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AN INEQUALITY INVOLVING A TRIANGLE AND AN INTERIOR POINT AND ITS APPLICATION

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Abstract. Let $\mathbf{x}_0$ be an interior split point in the triangle $T := [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3]$. By α_{ij} we denote the angle $\widehat{\mathbf{x}_0, \mathbf{x}_i, \mathbf{x}_j}$, $i \neq j$. We show that

$$\cos\alpha_{12}\cos\alpha_{23}\cos\alpha_{31} + \cos\alpha_{21}\cos\alpha_{32}\cos\alpha_{13} > 0.$$

Additionally, we use this inequality to prove uniqueness and existence of a conforming quadratic piecewise harmonic finite element on the Clough-Tocher split of a triangle.

1. Introduction

In the process of proving existence and uniqueness of a certain finite element for the Laplace equation in two variables we came across an interesting geometric inequality that seemingly has nothing to do with harmonic functions. This inequality does not appear in the most comprehensive reference [4] or, to the best of our knowledge, in any other sources. Chapter XI of [4] entitled “Triangle and Point” would have been an appropriate emplacement for this inequality. In the fields of multivariate piecewise polynomials, also known as multivariate splines, and finite element analysis related to numerical solutions of partial differential equations, the split of a triangle into three subtriangles obtained by coning off an arbitrary interior point is known as *the Cough-Tocher split*, see Fig. 1.1(a). This paper is organized as follows. In the introduction, we review barycentric coordinates and Bernstein-Bézier framework for quadratic bivariate polynomial splines. We also state the boundary value problem whose solution can be approximated by our new finite element. The inequality itself is proved in Section 2. It is used to show existence and uniqueness of a conforming finite element constructed in Section 3. Comprehensive references for multivariate spines and finite element analysis are [3] and [2], respectively.

Bernstein-Bézier techniques have become standard tools in analyzing multivariate splines. We recall some of them in this section. Let

$$b_i = b_i(\mathbf{x}) \quad \text{where} \quad i = 1, 2, 3,$$

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denote the *barycentric coordinates* of a point $\mathbf{x} = (x^1, x^2)$ relative to a triangle $T = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3] \in \mathbb{R}^2$, where $\mathbf{x}_i = (x_i^1, x_i^2)$. The barycentric coordinates are defined by the equation

$$\begin{bmatrix} x_1^1 & x_2^1 & x_3^1 \\ x_1^2 & x_2^2 & x_3^2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} x^1 \\ x^2 \\ 1 \end{bmatrix}.$$

In this paper we only use quadratic polynomials. Every such polynomial p can be written uniquely in its *Bernstein-Bézier (BB-) form* as

$$p(\mathbf{x}) = \sum_{i+j+k=2} c_{ijk} B_{ijk}, \quad \text{where} \quad B_{ijk} = \frac{2!}{i!j!k!} b_1^i b_2^j b_3^k. \tag{1}$$

The six coefficients c_{ijk} are called the *BB-coefficients* of p. Each such coefficient is uniquely associated with its *domain point* $\xi_{ijk} = (i\mathbf{x}_1 + j\mathbf{x}_2 + k\mathbf{x}_3)/2$, $i + j + k = 2$, located either at a vertex or at a mid-edge of T. The points (ξ_{ijk}, c_{ijk}) are called the *control points* of p.

Let $\Delta_{CT}(T)$ be the Clough-Tocher split of T into three subtriangles T_1, T_2, and T_3 as in Fig. 1.1(b). On each triangle T_i, $i = 1,2,3$, we define a quadratic polynomial p_i, $i = 1,2,3$, in two variables. Each p_i has six domain points depicted as either black dots or empty circles in Fig. 1.1(b). The black dots correspond to the BB-coefficients that will be set to zero in the proof of Theorem 3.1. Since the three control points associated with each edge $e_i := [\mathbf{x}_0, \mathbf{x}_i]$ are the same for the two polynomials p_j and p_k corresponding to the two triangles T_j and T_k sharing e_i, we conclude that $p_j|_{e_i} = p_k|_{e_i}$, where i, j, k are pairwise distinct. Indeed, each such restriction is a univariate quadratic polynomial uniquely defined by these three BB-coefficients. Therefore, the dimension of the continuous quadratic piecewise polynomial (or *spline*) space defined on $\Delta_{CT}(T)$ is equal to the total number of the control points, i.e., ten. In Section 3, we will impose an additional condition on each p_i, $i = 1,2,3$, namely $\Delta p_i = 0$, see [1]. The new space of piecewise harmonic splines is used in Section 3, where we show that its dimension is seven.

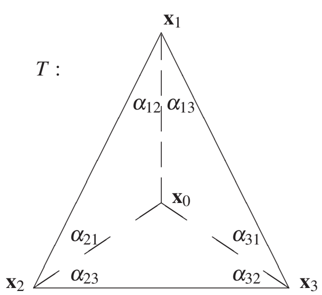

Figure 1.1(a)

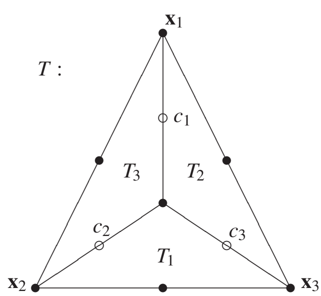

Figure 1.1(b)

The finite element constructed in Theorem 3.1 can be used to solve the following boundary value problem that has a harmonic solution:

$$\begin{aligned} -\Delta u &= 0, \quad \text{in } \Omega, \\ u &= f, \quad \text{on } \partial\Omega, \end{aligned} \tag{2}$$

where Ω is a bounded polygonal domain in $\mathbb{R}^2$. The proof of the optimal order of convergence for a special choice of the split point $\mathbf{x}_0$ can be found in [5]. It is easy to see that the proof in [5] does not depend on the exact location of the split point.

2. The inequality

THEOREM 2.1. *Let $\mathbf{x}_0$ be an interior point in the triangle $T := [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3]$. By α_{ij} we denote the angle $\widehat{\mathbf{x}_0, \mathbf{x}_i, \mathbf{x}_j}$, $i \neq j$, see Fig. 1.1(a). Then*

$$\Phi := \cos\alpha_{12}\cos\alpha_{23}\cos\alpha_{31} + \cos\alpha_{21}\cos\alpha_{32}\cos\alpha_{13} > 0.$$

Proof. Let $\mathbf{v}_i := \mathbf{x}_i - \mathbf{x}_0$, and let $\mathbf{u}_i$ be the side of T opposite $\mathbf{x}_i$, oriented counterclockwise, $i = 1,2,3$. Then $\mathbf{u}_1 = \mathbf{v}_3 - \mathbf{v}_2$, $\mathbf{u}_2 = \mathbf{v}_1 - \mathbf{v}_3$, $\mathbf{u}_3 = \mathbf{v}_2 - \mathbf{v}_1$, and

$$\begin{aligned} \Phi \times \prod_{i=1}^{3} (\|\mathbf{v}_i\| \|\mathbf{u}_i\|) =& (\mathbf{v}_1 \cdot \mathbf{u}_2)(\mathbf{v}_2 \cdot \mathbf{u}_3)(\mathbf{v}_3 \cdot \mathbf{u}_1) - (\mathbf{v}_1 \cdot \mathbf{u}_3)(\mathbf{v}_3 \cdot \mathbf{u}_2)(\mathbf{v}_2 \cdot \mathbf{u}_1) \\ =& (\mathbf{v}_1 \cdot (\mathbf{v}_1 - \mathbf{v}_3))(\mathbf{v}_2 \cdot (\mathbf{v}_2 - \mathbf{v}_1))(\mathbf{v}_3 \cdot (\mathbf{v}_3 - \mathbf{v}_2)) \\ &- (\mathbf{v}_1 \cdot (\mathbf{v}_2 - \mathbf{v}_1))(\mathbf{v}_3 \cdot (\mathbf{v}_1 - \mathbf{v}_3))(\mathbf{v}_2 \cdot (\mathbf{v}_3 - \mathbf{v}_2)). \end{aligned}$$

Finally, we use barycentric coordinates to write $\mathbf{x}_0 = b_1\mathbf{x}_1 + b_2\mathbf{x}_2 + b_3\mathbf{x}_3$, where $b_1 + b_2 + b_3 = 1$, and $b_i > 0$, $i = 1,2,3$, since $\mathbf{x}_0$ is an interior point of T. Then direct substitution shows that $b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + b_3\mathbf{v}_3 = 0$. Thus, $\mathbf{v}_3 = -\alpha\mathbf{v}_1 - \beta\mathbf{v}_2$, where both α and β are positive. Substituting $\mathbf{v}_3$ into the formula above we obtain

$$\Phi \prod_{i=1}^{3} (\|\mathbf{v}_i\| \|\mathbf{u}_i\|) = (\alpha + \beta + 1) \left(\mathbf{v}_1^2\mathbf{v}_2^2 - (\mathbf{v}_1 \cdot \mathbf{v}_2)^2\right) \left((\alpha\mathbf{v}_1 + \beta\mathbf{v}_2)^2 + \alpha\mathbf{v}_1^2 + \beta\mathbf{v}_2^2)\right).$$

Thus, Φ is positive, and moreover

$$\Phi = \frac{(\alpha + \beta + 1) \left(\mathbf{v}_1^2\mathbf{v}_2^2 - (\mathbf{v}_1 \cdot \mathbf{v}_2)^2\right) \left((\alpha\mathbf{v}_1 + \beta\mathbf{v}_2)^2 + \alpha\mathbf{v}_1^2 + \beta\mathbf{v}_2^2)\right)}{\|\mathbf{v}_1\| \|\mathbf{v}_2\| \|\mathbf{v}_3\| \|\mathbf{u}_1\| \|\mathbf{u}_2\| \|\mathbf{u}_3\|}.$$

REMARK 2.1. Note that when $\mathbf{x}_0$ approaches the boundary of T, the Cauchy-Schwarz term in the numerator of Φ makes the inequality of Theorem 2.1 sharp.

REMARK 2.2. We conjecture that some analog of Theorem 2.1 holds for a simplex and an interior point in $\mathbb{R}^n$.

3. An application

Let $\Delta_{CT}(T)$ be the Clough-Tocher split of T into three subtriangles T_1, T_2, and T_3 as in Fig. 1.1(b). Let P_2 be the space of quadratic polynomials in two variables. On T the conforming harmonic P_2 finite element space is defined as a subspace of continuous quadratic splines:

$$P_T := \{s \in C^0(T) \mid \quad s|_{T_i} \in P_2 \quad \text{and} \quad \Delta p_i = 0, \quad i = 1,2,3\}.$$

THEOREM 3.1. *The Clough-Tocher conforming P_2 harmonic finite element space P_T has dimension seven, and is unisolvent by the seven nodal values:*

$$s(\mathbf{x}_i), \quad i = 0,\dots,3, \quad s((\mathbf{x}_i + \mathbf{x}_{i+1})/2), \quad i = 1,2,3, \quad \textit{where} \quad \mathbf{x}_4 := \mathbf{x}_1. \tag{3}$$

Proof. Let s be a continuous quadratic spline on $\Delta_{CT}(T)$. Using the Bernstein-Bézier (BB) basis for a continuous spline s, see (1), the seven nodal values in (3) can be used to compute the BB-coefficients of s associated with the back dots in Fig. 1.1(b). The remaining three coefficients c_1, c_2, and c_3 associated with the midpoints of the interior edges need to be computed using the fact that s is piecewise harmonic. This yields a system of three linear equations with three unknowns. By Lemma 4.1 in [1], the following three conditions are necessary and sufficient for s to be harmonic on each triangle T_i, $i = 1,2,3$, in the triangle T:

$$c_i\|\mathbf{x}_{i+1} - \mathbf{x}_0\|\cos\alpha_{i+1,i} + c_{i+1}\|\mathbf{x}_i - \mathbf{x}_0\|\cos\alpha_{i,i+1} = f_i, \quad \mathbf{x}_4 := \mathbf{x}_1, \tag{4}$$

where f_i is a known right-hand side determined by the nodal values in (3). The determinant of the system (4) is given by

$$\begin{aligned}
&Det\begin{bmatrix} \|\mathbf{x}_2 - \mathbf{x}_0\|\cos\alpha_{21} & \|\mathbf{x}_1 - \mathbf{x}_0\|\cos\alpha_{12} & 0 \\ 0 & \|\mathbf{x}_3 - \mathbf{x}_0\|\cos\alpha_{32} & \|\mathbf{x}_2 - \mathbf{x}_0\|\cos\alpha_{23} \\ \|\mathbf{x}_3 - \mathbf{x}_0\|\cos\alpha_{31} & 0 & \|\mathbf{x}_1 - \mathbf{x}_0\|\cos\alpha_{13} \end{bmatrix} \\
&= \prod_{i=1}^{3}\|\mathbf{x}_i - \mathbf{x}_0\| Det\begin{bmatrix} \cos\alpha_{21} & \cos\alpha_{12} & 0 \\ 0 & \cos\alpha_{32} & \cos\alpha_{23} \\ \cos\alpha_{31} & 0 & \cos\alpha_{13} \end{bmatrix} \\
&\prod_{i=1}^{3}\|\mathbf{x}_i - \mathbf{x}_0\|\left(\cos\alpha_{21}\cos\alpha_{32}\cos\alpha_{13} + \cos\alpha_{12}\cos\alpha_{23}\cos\alpha_{31}\right) \neq 0,
\end{aligned}$$

by Theorem 2.1. This completes the proof.

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