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A NOTE ON SUMS OF POWERS

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Abstract. We improve a result of Bennett concerning certain sequences involving sums of powers of positive integers.

1. Introduction

Estimations of sums of powers of positive integers have important applications in the study of l^p norms of weighted mean matrices, we leave interested readers [7] and [5] for more details in this direction. There are many inequalities for sequences involving sums of powers of positive integers in the literature and we shall also refer the interested readers to the papers [5], [6], [8] as well as the references therein for some results in this area.

In this note, we are interested in certain inequalities involving the following sequence: $\{P_n(r) | n = 1, 2, 3, \ldots\}$, where r is any real number and

$$P_n(r) = \left(\frac{1}{n} \sum_{i=1}^{n} i^r \Big/ \frac{1}{n+1} \sum_{i=1}^{n+1} i^r \right)^{1/r}, \quad r \neq 0; \quad P_n(0) = \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}.$$

We note that for $r > 0$, the following inequalities are valid:

$$\frac{n}{n+1} = \lim_{r \to +\infty} P_n(r) < P_n(r) < P_n(0). \tag{1}$$

The left-hand side inequality above is known as Alzer's inequality [1], and the right-hand side inequality above is known as Martins' inequality [10]. Alzer also considered inequalities satisfied by $P_n(r)$ for $r < 0$ in [2] and he showed [2, Theorem 2.3]:

$$P_n(0) \leqslant P_n(r) \leqslant \lim_{r \to -\infty} P_n(r) = 1. \tag{2}$$

Bennett [4] proved that for $r \geqslant 1$,

$$P_n(r) \leqslant P_n(1) = \frac{n+1}{n+2} \tag{3}$$

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with the above inequality reversed when $0 < r \leqslant 1$. This inequality and inequalities (1)-(2) suggest that $P_n(r)$ is a decreasing function of r. In [6], Bennett proved this for $r \leqslant 1$ and the author gave another proof in [8]. Bennett further asked, using his notation in [6], to decide whether the sequence $(1^r, 2^r, 3^r, \dots)$ is meaningful for any $r > 1$ or not ([6, Problem 1]), which is equivalent to determining whether $P_n(r)$ is a decreasing function of r for any $r > 1$ or not. It is our goal in this note to give a weaker result related to Bennett's question above by proving the following:

THEOREM 1. *The sequence*

$$\frac{\left(\sum_{i=1}^{n} i^r\right)^{\alpha}}{\sum_{i=1}^{n} i^{\alpha(r+1)-1}}, \quad n = 1, 2, 3, \dots,$$

is decreasing for $r \geqslant 1, \alpha \geqslant 2$.

We note here that Theorem 1 improves a result of Bennett [5, Theorem 12], which established the case $\alpha = 2$ of Theorem 1. We also note that one can readily deduce from Theorem 1 using an argument similar to the discussion in the paragraph below Corollary 3.1 in [8] the following

COROLLARY 1. *For any fixed integer* $n \geqslant 1$, $P_n(r) \geqslant P_n(r')$ *for* $r' \geqslant 2r+1, r \geqslant 1$.

2. Lemmas

LEMMA 1. ([11, Lemma 2.1]) *Let* $\{B_n\}_{n=1}^{\infty}$ *and* $\{C_n\}_{n=1}^{\infty}$ *be strictly increasing positive sequences with* $B_1/B_2 \leqslant C_1/C_2$. *If for any integer* $n \geqslant 1$,

$$\frac{B_{n+1} - B_n}{B_{n+2} - B_{n+1}} \leqslant \frac{C_{n+1} - C_n}{C_{n+2} - C_{n+1}}.$$

Then $B_n/B_{n+1} \leqslant C_n/C_{n+1}$ *for any integer* $n \geqslant 1$.

LEMMA 2. *For* $r \geqslant 2, x > 0, y > 0$, *let*

$$D_r(x,y) = \frac{x^r - y^r}{x - y}, \quad x \neq y; \quad D_r(x,x) = r x^{r-1}.$$

Then for positive numbers a, b, c, d *satisfying* $a \geqslant \max(b, c, d)$ *and* $a + b \geqslant c + d$, *we have*

$$D_r(a,b) \geqslant D_r(c,d).$$

Proof. We may assume $c \geqslant d$ here and note that $D_r(x,y)$ is an increasing function of x (or y) for fixed y (or x). It follows from this that if $b \geqslant d$, then $D_r(a,b) \geqslant D_r(c,b) \geqslant D_r(c,d)$. Otherwise by our assumption, one can find a positive number a' such that $a \geqslant a' \geqslant \max(b, c, d)$ and $a' + b = c + d$.

We now recall from the theory of majorization that for two positive real finite sequences $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, $\mathbf{x}$ is said to be majorized by $\mathbf{y}$ if for all convex functions f, we have

$$\sum_{j=1}^{n} f(x_j) \leqslant \sum_{j=1}^{n} f(y_j).$$

We write $\mathbf{x} \leqslant_{maj} \mathbf{y}$ if this occurs and the majorization principle states that if (x_j) and (y_j) are decreasing, then $\mathbf{x} \leqslant_{maj} \mathbf{y}$ is equivalent to

$$\begin{aligned} &x_1 + x_2 + \ldots + x_j \leqslant y_1 + y_2 + \ldots + y_j \ \ (1 \leqslant j \leqslant n-1), \\ &x_1 + x_2 + \ldots + x_n = y_1 + y_2 + \ldots + y_n \ \ (n \geqslant 0). \end{aligned}$$

We refer the reader to [3, Sect. 1.30] for a simple proof of this.

Now let $I \subset (0, +\infty)$ be an open interval and denote $I^n = I \times I \times \cdots \times I$ (n copies). We recall a function $f : I^n \to R$ is said to be Schur convex if $f(\mathbf{x}) \leqslant f(\mathbf{y})$ for any two sequences $\mathbf{x}, \mathbf{y} \in I^n$ with $\mathbf{x} \leqslant_{maj} \mathbf{y}$. If f also has continuous partial derivatives on I^n, then f is Schur convex if and only if (see [9, p. 57])

$$(x_i - x_j)\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right) \geqslant 0. \tag{1}$$

Back to our situation, we apply the notion of majorization to write $(c,d) \leqslant_{maj} (a', b)$ and we next show that $D_r(x,y)$ satisfies the criterion (1) on $(0,+\infty) \times (0,+\infty)$. For this, we may assume $x > y$ here and then it is easy to see that it suffices to show

$$\frac{x^r - y^r}{x - y} = \frac{r}{x - y} \int_y^x t^{r-1} dt \leqslant \frac{x^{r-1} + y^{r-1}}{2}.$$

The inequality above now follows from Hadamard's inequality which asserts that for a continuous convex function $h(x)$ on an interval $[e, f]$,

$$h\left(\frac{e+f}{2}\right) \leqslant \frac{1}{f-e} \int_e^f h(x) dx \leqslant \frac{h(e) + h(f)}{2}.$$

It follows that $D_r(x,y)$ is Schur convex on $(0,+\infty) \times (0,+\infty)$ so that $D_r(a',b) \geqslant D_r(c,d)$. As $D_r(a,b) \geqslant D_r(a',b)$, this completes the proof.

LEMMA 3. *For* $r \geqslant 1, \alpha \geqslant 1$, *let*

$$g_r(\alpha) = 1 + 2^{\alpha(r+1)-1} - (1 + 2^r)^\alpha.$$

Then $g_r(\alpha) \geqslant 0$ *for* $r \geqslant 1, \alpha \geqslant 2$.

Proof. We may assume $r \geqslant 1$ is being fixed and regard $g_r(\alpha)$ as a function of α. Then

$$g_r'(\alpha) = (\ln 2^{r+1}) 2^{\alpha(r+1)-1} - \ln(1 + 2^r)(1 + 2^r)^\alpha.$$

From this we see that $g_r'(\alpha)=0$ has at most one positive root. Note that $g_r(2)\geqslant 0$ and $\lim_{\alpha\to+\infty} g_r(\alpha)=+\infty$, it thus suffices to show that $g_r'(2)>0$. Note that $g_r'(2)=f(2^r)$, where

$$f(x)=\ln(2x)(2x^2)-\ln(1+x)(1+x)^2.$$

As it is easy to check that $f(2)>0, f'(2)>0$, it suffices to show that $f''(x)\geqslant 0$ for $x\geqslant 2$. Calculation yields:

$$f''(x)=3+4\ln 2+2\Big(\ln x^2-\ln(1+x)\Big)>0.$$

The last inequality follows from $x^2>1+x$ when $x\geqslant 2$ and this completes the proof.

3. Proof of Theorem 1

We need to show that for $n\geqslant 1$, $r\geqslant 1, \alpha\geqslant 2$,

$$\frac{\left(\sum_{i=1}^{n} i^r\right)^{\alpha}}{\sum_{i=1}^{n} i^{\alpha(r+1)-1}}\geqslant\frac{\left(\sum_{i=1}^{n+1} i^r\right)^{\alpha}}{\sum_{i=1}^{n+1} i^{\alpha(r+1)-1}}.$$

When $n=1$, this follows from Lemma 3. Now by Lemma 1, it suffices to show for $n\geqslant 1$, $r\geqslant 1, \alpha\geqslant 2$,

$$\frac{\left(\sum_{i=1}^{n+1} i^r\right)^{\alpha}-\left(\sum_{i=1}^{n} i^r\right)^{\alpha}}{(n+1)^{\alpha(r+1)-1}}\geqslant\frac{\left(\sum_{i=1}^{n+2} i^r\right)^{\alpha}-\left(\sum_{i=1}^{n+1} i^r\right)^{\alpha}}{(n+2)^{\alpha(r+1)-1}}.$$

We can rewrite the above inequality as $D_\alpha(a,b)\geqslant D_\alpha(c,d)$, where

$$a=\frac{\sum_{i=1}^{n+1} i^r}{(n+1)^{r+1}}, b=\frac{\sum_{i=1}^{n} i^r}{(n+1)^{r+1}}, c=\frac{\sum_{i=1}^{n+2} i^r}{(n+2)^{r+1}}, d=\frac{\sum_{i=1}^{n+1} i^r}{(n+2)^{r+1}}.$$

It is easy to see that $a\geqslant\max(b,d)$ and $a\geqslant c$ is equivalent to $P_n(r)\geqslant P_n(0)$, which follows from (1). Thus our theorem will follow from Lemma 2 provided that we show $a+b\geqslant c+d$ here, which is

$$\frac{\sum_{i=1}^{n+1} i^r+\sum_{i=1}^{n} i^r}{(n+1)^{r+1}}\geqslant\frac{\sum_{i=1}^{n+2} i^r+\sum_{i=1}^{n+1} i^r}{(n+2)^{r+1}}. \tag{1}$$

On setting $B_n=n^{r+1}$ and $C_n=\sum_{i=1}^{n} i^r+\sum_{i=1}^{n-1} i^r$ (where we take the empty sum to be 0) in Lemma 1, it is easy to see that $B_1/B_2\leqslant C_1/C_2$. Hence inequality (1) will follow from Lemma 1 if we can show for $n\geqslant 1$,

$$\frac{(n+1)^r+n^r}{(n+1)^{r+1}-n^{r+1}}\geqslant\frac{(n+2)^r+(n+1)^r}{(n+2)^{r+1}-(n+1)^{r+1}}.$$

On setting $x=n/(n+1)$, it is easy to see that one can deduce the above inequality by showing the following function is decreasing for $0<x<1$:

$$f(x)=\frac{(1-x)(1+x^r)}{1-x^{r+1}}.$$

Calculation yields

$$f'(x) = \frac{x^{2r} - rx^{r+1} + rx^{r-1} - 1}{(1 - x^{r+1})^2}.$$

It is easy to see that the function $x \mapsto x^{2r} - rx^{r+1} + rx^{r-1} - 1$ is an increasing function of $0 < x < 1$ with value 0 when $x = 1$ for any fixed $r \geqslant 1$. This implies that $f'(x) \leqslant 0$ for $0 < x < 1$ and this completes the proof.

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