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ON A CONVEXITY PROBLEM WITH APPLICATIONS TO MASTROIANNI TYPE OPERATORS

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Abstract. This work has as starting point an inequality involving Bernstein polynomials and convex functions. Applications of the main results are given for Mastroianni type operators. The results obtained here represent a continuation of what was done in [3] and are strongly connected to the work done in [1].

1. Introduction

Let $n \in \mathbb{N}$ and let Π_n denote the set of all polynomials of degree $\leqslant n$. The fundamental Bernstein polynomials of degree n are given by:

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k},\ k = 0,1,\dots,n.$$

In ([8], Problem 2, pp. 164), I. Raşa ([8], Problem 2, pp. 164), came up with the following problem: *Prove or disprove the following inequality:*

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\left[b_{n,i}(x)b_{n,j}(x) + b_{n,i}(y)b_{n,j}(y) - 2b_{n,i}(x)b_{n,j}(y)\right] f\left(\frac{i+j}{2n}\right) \geqslant 0, \tag{1}$$

for any convex function $f \in C[0,1]$ and any $x,y \in [0,1]$. In [7], by using a probabilistic approach, J. Mrowiec, T. Rajba and S. Wasowicz, gave a positive answer to the above problem and proved the following generalization of inequality (1).

THEOREM 1. ([7], Theorem 12) *Let $m,n \in \mathbb{N}$ with $m \geqslant 2$. Then,*

$$\sum_{i_1,\dots,i_m=0}^{n}\left[b_{n,i_1}(x_1)\dots b_{n,i_m}(x_1) + \dots + b_{n,i_1}(x_m)\dots b_{n,i_m}(x_m)\right.$$
$$\left. - m b_{n,i_1}(x_1)\dots b_{n,i_m}(x_m)\right] f\left(\frac{i_1+\dots+i_m}{mn}\right) \geqslant 0, \tag{2}$$

for any convex function $f \in C[0,1]$ and any $x_1,\dots,x_m \in [0,1]$.

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An elementary proof of (1), was given recently by Abel in [2], where it is shown that a type (1) inequality holds also for the Mirakyan-Favard-Szász ([2], Theorem 5) and for the Baskakov operators ([2], Theorem 6).

In [3], we proved a type (1) inequality for a large class of operators defined in the following way. Let I be one of the intervals $[0,\infty)$ or $[0,1]$. Let $g_n : I \times D \to \mathbb{C}$, $D = \{z \in \mathbb{C} \mid |z| \leqslant R\}$, $R > 1$ be a function with the property that for any fixed $x \in I$, the function $g_n(x,\cdot)$ is an analytic function on D,

$$\begin{aligned} g_n(x,z) &= \sum_{k=0}^{\infty} a_{n,k}(x) z^k \\ a_{n,k}(x) &\geqslant 0, \forall k \geqslant 0 \\ g_n(x,1) &= 1, \forall x \in I. \end{aligned} \tag{3}$$

In what follows, let $I = [0,\infty)$. The case $I = [0,1]$ follows in the same way. Let $\mathscr{F}$ be a linear set of functions defined on the interval I and let $\{A_t\}_{t\in I}$ be a set of real linear positive functionals defined on $\mathscr{F}$ with the property that for any $f \in \mathscr{F}$, the series

$$L_{n,A}(f)(x) := \sum_{k=0}^{\infty} a_{n,k}(x) A_{\frac{k}{n}}(f). \tag{4}$$

is convergent for any $x \in I$. The identity (4) defines a *positive linear operator*. The function g_n will be referred to as the *generating function* for the operator $L_{n,A}$ relative to the set of functionals $\{A_t\}_{t\in I}$.

In what follows , we assume that the linear positive functionals $\{A_t\}_{t\in I}$ are such that $L_{n,A}$ is well defined for any $f \in \mathscr{F}$ and any $x \in I$, the set of all real polynomials $\Pi \subseteq \mathscr{F}$ and every functional A_t has the following properties:

i) $A_t(e_0) = 1, t \in I$.

ii) $A_t(e_1) = at + b, t \in I$, where a and b are two real numbers independent of t and $e_i(x) = x^i$, $x \in I$, $i \in \mathbb{N}$.

In [3], we obtained the following result: *if*

$$\left[\frac{k}{n}, \frac{k+1}{n}, \frac{k+2}{n}; A_t(f)\right] \geqslant 0 \tag{5}$$

and

$$\frac{d^k}{dz^k} \left[\frac{g_n(x,z) - g_n(y,z)}{z-1}\right]^2 \Bigg|_{z=0} \geqslant 0, \tag{6}$$

for any $k \in \mathbb{N}$ and all $x, y \in I$, then $A(f) \geqslant 0$. Here, for $x, y \in I$ fixed, the functional A is defined by

$$A(f) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} [a_{n,i}(x) a_{n,j}(x) + a_{n,i}(y) a_{n,j}(y) - 2a_{n,i}(x) a_{n,j}(y)] A_{\frac{i+j}{2n}}(f).$$

The following result ([3], Corollary 3.2) is useful to verify inequality (6).
Let $x,y \in I$ be two distinct numbers. Assume that conditions i) and ii) above hold,

$$\frac{g_n(x,z) - g_n(y,z)}{z-1} = \sum_{k=0}^{\infty} \beta_{n,k}(x,y) z^k \tag{7}$$

and sgn $\beta_{n,k}(x,y)$ is the same for all $k \in \mathbb{N}$, then (6) is satisfied.

For $m \in \mathbb{N}$, $m \geqslant 2$ and $x \in I^m$, $x = (x_1, \dots, x_m)$, we define the functionals:

$$\begin{aligned} C_m(f) = & \sum_{i_1,\dots,i_m=0}^{\infty} [a_{n,i_1}(x_1)\dots a_{n,i_m}(x_1) + \dots + a_{n,i_1}(x_m)\dots a_{n,i_m}(x_m) \\ & - m a_{n,i_1}(x_1)\dots a_{n,i_m}(x_m)] A_{\frac{i_1+\dots+i_m}{mn}}(f). \end{aligned} \tag{8}$$

In [3], Theorem 4.1, we have proved the following result:
If (5) and (6) hold, then

$$C_m(f) \geqslant 0$$

for any $m \in \mathbb{N}$, $m \geqslant 2$.

Applications, such as Bernstein type operators, Mirakyan-Favard-Szász type operators, Meyer-König and Zeller type operators, were considered in [3].

Let us assume that the generating functions g_n, $n \in \mathbb{N}^*$ are of the form

$$g_n(t,z) = \phi^n(t,z), \tag{9}$$

where $\phi : I \times D \to \mathbb{C}$ is such that $\phi(t,\cdot)$ is an analytic function and the function g_n given by (9) satisfies conditions (3). Under these assumptions, we have

$$\sum_{i_1+\dots+i_m=k} a_{n,i_1}(t)\dots a_{n,i_m}(t) = a_{nm,k}(t). \tag{10}$$

The above identity implies that

$$C_m(f) = \sum_{k=1}^{m} L_{mn,A}(f)(x_k) - m \sum_{i_1,\dots,i_m=0}^{\infty} a_{n,i_1}(x_1)\dots a_{n,i_m}(x_m) A_{\frac{i_1+\dots+i_m}{mn}}(f). \tag{11}$$

Let us assume that the sequence $(L_{n,A})_{n\in\mathbb{N}^*}$ preserves convexity. More precisely, we assume that for every convex function $f \in \mathscr{F}$, $L_{n,A}(f)$, $n \in \mathbb{N}^*$ is convex too. Under this assumption, we have

$$L_{nm,A}(f)\left(\frac{x_1+\dots+x_m}{m}\right) \leqslant \sum_{k=1}^{m} \frac{L_{nm,A}(f)(x_k)}{m}. \tag{12}$$

For the Bernstein operators, in [1], the following problem was studied:
Prove that

$$B_{2n}(f)\left(\frac{x_1+x_2}{2}\right) \geqslant \sum_{i=0}^{n}\sum_{j=0}^{n} b_{n,i}(x_1) b_{n,j}(x_2) f\left(\frac{i+j}{2n}\right), \tag{13}$$

for all convex $f \in C[0,1]$ and $x_1, x_2 \in [0,1]$.

A probabilistic solution was found by A. Komisarski and T. Rajba, [5]. In [1], U. Abel and I. Raşa gave an analytic proof to the following theorem.

THEOREM 2. ([1], Theorem 1) *Let $n,m \in \mathbb{N}$. If $f \in C[0,1]$ is a convex function, then the inequality*

$$B_{mn}(f)\left(\frac{1}{m}\sum_{\nu=1}^{m} x_\nu\right) \geqslant \sum_{i_1=0}^{n} \cdots \sum_{i_m=0}^{n}\left(\prod_{\nu=1}^{m} b_{n,i_\nu}(x_\nu)\right) f\left(\frac{1}{mn}\sum_{\nu=1}^{m} i_\nu\right)$$

is valid for all $x_1,\ldots,x_m \in [0,1]$.

The purpose of this paper is to give sufficient conditions for the generating functions g_n, $n \in \mathbb{N}$, such that the functional $\mathbb{B}_m : \mathscr{F} \to \mathbb{R}$,

$$\mathbb{B}_m(f) = L_{mn,A}(f)\left(\frac{x_1+\ldots+x_m}{m}\right) - \sum_{i_1,\ldots,i_m=0}^{\infty} a_{n,i_1}(x_1)\ldots a_{n,i_m}(x_m) A_{\frac{i_1+\ldots+i_m}{mn}}(f) \quad (14)$$

is nonnegative for any function $f \in \mathscr{F}$ for which

$$\left[\frac{k}{n}, \frac{k+1}{n}, \frac{k+2}{n}; A_t(f)\right] \geqslant 0 \quad (15)$$

and for any $x = (x_1,\ldots,x_m) \in I^m$ and any $k \in \mathbb{N}$. It is immediate to see, from (11), that if $\mathbb{B}_m(f) \geqslant 0$, then $C_m(f) \geqslant 0$ as well.

2. Main results

THEOREM 3. *Let $f \in \mathscr{F}$ be such that inequality (15) holds. If*

$$\frac{d^k}{dz^k}\left[\frac{g_{nm}\left(\frac{x_1+\ldots+x_m}{m}, z\right) - g_n(x_1,z)\ldots g_n(x_m,z)}{z-1}\right]^2 \Bigg|_{z=0} \geqslant 0 \quad (16)$$

for any $k \in \mathbb{N}$ and any $x = (x_1,\ldots,x_m) \in I^m$, then

$$\mathbb{B}_m(f) \geqslant 0.$$

If (16) holds with opposite sign for any $k \in \mathbb{N}$ and any $x = (x_1,\ldots,x_m) \in I^m$, then

$$\mathbb{B}_m(f) \leqslant 0.$$

Proof. We note that

$$\mathbb{B}_m(e_0) = \mathbb{B}_m(e_1) = 0.$$

On the other hand, we have

$$\mathbb{B}_m(f) = L_{mn,A}(f)\left(\frac{x_1+\ldots.+x_m}{m}\right) - \sum_{k=0}^{\infty} \alpha_{n,k}(x) A_{\frac{k}{mn}}(f),$$

where

$$\alpha_{n,k}(x) = \sum_{i_1+\ldots+i_m=k} a_{n,i_1}(x_1)\ldots a_{n,i_m}(x_m).$$

So

$$\mathbb{B}_m(f)=\sum_{k=0}^{\infty}\left[a_{mn,k}\left(\frac{x_1+\ldots+x_m}{m}\right)-\sum_{i_1+\ldots+i_m=k}a_{n,i_1}(x_1)\ldots a_{n,i_m}(x_m)\right]f\left(\frac{k}{mn}\right).$$

We note that

$$\begin{aligned}&g_{mn}\left(\frac{x_1+\ldots+x_m}{m},z\right)-g_n(x_1,z)\ldots g_n(x_m,z)\\&=\sum_{k=0}^{\infty}\left[a_{mn,k}\left(\frac{x_1+\ldots+x_m}{m}\right)-\sum_{i_1+\ldots+i_m=k}a_{n,i_1}(x_1)\ldots a_{n,i_m}(x_m)\right]z^k.\end{aligned}\tag{17}$$

From (17), we get

$$\begin{aligned}&a_{mn,k}\left(\frac{x_1+\ldots+x_m}{m}\right)-\sum_{i_1+\ldots+i_m=k}a_{n,i_1}(x_1)\ldots a_{n,i_m}(x_m)\\&=\frac{1}{2\pi}\int_0^{2\pi}\left[g_{mn}\left(\frac{x_1+\ldots+x_m}{m},e^{i\theta}\right)-g_n\left(x_1,e^{i\theta}\right)\ldots g_n\left(x_m,e^{i\theta}\right)\right]e^{-ik\theta}d\theta,\end{aligned}\tag{18}$$

for any $k\in\mathbb{N}$. From (18), by using the same technique as in the proof of Theorem 4.1, [3], we get

$$\mathbb{B}_m(f)=\frac{2}{nm}\sum_{k=2}^{\infty}\mathbb{B}_m\left(\left|\cdot-\frac{k-1}{nm}\right|\right)\left[\frac{k-2}{mn},\frac{k-1}{mn},\frac{k}{mn};A_t(f)\right],\tag{19}$$

where

$$\mathbb{B}_m\left(\left|\cdot-\frac{k-1}{mn}\right|\right)=\frac{1}{nm}\frac{1}{(k-2)!}\frac{d^{k-2}}{dz^{k-2}}\frac{E_m^2(x,z)}{(z-1)^2}\bigg|_{z=0}\tag{20}$$

and

$$E_m(x,z)=g_{mn}\left(\frac{x_1+\ldots+x_m}{m},z\right)-g_n(x_1,z)\ldots g_n(x_m,z).\tag{21}$$

Equations (19), (20) and (7) conclude our proof. □

In what follows we are interested in whether there exists a large class of linear positive operators for which $A(f)\geqslant 0$, whenever (5) and (6) are satisfied and $\mathbb{B}_m(f)\geqslant 0$ or $\mathbb{B}_m(f)\leqslant 0$.

Mastroianni type operators

We denote by $C_2([0,\infty))$ the function space

$$C_2([0,\infty)):=\left\{f\in C([0,\infty))\ :\ \exists\lim_{x\to\infty}\frac{f(x)}{1+x^2}<\infty\right\}.$$

Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of real functions defined on $[0,\infty)$, $\varphi_n\in C^\infty[0,\infty)$, $n\in\mathbb{N}$ that are strictly monotone and satisfy the following conditions:

$$\varphi_n(0)=1,n\in\mathbb{N}\text{ and }(-1)^n\varphi_n^{(k)}(x)\geqslant 0,n\in\mathbb{N}^*,k\in\mathbb{N},x\geqslant 0,$$

$$\forall (n,k) \in \mathbb{N}\times\mathbb{N}, \exists\, p(n,k) \in \mathbb{N}, \exists \alpha_{n,k} : [0,\infty) \to \mathbb{R} \text{ such that } \forall x \geqslant 0, \forall i \in \mathbb{N}^*,$$

$$\varphi_n^{(i+k)}(x) = (-1)^k \varphi_{p(n,k)}^{(i)}(x)\alpha_{n,k}(x) \text{ and } \lim_{n\to\infty} \frac{n}{p(n,k)} = \lim_{n\to\infty} \frac{\alpha_{n,k}(x)}{n^k} = 1.$$

G. Mastroianni, in [6], introduced for any $n \in \mathbb{N}^*$, the operators $M_n : C_2([0,\infty)) \to C([0,\infty))$, defined by

$$M_n(f)(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k \varphi_n^{(k)}(x) f\left(\frac{k}{n}\right).$$

Let $(A_t)_{t\in I}$ be a set of linear positive functionals defined on the linear set of functions $\mathscr{F}$, satisfying conditions i) and ii) above and such that for every $f \in \mathscr{F}$, the series

$$M_{n,A}(f)(x) := \sum_{k=0}^{\infty} (-1)^k \frac{x^k \varphi_n^{(k)}(x)}{k!} A_{\frac{k}{n}}(f) \tag{22}$$

converges. We will assume that $\Pi_2 \subseteq \mathscr{F}$.

REMARK 1. If $\mathscr{F} = C_2([0,\infty))$, then $M_{n,A}(f)$ is well defined, [6].

LEMMA 1. *If for any $x \in [0,\infty)$, the function $g_n(x,\cdot) = \varphi_n(x(1-\cdot))$ is analytic in $D = \{z \in \mathbb{C} : |z| < R\}$, $R > 1$, then g_n is a generating function for $M_{n,A}$.*

Proof. We have

$$\frac{d^k}{dz^k} g_n(x,z) = (-1)^k x^k \varphi_n^{(k)}(x(1-z))$$

and therefore

$$g_n(x,z) = \sum_{k=0}^{\infty} (-1)^k x^k \frac{\varphi_n^{(k)}(x)}{k!} z^k. \quad \square$$

THEOREM 4. *Let $x, y \in [0,\infty)$, $x \neq y$. If*

$$\frac{g_n(x,z) - g_n(y,z)}{z-1} = \sum_{k=0}^{\infty} \beta_{n,k}(x,y) z^k,$$

then sgn $\beta_{n,k}(x,y)$ is the same for all $k \in \mathbb{N}$.

Proof. We have

$$\frac{g_n(x,z) - g_n(y,z)}{z-1} = -\sum_{p=0}^{\infty} \frac{(-1)^p x^p \varphi_n^{(p)}(x) - (-1)^p y^p \varphi_n^{(p)}(y)}{p!} z^p \sum_{m=0}^{\infty} z^m.$$

It follows that

$$\beta_{n,k}(x,y) = -\sum_{p=0}^{k} \frac{(-1)^p x^p \varphi_n^{(p)}(x) - (-1)^p y^p \varphi_n^{(p)}(y)}{p!}. \tag{23}$$

Let us consider the function $h_{n,k} : [0,\infty) \to \mathbb{R}$ defined by

$$h_{n,k}(t) = -\sum_{p=0}^{k} \frac{(-1)^p t^p \varphi_n^{(p)}(t)}{p!}.$$

We have

$$\begin{aligned} h'_{n,k}(t) &= -\sum_{p=0}^{k} \frac{(-1)^p p t^{p-1} \varphi_n^{(p)}(t)}{p!} - \sum_{p=0}^{k} \frac{(-1)^p t^p \varphi_n^{(p+1)}(t)}{p!} \\ &= \sum_{p=0}^{k-1} \frac{(-1)^p t^p \varphi_n^{(p+1)}(t)}{p!} - -\sum_{p=0}^{k} \frac{(-1)^p t^p \varphi_n^{(p+1)}(t)}{p!} \\ &= \frac{(-1)^{p+1} t^p \varphi_n^{(p+1)}(t)}{p!} \geqslant 0, \forall t \in [0,\infty), \forall p \in \mathbb{N}. \end{aligned}$$

But

$$\beta_{n,k}(x,y) = h_{n,k}(x) - h_{n,k}(y)$$

and therefore

$$sgn\beta_{n,k}(x,y) = sgn(x-y), \forall x,y \in [0,\infty),$$

which concludes our proof. □

COROLLARY 1. *Let $M_{n,A}$ be a family of Mastroianni type operators and let $f \in \mathscr{F}$. If*

$$\left[\frac{k}{n}, \frac{k+1}{n}, \frac{k+2}{n}; A_t(f)\right] \geqslant 0, \forall k \in \mathbb{N},$$

then for all the functionals C_m, given by (8), with

$$a_{n,i_k}(x_i) = \frac{(-1)^{i_k} x_i^{i_k} \varphi_n^{(i_k)}(x_i)}{i_k!},$$

we have $C_m(f) \geqslant 0$.

Examples

1. **Bernstein type operators** are Mastroianni type operators with the functions $(\varphi_n)_{n\in\mathbb{N}}$ defined by $\varphi_n(x) = (1-x)^n$ and the generating functions $g_n(x,t)$ given by

$$g_n(x,t) = (1-x+tx)^n.$$

2. **Mirakyan-Favard-Szász type operators**, $S_{n,A}$,

$$S_{n,A}(f)(x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}A_{\frac{k}{n}}(f)$$

are obtained for $\varphi_n(x) = e^{-nx}$, $x \geqslant 0$ and $g_n(x,z) = e^{-nx(1-z)}$.

3. **Baskakov type operators**, $V_{n,A}$,

$$V_{n,A}(f)(x) = (1+x)^{-n}\sum_{k=0}^{\infty}\binom{n+k-1}{k}\left(\frac{x}{1+x}\right)^k A_{\frac{k}{n}}(f)$$

are obtained for $\varphi_n(x) = (1+x)^{-n}$, $n \in \mathbb{N}^*$ and $g_n(x,z) = (1+x-xz)^{-n}$.

4. **Szász-Schurer type operators**, $S_{n,p,A}$,

$$S_{n,p,A}(f)(x) = e^{-(n+p)x}\sum_{k=0}^{\infty}\frac{(n+p)^k x^k}{k!}A_{\frac{k}{n}}(f)$$

are obtained for $\varphi_n(x) = e^{-(n+p)x}$ and $g_n(x,z) = e^{-(n+p)x(1-z)}$.

We note that in the above examples the generating functions are of the following form:

$$g_n(x,z) = \phi^{n+p}(x,z),$$

where $\phi(x,z) = e^{-x(1-z)}$ is the same with the g_1–function corresponding to the Mirakyan-Favard-Szász type operators detailed above. Let p be a fixed natural number. Using now Theorem 3, with $n := n+p$ and the results from [3] related to Mirakyan-Favard-Szász operators, the next theorem follows.

THEOREM 5. *Let $f \in \mathscr{F}$ be a function with the property that*

$$\left[\frac{k}{n}, \frac{k+1}{n}, \frac{k+2}{n}; A_t(f)\right] \geqslant 0, \forall k \in \mathbb{N}.$$

Then $\mathbb{B}_m(f) \geqslant 0$.

Concluding remarks

We mention below a few consequences of Theorem 3.

1. The Bernstein type operators verify (16). In this case $g_1(x,z) = 1 - x + zx$ and inequality (16) follows from Gusić, [4], Theorem 1 (see also [9], Equation (2)), where the following representation is given

$$\left(\sum_{\nu=1}^{m} a_\nu\right)^m - m^m \sum_{\nu=1}^{m} a_\nu = \sum_{1 \leqslant i < j \leqslant m} (a_i - a_j)^2 P_{i,j}(a_1, \ldots, a_m). \qquad (24)$$

In (24), $P_{i,j}$ are some homogeneous polynomials of degree $n-2$ with non-negative coefficients. Identity (24) was used by Abel and Raşa in [1] for the classical Bernstein operators.

2. For $g_1(x,z) = e^{-x(1-z)}$, we get

$$\mathbb{B}_m(f) = C_m(f),\ m \in \mathbb{N}^*.$$

3. In the case of Baskakov type operators, we have

$$g_1(x,z) = \frac{1}{1+x-xz}.$$

Using now (24), it follows that the reverse of inequality (16) is satisfied. Therefore, if $f \in \mathscr{F}$ and

$$\left[\frac{k}{n}, \frac{k+1}{n}, \frac{k+2}{n}; A_t(f)\right] \geqslant 0, \forall k \in \mathbb{N},$$

then the Baskakov type operators satisfy the following inequalities

$$V_{n,A}(f)\left(\frac{x_1+\ldots+x_m}{m}\right) \leqslant \sum_{i_1=0}^{\infty} \ldots \sum_{i_m=0}^{\infty} \prod_{\nu=1}^{m} a_{n,i_\nu}(x_\nu) A_{\sum_{\nu=1}^{m} i_\nu/m}$$

and

$$\sum_{i_1,\ldots,i_m=0}^{\infty} [a_{n,i_1}(x_1)..a_{n,i_m}(x_1). + ... + a_{n,i_1}(x_1 m..a_{n,i_m}(x_m)] A_{\frac{i_1+\ldots+i_m}{nm}}(f)$$

$$\geqslant m \sum_{i_1,\ldots,i_m=0}^{\infty} a_{n,i_1}(x_1)....a_{n,i_m}(x_m) A_{\frac{i_1+\ldots+i_m}{nm}}(f).$$

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