{2}XBX^*\right)^r\right|\right|\right|+\left|\left|\left|\left(\frac{1}{2}X^*AX\right)^r\oplus\left(\frac{1}{2}B\right)^r\right|\right|\right|$$

for every unitarily invariant norm.

Proof. The result follows from Theorem 3 by letting $f(t)=t^r$, $r\in(0,1]$. $\square$

The last inequality in this paper is a singular value inequality based on the following lemma, which can be found in [3]

LEMMA 5. *Let $A_1,\dots,A_m\in\mathbb{M}_n(\mathbb{C})$ be Hermitian and let f be increasing convex function on an interval $[a,b]$ containing the spectra of A_i, $i=1,\dots,m$. If Z_i , $i=1,\dots,m$, is an isometric column, (i.e., if $\sum\limits_{i=1}^{m}Z_i^*Z_i=I$), then there exists a unitary matrix U such that*

$$f\left(\sum_{i=1}^{m}Z_i^*A_iZ_i\right)\leqslant U\left(\sum_{i=1}^{m}Z_i^*f(A_i)Z_i\right)U^*.$$

Our inequality can be seen in the following theorem.

THEOREM 4. *Let $A,B,X,Y\in\mathbb{M}_n(\mathbb{C})$ such that $|X|^2+|Y|^2=I$ and let f be a nonnegative increasing convex function on $[0,\infty)$. Then*

$$\max(s_j(f(X^*|A|X),s_j(f(Y^*|B|Y))\leqslant s_j\left(X^*f(|A|)X+Y^*f(|B|)Y\right)$$

for $j=1,2,\dots,n$.

Proof. For $j = 1,2,...,n,$ we have

$$\begin{aligned} s_j(f(X^*|A|X)) &= f(s_j(X^*|A|X)) && \text{(by Lemma 1)} \\ &\leqslant f(s_j(X^*|A|X+Y^*|B|Y)) && \text{(by the Weyl monotonocity principle)} \\ &= s_j(f(X^*|A|X+Y^*|B|Y)) \\ &\leqslant s_j\left(X^*f(|A|)X+Y^*f(|B|)Y\right) && \text{(by Lemma 5).} \end{aligned}$$

Similarly,

$$s_j(f(Y^*|B|Y)) \leqslant s_j(X^*f(|A|)X+Y^*f(|B|)Y),$$

and so

$$\max(s_j(f(X^*|A|X), s_j(f(Y^*|B|Y)) \leqslant s_j\left(X^*f(|A|)X+Y^*f(|B|)Y\right),$$

as required. □

COROLLARY 7. *Let* $A,B,X,Y \in \mathbb{M}_n(\mathbb{C})$ *such that* $|X|^2+|Y|^2 = I$. *Then, for* $r \in [1,\infty)$, *we have*

$$\max(s_j\left(X^*|A|X\right)^r, s_j\left(Y^*|B|Y\right)^r) \leqslant s_j\left(X^*|A|^r X+Y^*|B|^r Y\right)$$

for $j = 1,2,...,n.$

Proof. The result follows from Theorem 4 by letting $f(t) = t^r,\ r \in [1,\infty)$. □

COROLLARY 8. *Let* $A,B,X,Y \in \mathbb{M}_n(\mathbb{C})$ *such that* $|X|^2+|Y|^2 = I$. *Then*

$$\max(s_j\left(e^{X^*|A|X}-I\right), s_j\left(e^{Y^*|B|Y}-I\right)) \leqslant s_j\left(e^{X^*|A|X}+e^{Y^*|B|Y}-2I\right)$$

for $j = 1,2,...,n.$

Proof. The result follows from Theorem 4 by letting $f(t) = e^t-1$. □

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