Mathematical
Inequalities
& Applications
Volume 23, Number 2 (2020), 759–764 doi:10.7153/mia-2020-23-63

POLYNOMIAL INEQUALITIES ON MEASURABLE SETS IN LORENTZ SPACES AND THEIR APPLICATIONS

F. E. LEVIS

(*Communicated by J. Soria*)

Abstract. In this short note, we study inequalities for algebraic polynomials on measurable sets in Lorentz spaces and discuss their applications to best approximation.

1. Introduction

Polynomial inequalities on measurable sets play an important role in analysis (see for instance [2, 5, 9]). In particular, generalized Polya inequalities in L^p and its application to best approximation can be see in [4].

In this paper, we establish Remez and Polya-type inequalities on measurable sets in Lorentz spaces and apply them to problems of approximation theory.

Let $\mathscr{M}_0$ be the class of all real extended μ-measurable functions on $(0,\alpha)$, $0<\alpha<\infty$, where μ is the Lebesgue measure. As usual, for $f\in\mathscr{M}_0$ we denote its distribution function by $\mu_f(\lambda)=\mu(\{x\in(0,\alpha):|f(x)|>\lambda\})$, $\lambda\geqslant 0$, and its decreasing rearrangement by $f^*(t)=\inf\{\lambda:\mu_f(\lambda)\leqslant t\}$, $t\geqslant 0$.

Let $w:(0,\alpha)\to(0,\infty)$ be a weight function locally integrable with respect to μ. We denote by $W:[0,\alpha]\to[0,\infty)$ the function $W(r)=\int_0^r w\,d\mu$. For $f\in\mathscr{M}_0$ and $1<q<\infty$, let

$$\|f\|_{w,q}=\left(\int_0^\infty (f^*)^q w\,d\mu\right)^{\frac{1}{q}}.$$

Consider the Lorentz spaces $L^{w,q}=\{f\in\mathscr{M}_0:\|f\|_{w,q}<\infty\}$ (see [3]). These spaces have been widely studied in multiple contexts. In general, they are not Banach spaces; however, they are quasi-Banach spaces if and only if W satisfies the Δ_2-condition, that is, there exists $C>0$ such that $W(2t)\leqslant CW(t)$ for all $t\geqslant 0$. Moreover, $\|\cdot\|_{w,q}$ is a norm if and only if the weight w is decreasing. In particular, if $w(t)=t^{\frac{q}{p}-1}$ with $1\leqslant q\leqslant p<\infty$, they are the classical Lorentz spaces $L^{p,q}$ and $\|f\|_{w,q}=\|f\|_{p,q}$ A good reference for a description of these spaces is [6, 7, 8].

Mathematics subject classification (2010): 46E30, 47A30, 41A10.
Keywords and phrases: Lorentz spaces, basic inequalities, approximation by polynomials, best local approximation.
This paper was partially supported by Universidad Nacional de Río Cuarto (grant PPI 18/C559), Universidad Nacional de La Pampa, Facultad de Ingeniería (grant Resol. Nro. 165/18), and CONICET (grant PIP 112-201501-00433CO).

Here and in the sequel Π^m denotes the real vector space of algebraic polynomials of degree at most m, and W satisfies the Δ_2-condition. In these cases, from [3, Corollary 2.2] we have

$$\|g+h\|_{w,q}^q \leqslant C(\|g\|_{w,q}^q + \|h\|_{w,q}^q), \qquad g,h \in L^{w,q}, \tag{1}$$

The paper is organized as follows. In Section 2, we prove a lower estimate of decreasing rearrangement of algebraic polynomials and state a theorem about the equivalence between norms on measurable sets in Lorentz spaces. As a consequence, we obtain the Remez and Polya-type inequalities for algebraic polynomials. Section 3 is devoted to show inequalities of the best polynomial approximation on measurable sets in Lorentz spaces and their applications in best local approximation.

2. Basic inequalities

For a measurable set $E \subset (0,\alpha)$ and $P \in \Pi^m$, we denote $\mu(E) = |E|$ and

$$\|P\|_E = \sup_{x \in E} |P(x)|.$$

The result below is a key theorem in this section and shows a lower estimate of decreasing rearrangements of polynomials.

THEOREM 1. *For a measurable set $E \subset (0,\alpha)$, $|E| > 0$, $t \in (0,|E|)$, and a polynomial $P \in \Pi^m$:*

$$\frac{(|E|-t)^m}{4^m |E|^m} \|P\|_E \leqslant (P\chi_E)^*(t), \tag{2}$$

where χ_E is the characteristic function of E.

Proof. Let $F_t = \{x \in E : |P(x)| \leqslant (P\chi_E)^*(t)\}$. We observe that

$$\begin{aligned} |F_t| &= |\{x \in (0,\alpha) : |(P\chi_E)(x)| \leqslant (P\chi_E)^*(t)\}| - \alpha + |E| \\ &= |E| - \mu_{P\chi_E}((P\chi_E)^*(t)) = |E| - \mu_{(P\chi_E)^*}((P\chi_E)^*(t)) = |E| - t. \end{aligned}$$

On the other hand, it follows that $\mu_{P\chi_{F_t}}((P\chi_E)^*(t)) = 0$. So $\|P\|_{F_t} = \|P\chi_{F_t}\|_{(0,\alpha)} \leqslant (P\chi_E)^*(t)$ by [1, Proposition 1.8]. According to [5, Formula (2.2)] we have

$$\|P\|_E \leqslant T_m\left(\frac{2|E|}{|F_t|} - 1\right) \|P\|_{F_t} \leqslant T_m\left(\frac{|E|+t}{|E|-t}\right) (P\chi_E)^*(t), \tag{3}$$

where

$$T_m(x) = \frac{1}{2}\left(\left(x + \sqrt{x^2-1}\right)^m + \left(x - \sqrt{x^2-1}\right)^m\right)$$

is the Chebyshev polynomial of degree m. Clearly

$$0 \leqslant \frac{T_m(x)}{x^m} \leqslant 2^m, \tag{4}$$

for every $x \geqslant 1$. Now (3) becomes

$$\|P\|_E \leqslant 2^m \left(\frac{|E|+t}{|E|-t}\right)^m (P\chi_E)^*(t) \leqslant \frac{4^m|E|^m}{(|E|-t)^m}(P\chi_E)^*(t).$$

This completes the proof. □

Now, we discuss some inequalities among norms in Lorentz spaces.

THEOREM 2. *For a measurable set $E \subset (0,\alpha)$, $|E|>0$, and a polynomial $P \in \Pi^m$:*

$$\frac{1}{8^m C^{\frac{1}{q}}} W(|E|)^{\frac{1}{q}} \|P\|_E \leqslant \|P\chi_E\|_{w,q} \leqslant W(|E|)^{\frac{1}{q}} \|P\|_E.$$

Proof. According to [1, Proposition 1.8] we have

$$(P\chi_E)^*(t) \leqslant (P\chi_E)^*(0) = \sup_{x\in(0,\alpha)} |P(x)\chi_E(x)| = \|P\|_E,$$

for every $t \in (0,\alpha)$. Then

$$\|P\chi_E\|_{w,q}^q = \int_0^{|E|} ((P\chi_E)^*)^q w d\mu \leqslant \|P\|_E^q \int_0^{|E|} w d\mu = W(|E|)\|P\|_E^q.$$

On the other hand, as W satisfies the Δ_2-condition, (2) leads to

$$\begin{aligned}\|P\|_E^q \frac{1}{8^{mq}C} W(|E|) &\leqslant \|P\|_E^q \frac{1}{8^{mq}} W\left(\frac{|E|}{2}\right) \leqslant \|P\|_E^q \int_0^{\frac{|E|}{2}} \frac{(|E|-t)^{mq}}{4^{mq}|E|^{mq}} w d\mu \\ &\leqslant \int_0^{\alpha} ((P\chi_E)^*)^q w d\mu,\end{aligned}$$

and so the proof is complete. □

COROLLARY 1. *For a measurable set $E \subset (0,\alpha)$, $|E|>0$, and a polynomial $P \in \Pi^m$:*

$$\frac{1}{2^{3m+\frac{1}{p}}} \left(\frac{p}{q}\right)^{\frac{1}{q}} |E|^{\frac{1}{p}} \|P\|_E \leqslant \|P\chi_E\|_{p,q} \leqslant \left(\frac{p}{q}\right)^{\frac{1}{q}} |E|^{\frac{1}{p}} \|P\|_E.$$

The following theorem provides us a Remez-type inequality.

THEOREM 3. *For a measurable set $E \subset (0,\alpha)$, $|E|>0$, and a polynomial $P \in \Pi^m$:*

$$\|P\|_{w,q} \leqslant \frac{16^m(2\alpha-|E|)^m C^{\frac{1}{q}} W(\alpha)^{\frac{1}{q}}}{|E|^m W(|E|)^{\frac{1}{q}}} \|P\chi_E\|_{w,q}.$$

Proof. According to Theorem 2, [5, Theorem 1.2] and (4) we have

$$\|P\|_{w,q} \leqslant W(\alpha)^{\frac{1}{q}} \|P\|_{(0,\alpha)} \leqslant W(\alpha)^{\frac{1}{q}} T_m\left(\frac{2\alpha}{|E|} - 1\right) \|P\|_E$$
$$\leqslant \frac{W(\alpha)^{\frac{1}{q}} 2^m (2\alpha - |E|)^m}{|E|^m} \|P\|_E \leqslant \frac{16^m (2\alpha - |E|)^m C^{\frac{1}{q}} W(\alpha)^{\frac{1}{q}}}{|E|^m W(|E|)^{\frac{1}{q}}} \|P\chi_E\|_{w,q}. \quad \square$$

Now, a generalized Polya-type inequality is also presented.

THEOREM 4. *For a measurable set $E \subset (0,\alpha)$, $|E| > 0$, and a polynomial $P(x) = \sum_{k=0}^{m} a_k x^k \in \Pi^m$:*

$$|a_k| \leqslant \frac{A_k}{|E|^m W(|E|)^{\frac{1}{q}}} \|P\chi_E\|_{w,q},$$

for $0 \leqslant k \leqslant m$, and where $A_k = 2^{8m+k} \alpha^{m-k} k! \binom{m}{k}^2 C^{\frac{2}{q}}$, $0 \leqslant k \leqslant m$.

Proof. Set $Q(x) = P\left(\frac{\alpha}{2}(x-1) + \alpha\right)$. By the Markov's inequality for higher derivatives ([2, p. 248]),

$$\frac{k!\alpha^k}{2^k} |a_k| \leqslant \frac{\alpha^k}{2^k} \left\|P^{(k)}\right\|_{[0,\alpha]} = \left\|Q^{(k)}\right\|_{[-1,1]} \leqslant \frac{m!^2}{(m-k)!^2} \|Q\|_{[-1,1]} = \frac{m!^2}{(m-k)!^2} \|P\|_{(0,\alpha)}.$$

On account to the above inequality and Theorems 2 and 3, we obtain

$$\frac{k!\alpha^k}{2^k} |a_k| \leqslant \frac{m!^2 8^m C^{\frac{1}{q}}}{(m-k)!^2 W(\alpha)^{\frac{1}{q}}} \|P\|_{w,q} \leqslant \frac{m!^2 2^{8m} \alpha^m C^{\frac{2}{q}}}{(m-k)!^2 |E|^m W(|E|)^{\frac{1}{q}}} \|P\chi_E\|_{w,q}.$$

This completes the proof. $\square$

3. Applications to best approximation

For $f \in L^{w,q}$, we consider the problem to find $P \in \Pi^m$, such that

$$\|(f-P)\chi_E\|_{w,q} = \inf_{Q \in \Pi^m} \|(f-Q)\chi_E\|_{w,q},$$

where $E \subset (0,\alpha)$, $|E| > 0$. Such a polynomial $P = P_{w,q}(f,E)$ there always exists and it is unique. It is called the best approximation of f on E from Π^m.

The uniform norm of the best approximation of f on E from Π^m can be bounded in terms of the measure of E and the Lorentz norm of f on E.

THEOREM 5. *Let $K = 2^{8m+\frac{1}{q}} C^{\frac{3}{q}} \alpha^m$. For a measurable set $E \subset (0,\alpha)$, $|E| > 0$, and $f \in L^{w,q}$:*

$$\left\|P_{w,q}(f,E)\right\|_{(0,\alpha)} \leqslant \frac{K}{|E|^m W(|E|)^{\frac{1}{q}}} \|f\chi_E\|_{w,q}.$$

Proof. From (1) we have

$$\left\|P_{w,q}(f,E)\chi_E\right\|_{w,q}^q \leqslant 2C\left\|f\chi_E\right\|_{w,q}^q. \tag{5}$$

Now, Theorems 2 and 3 imply

$$\begin{aligned}\left\|P_{w,q}(f,E)\right\|_{(0,\alpha)} &\leqslant \frac{8^m C^{\frac{1}{q}}}{W(\alpha)^{\frac{1}{q}}}\left\|P_{w,q}(f,E)\right\|_{w,q} \leqslant \frac{2^{8m}C^{\frac{2}{q}}\alpha^m}{|E|^m W(|E|)^{\frac{1}{q}}}\left\|P_{w,q}(f,E)\chi_E\right\|_{w,q}\\ &\leqslant \frac{K}{|E|^m W(|E|)^{\frac{1}{q}}}\left\|f\chi_E\right\|_{w,q}. \quad \square\end{aligned}$$

As an immediate consequence of Theorem 4 and (5), we estimate the coefficients of $P_{w,q}(f,E)$.

THEOREM 6. *For a measurable set $E\subset(0,\alpha)$, $|E|>0$, and $P_{w,q}(f,E)=\sum_{k=0}^m a_k x^k\in\Pi^m$:*

$$|a_k| \leqslant \frac{2^{\frac{1}{q}}C^{\frac{1}{q}}A_k}{|E|^m W(|E|)^{\frac{1}{q}}}\left\|f\chi_E\right\|_{w,q}, \tag{6}$$

for $0\leqslant k\leqslant m$, and where A_k is given in Theorem 4.

Next, we give more estimates for the coefficients of $P_{w,q}(f,E)$ that will be sharper than (6).

COROLLARY 2. *For a measurable set $E\subset(0,\alpha)$, $|E|>0$, and $P_{w,q}(f,E)=\sum_{k=0}^m a_k x^k\in\Pi^m$:*

$$|a_k| \leqslant \frac{2^{\frac{1}{q}}C^{\frac{1}{q}}A_k}{|E|^m W(|E|)^{\frac{1}{q}}}\left\|\left(f-\sum_{j=0}^{k-1}a_j x^j\right)\chi_E\right\|_{w,q},$$

for $1\leqslant k\leqslant m$, and where A_k is given in Theorem 4.

Proof. As

$$P_{w,q}(f-P,E_s)=P_{w,q}(f,E_s)-P, \tag{7}$$

for every $P\in\Pi^m$, the proof is immediately followed by (6). $\square$

For $x\in(0,\alpha)$, we consider a family of measurable subsets of $(0,\alpha)$, $\{E_s\}$, such that $|E_s|>0$ and $\sup_{y\in E_s}|y-x|\to 0$ as $|E_s|\to 0$. If there exists the limit of $P_{w,q}(f,E_s)$ as $|E_s|\to 0$, it is called the best local approximation of f at x from Π^m.

We say that that a function f belongs to the class $t_{w,q}^m(x)$ if $f\in L^{w,q}$ and there exists a polynomial $P_x(f)\in\Pi^m$ such that

$$\|(f-P_x(f))\chi_{E_s}\|_{w,q}=o\left(|E_s|^m W(|E_s|)^{\frac{1}{q}}\right) \quad \text{as} \quad |E_s|\to 0. \tag{8}$$

It is possible to see that the condition (8) determines uniquely the polynomial $P_x(f)$.

PROPOSITION 1. *Let $f \in L^{w,q}$. If there exists $P_x(f) \in \Pi^m$ that satisfies (8), then it is unique.*

Proof. Suppose there are polynomials $P_x^1(f)$ and $P_x^2(f)$ such that (8) holds. From (7) and Theorem 5 we deduce that

$$\left\|P_{w,q}(f,|E_s|) - P_x^i(f)\right\|_{(0,\alpha)} = o(1) \quad \text{as} \quad |E_s| \to 0, \quad i = 1,2.$$

So by uniqueness of limit we have $P_x^1(f) = P_x^2(f)$. $\square$

In the same manner as in the proof of Proposition 1, we can prove the following existence theorem of the best local approximation.

THEOREM 7. *If $f \in t_{w,q}^m(x)$ then $P_x(f)$ is the best local approximation of f at x from Π^m.*

Acknowledgement. I would like to thank the Referees for their valuable comments and suggestions which helped me to improve the presentation of the paper.

REFERENCES

[1] C. BENNET, R. SHARPLEY, *Interpolation of Operators*, Academic Press,San Diego, 1988.
[2] P. BORWEIN, T. ERDELYI, *Polynomials and Polynomial Inequalities*, Springer, New York, 1995.
[3] M. J. CARRO, J. SORIA, *Weighted Lorentz Spaces and the Hardy Operator* J. Funct. Anal., 112, 2, 1993, 480–494.
[4] H. CUENYA, F. LEVIS, *Pólya-type polynomial inequalities in Lp spaces and best local approximation*, Numer. Funct. Anal. Optim., 26, 7-8, 2005, 813–827.
[5] M.I. GANZBURG, *Polynomial Inequalities on Measurable Sets and Their Applications*, Constr. Approx., 17, 2, 2001, 275–306.
[6] R. HUNT, *On $L(p,q)$ spaces*, Enseign. Math., 12, 2, 1966, 249–276.
[7] G. G. LORENTZ, *Some new functional spaces*, Ann. of Math., 51, 1, 1950, 37–55.
[8] G. G. LORENTZ, *On the theory of spaces*, A, Pacific J. Math., 1, 3, 1951, 411–429.
[9] A. F. TIMAN, *Theory of Approximation of Functions of a Real Variable*, Pergamon Press, New York, 1963.

(Received August 8, 2019)

F. E. Levis
Universidad Nacional de Río Cuarto
CONICET, FCEFQyN
RN 36 Km 601, Río Cuarto, 5800, Argentina
e-mail: flevis@exa.unrc.edu.ar