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ON EXTREMALS FOR THE TRUDINGER–MOSER INEQUALITY WITH VANISHING WEIGHT IN THE N–DIMENSIONAL UNIT BALL

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Abstract. In this paper, we study the extremal function for the Trudinger-Moser inequality with vanishing weight in the unit ball $\mathbb{B} \subset \mathbb{R}^N$ $(N \geqslant 3)$. To be exact, let $\mathscr{S}$ be the set of all decreasing radially symmetrical functions and $\alpha_N = N\omega_{N-1}^{1/(N-1)}$, where ω_{N-1} is the area of the unit sphere in $\mathbb{R}^N$. Suppose h is a nonnegative radially symmetrical function belonging to $C^0(\overline{\mathbb{B}})$ satisfying $h(x) > 0$ in $\overline{\mathbb{B}} \setminus \{0\}$ and $h(x)|x|^{-N\beta} \to 1$ as $x \to 0$ for some real number $\beta \geqslant 0$. By means of blow-up analysis, we prove that the supremum

$$\Lambda_\beta := \sup_{u \in W_0^{1,N}(\mathbb{B}) \cap \mathscr{S},\, \|\nabla u\|_N \leqslant 1} \int_{\mathbb{B}} \exp\left\{\alpha_N (1+\beta) |u|^{\frac{N}{N-1}}\right\} h(x)\, dx$$

can be attained by some $u_0 \in W_0^{1,N}(\mathbb{B}) \cap \mathscr{S}$ with $\|\nabla u_0\|_N = 1$. This improves a recent result of Yang-Zhu [39].

1. Introduction and main results

Let $\Omega \subseteq \mathbb{R}^N (N \geqslant 2)$ be a smooth bounded domain and $W_0^{1,N}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ under the Sobolev norm

$$\|\nabla u\|_N = \left(\int_\Omega |\nabla u|^N dx\right)^{1/N},$$

where $\|\cdot\|_N$ denotes the standard L^N-norm and ∇ denotes the gradient operator. Let $\alpha_N = N\omega_{N-1}^{1/(N-1)}$, where ω_{N-1} represents the area of the unit sphere in $\mathbb{R}^N$. Then the classical Trudinger-Moser inequality [26, 30, 31, 33, 40], as a limit case of the Sobolev embeddings, says

$$\sup_{u \in W_0^{1,N}(\Omega),\, \|\nabla u\|_N \leqslant 1} \int_\Omega \exp\left\{\alpha |u|^{\frac{N}{N-1}}\right\} dx < \infty, \quad \forall\, \alpha \leqslant \alpha_N. \tag{1}$$

When $\alpha > \alpha_N$, all integrals in (1) are still finite, but the supremum is infinite. In this sense, α_N is called the best constant of this inequality. While the existence of extremal functions for it was solved by Carleson-Chang [6], Flucher [15], Lin [23].

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Through a change of variables and a symmetrization argument, the Trudinger-Moser inequality (1) was extended by Adimurthi-Sandeep [2] to a singular version, namely

$$\sup_{u\in W_0^{1,N}(\Omega),\|\nabla u\|_N\leqslant 1}\int_\Omega \exp\left\{\alpha_N\gamma|u|^{\frac{N}{N-1}}\right\}|x|^{N\beta}dx<\infty,\ \forall\ -1<\beta\leqslant 0,\ 0<\gamma\leqslant 1+\beta. \tag{2}$$

When $\gamma > 1+\beta$, all integrals in (2) are still finite, but the supremum is infinite. Thus $\alpha_N(1+\beta)$ is the best constant of (2). Later, (2) was generalized to the case of whole Euclidean space $\mathbb{R}^N$ by Adimurthi-Yang [3] via the Young inequality and the Hardy-Littlewood inequality. When $N=2$, the existence of extremals for (2) was obtained by Casto-Roy [7], Iula-Mancini [17], Li-Yang [19] and Yang-Zhu [38]. Note that (2) reduces to (1) when $\beta=0$, but (2) does not hold any more when $\beta>0$.

Let $\mathscr{S}$ be the set of all decreasing radially symmetric functions and $\mathbb{B}$ be the unit ball in $\mathbb{R}^N$. For the case of $N=2$, de Figueiredo-do Ó-dos Santos [10] replaced the function space $W_0^{1,N}(\mathbb{B})\cap\mathscr{S}$ with $W_0^{1,N}(\Omega)$ in (2), and obtained

$$\sup_{u\in W_0^{1,2}(\mathbb{B})\cap\mathscr{S},\|\nabla u\|_2\leqslant 1}\int_{\mathbb{B}} \exp\left\{4\pi(1+\beta)u^2\right\}|x|^{2\beta}dx<\infty,\ \forall\ \beta\geqslant 0. \tag{3}$$

Moreover, extremals of the above supremum exist. It was generalized by Yang-Zhu [39] to higher dimensional case. In particular, for any $\beta\geqslant 0$, the supremum

$$\sup_{u\in W_0^{1,N}(\mathbb{B})\cap\mathscr{S},\|\nabla u\|_N\leqslant 1}\int_{\mathbb{B}} \exp\left\{\alpha_N(1+\beta)|u|^{\frac{N}{N-1}}\right\}|x|^{N\beta}dx \tag{4}$$

can be attained.

We suppose that h is a nonnegative and radially symmetrical function belonging to $C^0(\overline{\mathbb{B}})$ satisfying $h(x)>0$ in $\overline{\mathbb{B}}\setminus\{0\}$ and $h(x)|x|^{-N\beta}\to 0$ as $x\to 0$ for some $\beta\geqslant 0$. In this paper, we consider more general weight $h(x)$ instead of $|x|^{N\beta}$ in (4). Our main result reads

THEOREM 1. *Let $N\geqslant 3$, $\beta\geqslant 0$, $\mathscr{S}$ be the set of all decreasing radially symmetrical functions, $\mathbb{B}$ be the unit ball in $\mathbb{R}^N$ and $\alpha_N=N\omega_{N-1}^{1/(N-1)}$, where ω_{N-1} is the area of the unit sphere in $\mathbb{R}^N$. Suppose h is a nonnegative and radially symmetrical function belonging to $C^0(\overline{\mathbb{B}})$ satisfying $h(x)>0$ in $\overline{\mathbb{B}}\setminus\{0\}$ and*

$$\lim_{x\to 0}\frac{h(x)}{|x|^{N\beta}}=1. \tag{5}$$

Then the supremum

$$\Lambda_\beta:=\sup_{u\in W_0^{1,N}(\mathbb{B})\cap\mathscr{S},\|\nabla u\|_N\leqslant 1}\int_{\mathbb{B}} \exp\left\{\alpha_N(1+\beta)|u|^{\frac{N}{N-1}}\right\}h(x)\,dx \tag{6}$$

can be attained by some nonnegative function $u_0\in W_0^{1,N}(\mathbb{B})\cap\mathscr{S}$ with $\|\nabla u_0\|_N=1$.

The structure of the proof of Theorem 1 is as follows: Firstly, we discuss the asymptotic behavior of maximizers for subcritical Trudinger-Moser functionals by means of blow-up analysis, which was originally used by Adimurthi-Struwe [2], Carleson-Chang [6], Ding-Jost-Li-Wang [11], Li [20, 21], and widely used by do Ó-de Souza [12, 13], Li [18], Li-Yang [19], Li-Ruf [22], Lu-Yang [25], Nguyen [27, 28], Yang [34, 35, 36, 37], Zhu [41], Fang-Zhang [14] and others. Secondly, we derive an upper bound of Λ_β defined as in (6). Finally, we construct a sequence of functions to reach a contradiction. Throughout this paper, we do not distinguish sequence and subsequence.

2. Blow-up analysis

In this section, we first consider the existence of extremals and its Euler-Lagrange equation. Let $N \geqslant 3$, $\beta \geqslant 0$ be fixed, and $\mathbb{B}$ be the unit ball in $\mathbb{R}^N$. According to ([34], Lemma 3.1) and ([38], Lemma 4), for any $\varepsilon > 0$, the supremum

$$\Lambda_{\beta-\varepsilon} := \sup_{u \in W_0^{1,N}(\mathbb{B}) \cap \mathscr{S},\, \|\nabla u\|_N = 1} \int_{\mathbb{B}} \exp\left\{\alpha_N (1+\beta-\varepsilon) |u|^{\frac{N}{N-1}}\right\} h(x)\, dx$$

can be attained by some nonnegative function $u_\varepsilon \in W_0^{1,N}(\mathbb{B}) \cap \mathscr{S}$ with $\|\nabla u_\varepsilon\|_N = 1$ and

$$\lim_{\varepsilon \to 0} \Lambda_{\beta-\varepsilon} = \Lambda_\beta. \tag{7}$$

The maximizers u_ε satisfies the Euler-Lagrange equation

$$-\Delta_N u_\varepsilon = \frac{1}{\lambda_\varepsilon} u_\varepsilon^{\frac{1}{N-1}} \exp\left\{\alpha_N (1+\beta-\varepsilon) u_\varepsilon^{\frac{N}{N-1}}\right\} h(x) \text{ in } \mathbb{B}, \tag{8}$$

where $\Delta_N u_\varepsilon = \operatorname{div}\left(|\nabla u_\varepsilon|^{N-2} \nabla u_\varepsilon\right)$ and

$$\lambda_\varepsilon := \int_{\mathbb{B}} u_\varepsilon^{\frac{N}{N-1}} \exp\left\{\alpha_N (1+\beta-\varepsilon) u_\varepsilon^{\frac{N}{N-1}}\right\} h(x)\, dx. \tag{9}$$

Moreover, there holds

$$\liminf_{\varepsilon \to 0} \lambda_\varepsilon > 0. \tag{10}$$

Since $\|\nabla u_\varepsilon\|_N = 1$, there exists some nonnegative function u_0 in $W_0^{1,N}(\mathbb{B}) \cap \mathscr{S}$ with

$$\begin{cases} u_\varepsilon \rightharpoonup u_0 \text{ weakly in } W_0^{1,N}(\mathbb{B}), \\ u_\varepsilon \to u_0 \text{ strongly in } L^p(\mathbb{B}), \forall p > 1, \\ u_\varepsilon \to u_0 \text{ a.e. in } \mathbb{B}. \end{cases} \tag{11}$$

Without loss of generality, we assume in the following,

$$c_\varepsilon = \max_{\mathbb{B}} u_\varepsilon = u_\varepsilon(0) \to +\infty \quad \text{as} \quad \varepsilon \to 0. \tag{12}$$

LEMMA 1. *Assume* $u_\varepsilon \in W_0^{1,N}(\mathbb{B})$ *with* $\|\nabla u_\varepsilon\|_N = 1$ *and* $u_\varepsilon \rightharpoonup u_0$ *weakly in* $W_0^{1,N}(\mathbb{B})$. *Then for any* $q < 1/\left(1 - \|\nabla u_0\|_N^N\right)^{1/(N-1)}$, *we have*

$$\limsup_{\varepsilon \to 0} \int_{\mathbb{B}} \exp\left\{\alpha_N(1+\beta)p|u_\varepsilon|^{\frac{N}{N-1}}\right\}|x|^{N\beta}\,dx < +\infty. \tag{13}$$

Proof. By a change of variables, we define a function sequence

$$m_\varepsilon(r) = (1+\beta)^{\frac{N-1}{N}} u_\varepsilon(r^{\frac{1}{1+\beta}}).$$

In view of (11), we get $m_\varepsilon \in W_0^{1,N}(\mathbb{B})$, $m_\varepsilon \rightharpoonup m_0$ weakly in $W_0^{1,N}(\mathbb{B})$. A straightforward calculation shows

$$\int_{\mathbb{B}} |\nabla m_\varepsilon|^N dx = \int_{\mathbb{B}} |\nabla u_\varepsilon|^N dx = 1.$$

According to P. L. Lions [24], we have

$$\limsup_{\varepsilon \to 0} \int_{\mathbb{B}} \exp\left\{\alpha_N p|m_\varepsilon|^{\frac{N}{N-1}}\right\} dx < +\infty$$

for any $q < 1/\left(1 - \|\nabla m_0\|_N^N\right)^{1/(N-1)}$. This together with the fact

$$\begin{aligned}
&\int_{\mathbb{B}} \exp\left\{\alpha_N(1+\beta)p|u_\varepsilon|^{\frac{N}{N-1}}\right\}|x|^{N\beta}dx\\
=&\omega_{N-1}\int_0^1 \exp\left\{\alpha_N p(1+\beta)|u_\varepsilon(r)|^{\frac{N}{N-1}}\right\} r^{N-1+N\beta}dr\\
=&\omega_{N-1}\int_0^1 \exp\left\{\alpha_N p\left|m_\varepsilon\left(r^{1+\beta}\right)\right|^{\frac{N}{N-1}}\right\} r^{N-1+N\beta}dr\\
=&\frac{\omega_{N-1}}{1+\beta}\int_0^1 \exp\left\{\alpha_N p|m_\varepsilon(t)|^{\frac{N}{N-1}}\right\} t^{N-1}dt\\
=&\frac{1}{1+\beta}\int_{\mathbb{B}} \exp\left\{\alpha_N p|m_\varepsilon|^{\frac{N}{N-1}}\right\} dx
\end{aligned}$$

leads to (13). $\square$

Then we have the following:

LEMMA 2. $u_0 \equiv 0$ *in* $\mathbb{B}$ *and* $|\nabla u_\varepsilon|^N dx \rightharpoonup \delta_0$ *in sense of measure, where* δ_0 *is the usual Dirac measure centered at the origin.*

Proof. Suppose $u_0 \not\equiv 0$. By (5) and Lemma 1, $\exp\left\{\alpha_N(1+\beta-\varepsilon)u_\varepsilon^{N/(N-1)}\right\}h(x)$ is bounded in $L^q(\mathbb{B})$ for $1 < q < 1/\left(1 - \|\nabla u_0\|_N^N\right)^{1/(N-1)}$. Combining this and (10), we know that $\Delta_N u_\varepsilon$ is bounded in $L^q(\mathbb{B})$. Then applying elliptic estimates to (8), we conclude that u_ε is uniformly bounded in $\mathbb{B}$, which contradicts (12). Therefore $u_0 \equiv 0$.

Suppose $|\nabla u_\varepsilon|^N dx$ does not weakly converge to δ_0 in sense of measure. There exists a constant $r_0 > 0$ such that $\mathbb{B}_{r_0} \subset \mathbb{B}$ and

$$\lim_{\varepsilon\to 0}\int_{\mathbb{B}_{r_0}} |\nabla u_\varepsilon|^N dx = \eta < 1.$$

Since u_ε is nonnegative decreasing radially symmetric, we have

$$\int_{\mathbb{B}_{r_0}} u_\varepsilon^N(x)dx \geqslant \frac{u_\varepsilon^N(r_0)\, r_0^N \omega_{N-1}}{N}.$$

This together with $\|\nabla u_\varepsilon\|_N = 1$ and the Pocaré inequality gives

$$u_\varepsilon(r_0) \leqslant \left(\frac{N}{\omega_{N-1}}\right)^{\frac{1}{N}} \frac{C}{r_0}$$

for some constant C. Let $\bar{u}_\varepsilon(x) = u_\varepsilon(x) - u_\varepsilon(r_0)$ for $x \in \mathbb{B}_{r_0}$. Then $\bar{u}_\varepsilon(x) \in W_0^{1,N}(\mathbb{B}_{r_0})$ and $\int_{\mathbb{B}_{r_0}} |\nabla \bar{u}_\varepsilon|^N dx = \eta < 1$. For any real number $v > 0$, there exists some constant C depending only on N and v such that for all $x \in \mathbb{B}_{r_0}$,

$$u_\varepsilon^{\frac{N}{N-1}}(x) \leqslant (1+v)\,\bar{u}_\varepsilon^{\frac{N}{N-1}}(x) + C u_\varepsilon^{\frac{N}{N-1}}(r_0).$$

Here and in the sequel, we denote various constants by the same C. It follows that

$$\begin{aligned}
&\int_{\mathbb{B}_{r_0}} \exp\left\{\alpha_N(1+\beta-\varepsilon)\, p u_\varepsilon^{\frac{N}{N-1}}\right\} h(x)dx \\
\leqslant & C\int_{\mathbb{B}_{r_0}} \exp\left\{\alpha_N(1+\beta-\varepsilon)\, p(1+v)\,\eta^{\frac{1}{N-1}}\left(\frac{\bar{u}_\varepsilon}{\eta^{\frac{1}{N}}}\right)^{\frac{N}{N-1}}\right\} h(x)dx
\end{aligned}$$

where C is a constant depending only on N, v and r_0. Choose $p > 1$ sufficiently close to 1 and $v > 0$ sufficiently small such that $p(1+v)\,\eta^{1/(N-1)} \leqslant 1$. By the inequalities (4) and (5), $\exp\left\{\alpha_N(1+\beta-\varepsilon)\, u_\varepsilon^{N/(N-1)}\right\} h(x)$ is bounded in $L^p(\mathbb{B}_{r_0})$. Applying elliptic estimates to (8), we conclude that u_ε is uniformly bounded in $\mathbb{B}_{r_0/2}$, which contradicts (12) and completes the proof of the lemma. □

Let $r_\varepsilon > 0$ be such that

$$r_\varepsilon^N = \lambda_\varepsilon c_\varepsilon^{\frac{N}{1-N}} \exp\left\{-\alpha_N(1+\beta-\varepsilon)\, c_\varepsilon^{\frac{N}{1-N}}\right\}. \tag{14}$$

For any $0 < \delta < 1$, in view of (4), (5) and (12), there is a constant C depending only on δ such that

$$\begin{aligned}
\lambda_\varepsilon &= \int_{\mathbb{B}} u_\varepsilon^{\frac{N}{N-1}} \exp\left\{\alpha_N(1+\beta-\varepsilon)\, u_\varepsilon^{\frac{N}{N-1}}\right\} h(x)\,dx \\
&\leqslant c_\varepsilon^{\frac{N}{N-1}} \exp\left\{\delta\alpha_N(1+\beta-\varepsilon)\, c_\varepsilon^{\frac{N}{N-1}}\right\} \int_{\mathbb{B}} \exp\left\{(1-\delta)\,\alpha_N(1+\beta-\varepsilon)\, u_\varepsilon^{\frac{N}{N-1}}\right\} h(x)dx \\
&\leqslant C c_\varepsilon^{\frac{N}{N-1}} \exp\left\{\delta\alpha_N(1+\beta-\varepsilon)\, c_\varepsilon^{\frac{N}{N-1}}\right\}.
\end{aligned}$$

According to this and (14), we get $r_\varepsilon^N \leqslant C\exp\left\{(\delta-1)\alpha_N(1+\beta-\varepsilon)c_\varepsilon^{N/(N-1)}\right\}$. This immediately leads to $r_\varepsilon \to 0$ and $\mathbb{B}_{r_\varepsilon^{-1/(1+\beta)}} := \left\{x\in\mathbb{R}^N : r_\varepsilon^{1/(1+\beta)}x \in \mathbb{B}\right\} \to \mathbb{R}^N$ as $\varepsilon \to 0$. We now define on $\mathbb{B}_{r_\varepsilon^{-1/(1+\beta)}}$ two blow-up sequences of functions as

$$\psi_\varepsilon := c_\varepsilon^{-1} u_\varepsilon\left(r_\varepsilon^{\frac{1}{1+\beta}}x\right) \tag{15}$$

and

$$\phi_\varepsilon := c_\varepsilon^{\frac{1}{N-1}}\left(u_\varepsilon\left(r_\varepsilon^{\frac{1}{1+\beta}}x\right)-c_\varepsilon\right). \tag{16}$$

In view of (5), (8) and (14)-(16), a direct computation shows

$$-\Delta_N\psi_\varepsilon = c_\varepsilon^{-N}\psi_\varepsilon^{\frac{1}{N-1}}\exp\left\{\alpha_N(1+\beta-\varepsilon)\left(u_\varepsilon^{\frac{N}{N-1}}\left(r_\varepsilon^{\frac{1}{1+\beta}}x\right)-c_\varepsilon^{\frac{N}{N-1}}\right)\right\}r_\varepsilon^{\frac{-N\beta}{1+\beta}}h\left(r_\varepsilon^{\frac{1}{1+\beta}}x\right) \tag{17}$$

and

$$-\Delta_N\phi_\varepsilon = \psi_\varepsilon^{\frac{1}{N-1}}\exp\left\{\alpha_N(1+\beta-\varepsilon)\left(u_\varepsilon^{\frac{N}{N-1}}\left(r_\varepsilon^{\frac{1}{1+\beta}}x\right)-c_\varepsilon^{\frac{N}{N-1}}\right)\right\}r_\varepsilon^{\frac{-N\beta}{1+\beta}}h\left(r_\varepsilon^{\frac{1}{1+\beta}}x\right). \tag{18}$$

Now we study the convergence behavior of ψ_ε and ϕ_ε. Using the same argument as in the proof of ([19], Lemma 17), we conclude that

$$\psi_\varepsilon \to 1 \text{ in } C^1_{loc}\left(\mathbb{R}^N\right) \text{ as } \varepsilon\to 0 \tag{19}$$

and

$$\phi_\varepsilon \to \phi \text{ in } C^1_{loc}\left(\mathbb{R}^N\right) \text{ as } \varepsilon\to 0. \tag{20}$$

In view of the mean value theorem, we have

$$\begin{aligned} u_\varepsilon^{\frac{N}{N-1}}\left(r_\varepsilon^{\frac{1}{1+\beta}}x\right)-c_\varepsilon^{\frac{N}{N-1}} &= \frac{N}{N-1}\xi_\varepsilon^{\frac{N}{N-1}}\left(u_\varepsilon\left(r_\varepsilon^{\frac{1}{1+\beta}}x\right)-c_\varepsilon\right)\\ &= \frac{N}{N-1}\left(\frac{\xi_\varepsilon}{c_\varepsilon}\right)^{\frac{N}{N-1}}\phi_\varepsilon(x)\\ &= \frac{N}{N-1}\phi_\varepsilon(x)\left(1+o_\varepsilon(1)\right), \end{aligned} \tag{21}$$

where ξ_ε lies between $u_\varepsilon\left(r_\varepsilon^{1/(1+\beta)}x\right)$ and c_ε. According to (18)-(21), we can see that ϕ solves

$$\begin{cases} -\Delta_N\phi = \exp\left\{\alpha_N(1+\beta)\frac{N}{N-1}\phi\right\}|x|^{N\beta},\\ \phi(0)=0=\sup\limits_{\mathbb{R}^N}\phi \end{cases} \tag{22}$$

in the distributional sense. The unique solution of (22) can be written as

$$\phi(x) = -\frac{N-1}{\alpha_N(1+\beta)} \log\left(1 + C_N |x|^{\frac{N(1+\beta)}{N-1}}\right), \tag{23}$$

where $C_N = \left(\omega_{N-1}^{-1} N(1+\beta)\right)^{1-N}$. It follows that

$$\begin{aligned} \int_{\mathbb{R}^N} \exp\left\{\alpha_N(1+\beta)\frac{N}{N-1}\phi\right\} |x|^{N\beta} dx &= \int_0^{+\infty} \frac{\omega_{N-1} r^{N(1+\beta)-1}}{\left(1 + C_N |x|^{\frac{N}{N-1}(1+\beta)}\right)^N} dx \\ &= \frac{\omega_{N-1}}{C_N^{N-1} N(1+\beta)} = 1. \end{aligned} \tag{24}$$

Following ([19], Lemma 19), we have that the supremum Λ_β (with Λ_β given in (6)) satisfies

$$\begin{aligned} \Lambda_\beta &= \int_{\mathbb{B}} h(x)\, dx + \lim_{R \to +\infty} \lim_{\varepsilon \to 0} \int_{\mathbb{B}_{Rr_\varepsilon^{1/(1+\beta)}}} \exp\left\{\alpha_N(1+\beta-\varepsilon) u_\varepsilon^{\frac{N}{N-1}}\right\} h(x)\, dx \\ &= \int_{\mathbb{B}} h(x)\, dx + \lim_{\varepsilon \to 0} c_\varepsilon^{\frac{N}{1-N}} \lambda_\varepsilon. \end{aligned} \tag{25}$$

Moreover, using the same arguments of the proof of ([34], Lemma 4.11), we obtain

$$\begin{cases} c_\varepsilon^{\frac{1}{N-1}} u_\varepsilon \rightharpoonup G \text{ weakly in } W_0^{1,q}(\mathbb{B}), \forall 1 < q < N, \\ c_\varepsilon^{\frac{1}{N-1}} u_\varepsilon \to G \text{ strongly in } L^p(\mathbb{B}), \forall 1 < p < \dfrac{Nq}{N-q}, \\ c_\varepsilon^{\frac{1}{N-1}} u_\varepsilon \to G \text{ in } C^1_{loc}\left(\overline{\mathbb{B}} \setminus \{0\}\right), \end{cases} \tag{26}$$

where G is a distributional solution of $-\Delta_N G = \delta_0$ in $\mathbb{B}$. Explicitly G can be written as

$$G = -\frac{1}{2\pi} \log|x|. \tag{27}$$

3. Upper bound estimate

To estimate the supremum Λ_β defined as in (6), we need the following:

LEMMA 3. *When $c_\varepsilon \to +\infty$ in $\mathbb{B}$ as $\varepsilon \to 0$, there holds*

$$\lim_{\varepsilon \to 0} c_\varepsilon^{\frac{N}{1-N}} \lambda_\varepsilon \leqslant \frac{\omega_{N-1}}{N(1+\beta)} \exp\left\{\sum_{j=1}^{N-1} \frac{1}{j}\right\}. \tag{28}$$

Proof. We take small $\delta > 0$ such that $\mathbb{B}_\delta \subset \mathbb{B}$ and define a function space

$$W_{a,\,b} := \left\{u \in W^{1,N}(\mathbb{B}_\delta \setminus \mathbb{B}_{Rr_\varepsilon^{1/(1+\beta)}}) \cap \mathscr{S} :\ a = u(\delta),\ b = u\left(Rr_\varepsilon^{\frac{1}{1+\beta}}\right)\right\},$$

where

$$a := c_\varepsilon^{\frac{1}{1-N}}\left(\frac{N}{\alpha_N}\log\frac{1}{\delta}+o_\varepsilon(1)\right), \tag{29}$$

$$b := c_\varepsilon + c_\varepsilon^{\frac{1}{1-N}}\left(\frac{1-N}{\alpha_N(1+\beta)}\log\left(1+C_N R^{\frac{N(1+\beta)}{N-1}}\right)+o_\varepsilon(1)\right). \tag{30}$$

In view of the direct method of variation, we get $\inf_{u\in W_{a,b}}\int_{Rr_\varepsilon^{1/(1+\beta)}\leqslant|x|\leqslant\delta}|\nabla u|^N dx$ can be attained by

$$m(x) = \frac{a\left(\log|x|-\log\left(Rr_\varepsilon^{\frac{1}{1+\beta}}\right)\right)-b\left(\log\delta-\log|x|\right)}{\log\delta-\log\left(Rr_\varepsilon^{\frac{1}{1+\beta}}\right)}$$

belonging to $W_{a,\,b}$ with $\Delta_N m(x)=0$. After a direct calculation, one gets

$$\int_{Rr_\varepsilon^{1/(1+\beta)}\leqslant|x|\leqslant\delta}|\nabla m(x)|^N dx = \frac{\omega^{N-1}|a-b|^N}{\left(\log\delta-\log\left(Rr_\varepsilon^{\frac{1}{1+\beta}}\right)\right)^{N-1}}. \tag{31}$$

Recalling (14), we have

$$\log\delta-\log\left(Rr_\varepsilon^{\frac{1}{1+\beta}}\right)=\log\delta-\log R-\frac{1}{N(1+\beta)}\log\left(c_\varepsilon^{\frac{N}{1-N}}\lambda_\varepsilon\right)+\frac{\alpha_N(1+\beta-\varepsilon)}{N(1+\beta)}c_\varepsilon^{\frac{N}{N-1}}. \tag{32}$$

According to (29)-(32), we obtain

$$\begin{aligned}
&\int_{Rr_\varepsilon^{1/(1+\beta)}\leqslant|x|\leqslant\delta}|\nabla m(x)|^N dx\\
&=\left(\frac{1+\beta}{1+\beta-\varepsilon}\right)^{N-1}c_\varepsilon^{\frac{N}{1-N}}\times\left(\frac{N(1-N)}{\alpha_N(1+\beta)}\log\left(1+C_N R^{\frac{N(1+\beta)}{N-1}}\right)+\frac{N^2}{\alpha_N}\log\delta+c_\varepsilon^{\frac{N}{N-1}}\right.\\
&\qquad\left.-\frac{(1+\beta)(N-1)N}{(1+\beta-\varepsilon)\alpha_N}\log\frac{\delta c_\varepsilon^{\frac{1}{(1+\beta)(N-1)}}}{R\lambda_\varepsilon^{\frac{1}{(1+\beta)N}}}+o(1)\right),
\end{aligned} \tag{33}$$

where $o(1)\to 0$ as $\varepsilon\to 0$ first and then $\delta\to 0$. Denote $\bar{u}_\varepsilon := \max\{a,\ \min\{b,\ u_\varepsilon\}\}\in W_{a,b}$. For sufficiently small ε, $|\nabla\bar{u}_\varepsilon|\leqslant|\nabla u_\varepsilon|$ in $\mathbb{B}_\delta\backslash\mathbb{B}_{Rr_\varepsilon^{1/(1+\beta)}}$. It follows that

$$\begin{aligned}
\int_{Rr_\varepsilon^{1/(1+\beta)}<|x|\leqslant\delta}|\nabla m(x)|^N dx &\leqslant \int_{Rr_\varepsilon^{1/(1+\beta)}<|x|\leqslant\delta}|\nabla\bar{u}_\varepsilon(x)|^N dx\\
&\leqslant \int_{Rr_\varepsilon^{1/(1+\beta)}<|x|\leqslant\delta}|\nabla u_\varepsilon(x)|^N dx\\
&\leqslant 1-\int_{\delta<|x|\leqslant 1}|\nabla u_\varepsilon(x)|^N dx-\int_{|x|\leqslant Rr_\varepsilon^{1/(1+\beta)}}|\nabla u_\varepsilon(x)|^N dx.
\end{aligned} \tag{34}$$

We next compute $\int_{\delta<|x|\leqslant 1}|\nabla u_\varepsilon(x)|^N dx$ and $\int_{|x|\leqslant Rr_\varepsilon^{1/(1+\beta)}}|\nabla u_\varepsilon(x)|^N dx$. Integration by parts leads to

$$\int_{\delta<|x|\leqslant 1}|\nabla G|^N dx = G(\delta)\int_{|x|=\delta}|\nabla G|^{N-1}ds = G(\delta)\int_{\delta<|x|\leqslant 1}(-\Delta_N G)dx = -\frac{N}{\alpha_N}\log\delta.$$

In view of (26), we obtain

$$\int_{\delta<|x|\leqslant 1}|\nabla u_\varepsilon(x)|^N dx = c_\varepsilon^{\frac{N}{1-N}}\left(-\frac{N}{\alpha_N}\log\delta + o_\varepsilon(1)\right). \tag{35}$$

Let $t = r^{N(1+\beta)/(N-1)}$ and $A = R^{N(1+\beta)/(N-1)}$. Recalling (23), one gets

$$\int_{|x|\leqslant R}|\nabla\phi(x)|^N dx = \omega_{N-1}\int_0^R|\phi'(r)|^N r^{N-1}dr = \frac{(N-1)\,\omega_{N-1}}{(N(1+\beta))^{\frac{2N-1}{N-1}}}\int_0^A\frac{t^{N-1}}{(1+C_N t)^N}dt. \tag{36}$$

Note that

$$\begin{aligned} I_N := \int_0^T\frac{t^{N-1}}{(1+bt)^N}dt &= -\frac{1}{(N-1)b^N} + \frac{1}{b}I_{N-1} + O\left(T^{-1}\right) \\ &= -\frac{1}{b^N}\sum_{j=1}^{N-1}\frac{1}{j} + \frac{1}{b^{N-1}}I_1 + O\left(T^{-1}\right) = \frac{1}{b^N}\left(-\sum_{j=1}^{N-1}\frac{1}{j} + \log(1+bT) + O\left(T^{-1}\right)\right), \end{aligned} \tag{37}$$

for any $T,\ b>0$. According to (20), (36) and (37), we get

$$\int_{|x|\leqslant Rr_\varepsilon^{1/(1+\beta)}}|\nabla u_\varepsilon(x)|^N dx = \frac{c_\varepsilon^{\frac{N}{1-N}}(N-1)}{\alpha_N(1+\beta)}\left(\log(1+C_N A) - \sum_{j=1}^{N-1}\frac{1}{j} + O\left(A^{-1}\right) + o_\varepsilon(1)\right). \tag{38}$$

Combining (34), (35) and (38), we get

$$\begin{aligned} &\int_{Rr_\varepsilon^{1/(1+\beta)}<|x|\leqslant\delta}|\nabla m(x)|^N dx \\ &\leqslant \frac{c_\varepsilon^{\frac{N}{1-N}}}{\alpha_N}\left(\frac{N-1}{1+\beta}\left(\log(1+C_N A) - \sum_{j=1}^{N-1}\frac{1}{j}\right) - N\log\delta + O\left(A^{-1}\right) + o_\varepsilon(1)\right). \end{aligned} \tag{39}$$

In view of (33) and (39), we have

$$(1+o_\varepsilon(1))\log\left(c_\varepsilon^{\frac{N}{1-N}}\lambda_\varepsilon\right) \leqslant \sum_{j=1}^{N-1}\frac{1}{j} + \log\frac{\omega_{N-1}}{N(1+\beta)} + o_\varepsilon(1) + o_R(1).$$

Hence the lemma is followed. $\square$

According to (25) and (28), we conclude the supremum

$$\Lambda_\beta \leqslant \int_{\mathbb{B}}h(x)\,dx + \frac{\omega_{N-1}}{N(1+\beta)}\exp\left\{\sum_{j=1}^{N-1}\frac{1}{j}\right\}. \tag{40}$$

4. A blow-up sequence

Let $N \geqslant 3$, $\beta \geqslant 0$ be fixed. We construct a blow-up sequence of functions

$$v_\varepsilon := \begin{cases} c + c^{\frac{1}{1-N}} \left(\dfrac{1-N}{\alpha_N (1+\beta)} \log \left(1 + C_N \left(\dfrac{r}{\varepsilon} \right)^{\frac{N(1+\beta)}{N-1}} \right) + B \right), & \text{for } r \leqslant R\varepsilon, \\ c^{\frac{1}{1-N}} G, & \text{for } R\varepsilon < r \leqslant 1, \end{cases} \tag{41}$$

with $\|\nabla v_\varepsilon\|_N = 1$, where $C_N = \left(\omega_{N-1}^{-1} N (1+\beta)\right)^{1-N}$, $R = (-\log \varepsilon)^{1/(1+\beta)}$, G given in (27), B and c are constants depending only on ε and β. In order to assure that $v_\varepsilon \in W_0^{1,N}(\mathbb{B}) \cap \mathscr{S}$, we set

$$c + c^{\frac{1}{1-N}} \left(\frac{1-N}{\alpha_N (1+\beta)} \log \left(1 + C_N R^{\frac{N(1+\beta)}{N-1}} \right) + B \right) = c^{\frac{1}{1-N}} G(R\varepsilon).$$

This gives

$$c^{\frac{N}{N-1}} = -B - \frac{N}{\alpha_N} \log (R\varepsilon) + \frac{N-1}{\alpha_N (1+\beta)} \log \left(1 + C_N R^{\frac{N(1+\beta)}{N-1}} \right). \tag{42}$$

Combining a change of variable $t := C_N (r/\varepsilon)^{N(1+\beta)/(N-1)}$ and (37), we have

$$\begin{aligned} \int_{|x| \leqslant R\varepsilon} |\nabla v_\varepsilon|^N dx &= \omega_{N-1} \int_0^{R\varepsilon} \left| \frac{\partial v_\varepsilon}{\partial r} \right|^N r^{N-1} dr \\ &= \frac{N-1}{\alpha_N (1+\beta) c^{\frac{N}{N-1}}} \int_0^{C_N R^{\frac{N(1+\beta)}{N-1}}} \frac{t^{N-1}}{(1+t)^N} dt \\ &= \frac{N-1}{\alpha_N (1+\beta) c^{\frac{N}{N-1}}} \left(\log \left(1 + C_N R^{\frac{N(1+\beta)}{N-1}} \right) - \sum_{j=1}^{N-1} \frac{1}{j} + O\left(R^{\frac{N(1+\beta)}{1-N}} \right) \right). \end{aligned} \tag{43}$$

The divergence theorem leads to

$$\int_{R\varepsilon < |x| \leqslant 1} |\nabla v_\varepsilon|^N dx = c^{\frac{N}{1-N}} \int_{R\varepsilon < |x| \leqslant 1} |\nabla G|^N dx = c^{\frac{N}{1-N}} G(R\varepsilon) = \frac{-N}{\alpha_N c^{\frac{N}{N-1}}} \log (R\varepsilon). \tag{44}$$

Applying (43), (44) and $\|\nabla v_\varepsilon\|_N = 1$, we have

$$\frac{\alpha_N (1+\beta) c^{\frac{N}{N-1}}}{N-1} = \log \left(1 + C_N R^{\frac{N(1+\beta)}{N-1}} \right) - \frac{N(1+\beta)}{N-1} \log (R\varepsilon) - \sum_{j=1}^{N-1} \frac{1}{j} + O\left(R^{\frac{N(1+\beta)}{1-N}} \right). \tag{45}$$

Inserting (42) into (45), we get

$$\alpha_N (1+\beta) B = (N-1) \sum_{j=1}^{N-1} \frac{1}{j} + O\left(R^{\frac{N(1+\beta)}{1-N}} \right). \tag{46}$$

Let

$$B_{\varepsilon}(x) := \frac{1-N}{\alpha_N(1+\beta)} \log\left(1+C_N\left(\frac{r}{\varepsilon}\right)^{\frac{N(1+\beta)}{N-1}}\right)+B.$$

In view of the Taylor formula, one gets

$$\begin{aligned} v_{\varepsilon}^{\frac{N}{N-1}}(x) &= c^{\frac{N}{N-1}}\left(1+c^{\frac{N}{1-N}}B_{\varepsilon}(x)\right)^{\frac{N}{N-1}} \\ &= c^{\frac{N}{N-1}}\left(1+\frac{N}{N-1}c^{\frac{N}{1-N}}B_{\varepsilon}(x)+\frac{N}{2(N-1)^2}(1+\xi)^{\frac{2-N}{N-1}}\left(c^{\frac{N}{1-N}}B_{\varepsilon}(x)\right)^2\right) \\ &\geqslant c^{\frac{N}{N-1}}+\frac{N}{N-1}B_{\varepsilon}(x), \end{aligned} \tag{47}$$

where ξ lies between $c^{N/(1-N)}B_{\varepsilon}(x)$ and 0. By (42), (46) and (47), for all $x \in \mathbb{B}_{Rr_{\varepsilon}}$, we obtain

$$\begin{aligned} \alpha_N(1+\beta)v_{\varepsilon}^{\frac{N}{N-1}} \geqslant \sum_{j=1}^{N-1}\frac{1}{j} - N\log\left(1+C_N\left(\frac{r}{\varepsilon}\right)^{\frac{N(1+\beta)}{N-1}}\right) - N(1+\beta)\log(R\varepsilon) \\ +(N-1)\log\left(1+C_N R^{\frac{N(1+\beta)}{N-1}}\right)+O\left(R^{\frac{N(1+\beta)}{1-N}}\right). \end{aligned}$$

It follows that

$$\int_{\mathbb{B}_{R\varepsilon}} \exp\left\{\alpha_N(1+\beta)v_{\varepsilon}^{\frac{N}{N-1}}\right\}h(x)\,dx \geqslant \frac{\omega_{N-1}}{N(1+\beta)}\exp\left\{\sum_{j=1}^{N-1}\frac{1}{j}\right\}+O\left(R^{\frac{N(1+\beta)}{1-N}}\right). \tag{48}$$

Moreover, using the fact of $e^t \geqslant t+1$ for any $t>0$ and (41), we have

$$\begin{aligned} &\int_{\mathbb{B}\setminus\mathbb{B}_{R\varepsilon}} \exp\left\{\alpha_N(1+\beta)v_{\varepsilon}^{\frac{N}{N-1}}\right\}h(x)\,dx \\ &\geqslant \int_{\mathbb{B}\setminus\mathbb{B}_{R\varepsilon}}\left(1+\alpha_N(1+\beta)v_{\varepsilon}^{\frac{N}{N-1}}\right)h(x)dx \\ &\geqslant \int_{\mathbb{B}} h(x)\,dx+\alpha_N(1+\beta)c^{\frac{-N}{(N-1)^2}}\int_{\mathbb{B}} h(x)G^{\frac{N}{N-1}}\,dx+O\left(R^{\frac{N(1+\beta)}{1-N}}\right). \end{aligned} \tag{49}$$

Combining (48) and (49), we obtain

$$\begin{aligned} &\int_{\mathbb{B}} \exp\left\{\alpha_N(1+\beta)v_{\varepsilon}^{\frac{N}{N-1}}\right\}h(x)\,dx \\ &\geqslant \int_{\mathbb{B}} h(x)\,dx+\frac{\omega_{N-1}}{N(1+\beta)}\exp\left\{\sum_{j=1}^{N-1}\frac{1}{j}\right\} \\ &\quad +\alpha_N(1+\beta)c^{\frac{-N}{(N-1)^2}}\int_{\mathbb{B}} h(x)G^{\frac{N}{N-1}}\,dx+O\left(R^{\frac{N(1+\beta)}{1-N}}\right). \end{aligned} \tag{50}$$

From $R=(-\log\varepsilon)^{1/(1+\beta)}$, (42) and (46), we get $R^{N(1+\beta)/(1-N)}=o\left(c^{-N/(N-1)^2}\right)$. Then we have

$$\alpha_N(1+\beta)c^{\frac{-N}{(N-1)^2}}\int_{\mathbb{B}}h(x)G^{\frac{N}{N-1}}dx+O\left(R^{\frac{N(1+\beta)}{1-N}}\right)>0$$

for sufficiently small ε. In view of (50), one gets

$$\int_{\mathbb{B}}\exp\left\{\alpha_N(1+\beta-\varepsilon)v_\varepsilon^{\frac{N}{N-1}}\right\}h(x)\,dx>\int_{\mathbb{B}}h(x)\,dx+\frac{\omega_{N-1}}{N(1+\beta)}\exp\left\{\sum_{j=1}^{N-1}\frac{1}{j}\right\}.$$

This contradicts (40).

Hence c_ε must be bounded, and thus Theorem 1 follows immediately from elliptic estimates on (8). □

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