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BOUNDEDNESS AND COMPACTNESS OF THE HARDY TYPE OPERATOR WITH VARIABLE UPPER LIMIT IN WEIGHTED LEBESGUE SPACES

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Abstract. Let $0<\alpha<1$. The operator of the form

$$K_{\alpha,\varphi}f(x)=\int_a^{\varphi(x)}\frac{f(t)w(t)dt}{(W(x)-W(t))^{(1-\alpha)}},\ x>0,$$

is considered, where the real weight functions $v(x)$ and $w(x)$ are locally integrable on $I:=(a,b)$, $0\leqslant a<b\leqslant\infty$ and $\frac{dW(x)}{dx}\equiv w(x)$. In this paper we derive criteria for the operator $K_{\alpha,\varphi}$, $0<\alpha<1$, $0<p;q<\infty$, $p>\frac{1}{\alpha}$ to be bounded and compact from the spaces $L_{p,w}$ to the spaces $L_{q,v}$.

1. Introduction

Let $0<p,q<\infty$, $I=(a,b)$, $0\leqslant a<b\leqslant\infty$, $0<\alpha<1$ and $\frac{1}{p}+\frac{1}{p'}=1$. Let $W:I\to R$ be a strictly increasing and locally absolutely continuous function on I. Suppose that $\frac{dW(x)}{dx}\equiv w(x)$ almost every $x\in I$ and $W(a)=\lim\limits_{t\to a^+}W(t)>-\infty$.

Let $v:I\to I$ be a non-negative locally integrable function on I and $\varphi:I\to I$ be a strictly increasing locally absolutely continuous function with the property:

$$\lim_{x\to a^+}\varphi(x)=a,\ \lim_{x\to b^-}\varphi(x)=b,\ \varphi(x)\leqslant x,\ \forall x\in I.$$

$$K_{\alpha,\varphi}f(x)=\int_a^{\varphi(x)}\frac{f(s)w(s)ds}{(W(x)-W(s))^{1-\alpha}},\quad x\in I, \tag{1}$$

from $L_{p,w}=L_{p,w}(I)$ to $L_{q,v}=L_{q,v}(I)$, where $L_{p,w}$ is the space of measurable functions $f:I\to R$ for which the functional

$$\|f\|_{p,w}=\left(\int_a^b|f(x)|^pw(x)dx\right)^{\frac{1}{p}},\quad 0<p<\infty,$$

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is finite. Let

$$W_0(x) = W(x) - W(a). \tag{2}$$

Then $W_0(x) \geqslant 0$, $W_0(a) = 0$, and the operator (1) can be written as

$$K_{\alpha,\varphi} f(x) = \int_a^{\varphi(x)} \frac{f(s)w(s)ds}{(W_0(x) - W_0(s))^{1-\alpha}}, \quad x \in I.$$

Therefore, unless otherwise stated, further on we will assume that in (1) $W(\cdot) \geqslant 0$ and $W(a) = 0$.

In the case $\varphi(x) \equiv x$ the operator (1) is studied in the papers [1, 3], similar operators are also considered in the work [2] and in the case $\varphi(x) \equiv x$, $W(x) = x$ the operator (1) is the Riemann-Liouville operator and its various aspects are considered in many papers and books, for example in [4, 9, 10, 11, 12].

Together with operator (1) we consider the operator

$$K'_{\alpha,\varphi} g(s) = \int_{\varphi^{-1}(s)}^{b} \frac{g(x)v(x)dx}{(W(x) - W(s))^{1-\alpha}}, \quad s \in I \tag{3}$$

from $L_{p,w}$ to $L_{q,v}$, where φ^{-1} is an inverse function to φ.

Throughout this paper expressions of the form $\frac{0}{0}$, $0 \cdot \infty$ are supposed be equal to zero. The relation $A \ll B$ $(A \gg B)$ means that $A \leqslant CB$ $(B \leqslant CA)$ with a constant C depending only on p, q, α which can be different in different places. If $A \ll B$ and $A \gg B$, then we write $A \approx B$. By Z we denote the set of all integer numbers and χ_E denotes the characteristic function of the set E.

Besides the operator (1) we also consider the operator

$$H_\varphi f(x) = \frac{1}{W^{1-\alpha}(x)} \int_a^{\varphi(x)} f(s)w(s)ds, \quad x \in I. \tag{4}$$

From (1), (4) it is easy to see that

$$K_{\alpha,\varphi} f \geqslant H_\varphi f \tag{5}$$

for $f \geqslant 0$.

In assumptions about the function φ the boundedness of the operator (4) from $L_{p,w}$ to $L_{q,v}$ is equivalent (see [8]) to the boundedness of the Hardy type operator

$$Hf(x) = \frac{1}{W^{1-\alpha}(\varphi^{-1}(x))} \int_a^x f(s)w(s)ds, \quad x \in I,$$

from $L_{p,w}$ to $L_{q,\widetilde{v}}$, where $\widetilde{v}(t) = v(\varphi^{-1}(t))(\varphi^{-1}(t))'$. Therefore, from the results of the study the Hardy inequality (see, for example, [7]), we have

LEMMA 1. *Let* $1 < p \leqslant q < \infty$. *Then the operator* (4) *is bounded from* $L_{p,w}$ *to* $L_{q,v}$ *if and only if* $A = \sup_{t \in I} A(t) < \infty$, *where*

$$A(t) = \left(\int_t^b W^{q(\alpha-1)}(x) v(x) dx \right)^{\frac{1}{q}} W^{\frac{1}{p'}}(\varphi(t)).$$

Moreover, $\|H_\varphi\| \approx A$.

REMARK 1. Here and below $\|T\|$ denotes the norm of the operator $T : L_{p,w} \to L_{q,v}$, where the operator T either $T = H_\varphi$ or $T = K_{\alpha,\varphi}$.

LEMMA 2. *Let* $0 < q < p < \infty, \quad p > 1$. *Then the operator* (4) *is bounded from* $L_{p,w}$ *to* $L_{q,v}$ *if and only if*

$$B = \left(\int_a^b \left(\int_t^b W^{q(\alpha-1)}(x) v(x) dx \right)^{\frac{q}{p-q}} W^{\frac{q(p-1)}{p-q}}(\varphi(t)) \frac{v(t)dt}{W^{q(1-\alpha)}(t)} \right)^{\frac{p-q}{pq}} < \infty.$$

Moreover, $\|H_\varphi\| \approx B$.

We also need the following Lemma:

LEMMA 3. *Let* $0 < \beta < 1$ *and the function* $\gamma(\cdot)$ *defined on I, such that* $0 < \gamma(x) \leqslant 1, \quad \forall x \in I$. *Then*

$$\int_0^{\gamma(x)} \frac{dz}{(1-z)^{1-\beta}} \leqslant \frac{\gamma(x)}{\beta}, \quad \forall x \in I.$$

Indeed, using the inequality $(1-\gamma(x))^\beta \geqslant 1-\gamma(x)$, we have

$$\int_0^{\gamma(x)} \frac{dz}{(1-z)^{1-\beta}} = \frac{1}{\beta}[1-(1-\gamma(x))^\beta] \leqslant \frac{1}{\beta}[1-(1-\gamma(x))] = \frac{\gamma(x)}{\beta}.$$

2. The main results

Our first main result reads:

THEOREM 1. *Let* $1 < p \leqslant q < \infty$, $\frac{1}{p} < \alpha < 1$ *and* A *be defined as in Lemma* 1. *Then the operator* (1) *is bounded from* $L_{p,w}$ *to* $L_{q,v}$ *if and only if* $A < \infty$. *Moreover,*

$$\|K_{\alpha,\varphi}\| \approx A. \tag{6}$$

Our next main result reads:

THEOREM 2. *Let* $0 < q < p < \infty$, $p > \frac{1}{\alpha}$, $0 < \alpha < 1$ *and* B *be defined as in Lemma* 2. *Then the operator* (1) *is bounded from* $L_{p,w}$ *to* $L_{q,v}$ *if and only if* $B < \infty$. *Moreover,*

$$\|K_{\alpha,\varphi}\| \approx B. \tag{7}$$

In the case $0 \neq W(a) > -\infty$, in accordance with Remark 1 the following theorems follows from Theorems 1 and 2, respectively:

COROLLARY 1. *Let* $1 < p \leqslant q < \infty$, $\frac{1}{p} < \alpha < 1$ *and* W_0 *be defined by* (2). *Then the operator* (1) *is bounded from* $L_{p,w}$ *to* $L_{q,v}$ *if and only if*

$$A_0 = \sup_{a<z<b} \left(\int_z^b W_0^{q(\alpha-1)}(x) v(x) dx \right)^{\frac{1}{q}} W_0^{\frac{1}{p'}}(\varphi(z)) < \infty.$$

Moreover, $\|K_{\alpha,\varphi}\| \approx A_0$.

COROLLARY 2. *Let* $0 < q < p < \infty$, $p > \frac{1}{\alpha}$, $0 < \alpha < 1$ *and* W_0 *be defined by* (2). *Then the operator* (1) *is bounded from* $L_{p,w}$ *to* $L_{q,v}$ *if and only if*

$$B_0 = \left(\int_a^b \left(\int_t^b W_0^{q(\alpha-1)}(x) v(x) dx \right)^{\frac{q}{p-q}} W_0^{\frac{q(p-1)}{p-q}}(\varphi(t)) \frac{v(t) dt}{W_0^{q(1-\alpha)}(t)} \right)^{\frac{p-q}{pq}} < \infty.$$

Moreover, $\|K_{\alpha,\varphi}\| \approx B_0$.

For the operator (3) we have the following results:

THEOREM 3. *Let* $1 < p \leqslant q < \frac{1}{1-\alpha}$, $0 < \alpha < 1$ *and* W_0 *be defined by* (2). *Let* $W(a) > -\infty$. *Then the operator* $K'_{\alpha,\varphi}$ *defined by* (3) *is bounded from* $L_{p,w}$ *to* $L_{q,v}$ *if and only if*

$$A' = \sup_{a<z<b} \left(\int_z^b W_0^{p'(\alpha-1)}(x) v(x) dx \right)^{\frac{1}{p'}} W_0^{\frac{1}{q}}(\varphi(z)) < \infty.$$

Moreover, $\|K'_{\alpha,\varphi}\| \approx A'$.

THEOREM 4. *Let* $1 < q < min\{p, \frac{1}{1-\alpha}\}$, $0 < \alpha < 1$ *and* W_0 *be defined by* (2). *Let* $W(a) > -\infty$. *Then the operator* $K'_{\alpha,\varphi}$ *defined by* (3) *is bounded from* $L_{p,w}$ *to* $L_{q,v}$ *if and only if*

$$B' = \left(\int_a^b \left(\int_t^b W_0^{p'(\alpha-1)}(x) v(x) dx \right)^{\frac{p(q-1)}{p-q}} W_0^{\frac{p}{p-q}}(\varphi(t)) \frac{v(t) dt}{W_0^{p'(1-\alpha)}(t)} \right)^{\frac{p-q}{pq}} < \infty.$$

Moreover, $\|K'_{\alpha,\varphi}\| \approx B'$.

The boundedness of the operator (1) from $L_{p,w}$ to $L_{q,v}$ is equivalent to the boundedness of the adjoint operator

$$K^*_{\alpha,\varphi}g(s) = w(s)\int_{\varphi^{-1}(s)}^{b} \frac{g(x)dx}{(W(x)-W(s))^{1-\alpha}}, \quad s \in I$$

from $L_{q',v^{1-q'}}$ to $L_{p',w^{1-p'}}$, which in turn is equivalent to the boundedness of the operator $K'_{\alpha,\varphi}$ defined by (3) from $L_{q',w}$ to $L_{p',v}$. Therefore, by replacing q' and p' by p and q, respectively, in Theorems 3 and 4, we obtain the assertions of Corollaries 1 and 2, respectively.

Our main results concerning compactness of the operator $K_{\alpha,\varphi}$ reads:

THEOREM 5. *Let $0<\alpha<1$ and $\frac{1}{\alpha}<p\leqslant q<\infty$. Then the following statements are equivalent:*

i) $K_{\alpha,\varphi}: L_{p,w} \to L_{q,v}$ *is compact;*

ii) $A<\infty$ *and* $\lim\limits_{t\to a^+} A(t) = \lim\limits_{t\to b^-} A(t) = 0$.

THEOREM 6. *Let $b<\infty$, $0<\alpha<1$, $0<q<p<\infty$ and $p>\frac{1}{\alpha}$. Then the operator $K_{\alpha,\varphi}$ is compact from $L_{p,w}$ to $L_{q,v}$ if and only if $B<\infty$ holds.*

3. Proofs of the main results

Proof of Theorem 1.

Necessity. Let the operator (1) be bounded from $L_{p,w}$ to $L_{q,v}$. Then from (1), (4), (5) it follows that the operator H_φ boundedly maps from $L_{p,w}$ to $L_{q,v}$ and $\|K_{\alpha,\varphi}\| \geqslant \|H_\varphi\|$. Consequently, by virtue of Lemma 1,

$$\|K_{\alpha,\varphi}\| \gg A. \tag{8}$$

Sufficiency. Let $A<\infty$. Consider the function $W(\varphi(x))$. In view of the conditions imposed on the function φ and W we have that the function $W(\varphi(x))$ is continuous, strictly increasing and $W(\varphi(a)) = W(a) = 0$.

For any $k\in Z$ we define $x_k = \sup\{x \in I : W(\varphi(x)) \leqslant 2^k\}$. Hence, $a<x_k\leqslant x_{k+1}\leqslant b$ for any $k\in Z$ and $W(\varphi(x_k)) \equiv \lim\limits_{x\to x_k} W(\varphi(x)) \leqslant 2^k$, but if $x_k<b$, then $x_{k-1}<x_k$ and $W(\varphi(x_k)) = 2^k$.

Assume that $\varphi(x_k)=t_k$, $I_k=[x_k,x_{k+1})$, $J_k=[t_k,t_{k+1})$ and $Z_0=\{k\in Z: I_k\neq\emptyset\}$. Then

$$I = \bigcup_{k\in Z_0} I_k = \bigcup_{k\in Z_0} J_k, \tag{9}$$

$$W(\varphi(x_k)) = W(t_k) = 2^k, \quad k \in Z_0, \tag{10}$$

$$2^k \leqslant W(\varphi(x)) < 2^{k+1}, \text{ with } x \in I_k, \quad k \in Z_0. \tag{11}$$

Let $f \in L_{p,w}$. By using (9) and the relation $\varphi(x_{k-1}) \leqslant x_{k-1} < x_k$, $k \in Z_0$ we have

$$\begin{aligned}
&\int_a^b v(x)|K_{\alpha,\varphi} f(x)|^q dx \\
&\leqslant \sum_k \int_{x_k}^{x_{k+1}} v(x) \left(\int_a^{\varphi(x)} \frac{|f(s)|w(s)ds}{(W(x)-W(s))^{1-\alpha}} \right)^q dx \\
&\leqslant 2^{q-1} \left(\sum_k \int_{x_k}^{x_{k+1}} v(x) \left(\int_{\varphi(x_{k-1})}^{\varphi(x)} \frac{|f(s)|w(s)ds}{(W(x)-W(s))^{1-\alpha}} \right)^q dx \right. \\
&\quad \left. + \sum_k \int_{x_k}^{x_{k+1}} v(x) \left(\int_a^{\varphi(x_{k-1})} \frac{|f(s)|w(s)ds}{(W(x)-W(s))^{1-\alpha}} \right)^q dx \right) := 2^{q-1}(F_1 + F_2). \qquad (12)
\end{aligned}$$

Here and in the sequal, the summation is taken over the set Z_0 with respect to index k.

We estimate the expressions F_1 and F_2 separately. Applying Hölder's inequality, we obtain

$$\begin{aligned}
F_1 =& \sum_k \int_{x_k}^{x_{k+1}} v(x) \left(\int_{\varphi(x_{k-1})}^{\varphi(x)} \frac{|f(s)|w(s)ds}{(W(x)-W(s))^{1-\alpha}} \right)^q dx \\
\leqslant& \sum_k \int_{x_k}^{x_{k+1}} v(x) \left(\int_{\varphi(x_{k-1})}^{\varphi(x)} |f(s)|^p w(s)ds \right)^{\frac{q}{p}} \left(\int_{\varphi(x_{k-1})}^{\varphi(x)} \frac{w(s)ds}{(W(x)-W(s))^{p'(1-\alpha)}} \right)^{\frac{q}{p'}} dx \\
\leqslant& \sum_k \left(\int_{\varphi(x_{k-1})}^{\varphi(x_{k+1})} |f(s)|^p w(s)ds \right)^{\frac{q}{p}} \int_{x_k}^{x_{k+1}} v(x) \left(\int_a^{\varphi(x)} \frac{w(s)ds}{(W(x)-W(s))^{p'(1-\alpha)}} \right)^{\frac{q}{p'}} dx. \qquad (13)
\end{aligned}$$

Making the change of the variable $W(s) = W(x)z$ in the last integral and applying Lemma 3, we find that

$$\begin{aligned}
\int_a^{\varphi(x)} \frac{w(s)ds}{(W(x)-W(s))^{p'(1-\alpha)}} &\leqslant \frac{W(x)}{W^{p'(1-\alpha)}(x)} \int_0^{\frac{W(\varphi(x))}{W(x)}} \frac{dz}{(1-z)^{1-p'(\alpha-\frac{1}{p})}} \\
&\leqslant \frac{1}{p'(\alpha-\frac{1}{p})} \frac{W(\varphi(x))}{W^{p'(1-\alpha)}(x)}.
\end{aligned}$$

Substituting this in (13) and using (9) - (11), we obtain that:

$$\begin{aligned}
F_1 &\ll \sum_k \left(\int_{t_{k-1}}^{t_{k+1}} |f(s)|^p w(s)ds \right)^{\frac{q}{p}} \int_{x_k}^{x_{k+1}} W^{q(\alpha-1)}(x)v(x)W^{\frac{q}{p'}}(\varphi(x))dx \\
&\leqslant \sum_k \left(\int_{t_{k-1}}^{t_{k+1}} |f(s)|^p w(s)ds \right)^{\frac{q}{p}} 2^{\frac{q}{p'}(k+1)} \int_{x_k}^{x_{k+1}} W^{q(\alpha-1)}(x)v(x)dx \\
&\ll \sum_k \left(\int_{t_{k-1}}^{t_{k+1}} |f(s)|^p w(s)ds \right)^{\frac{q}{p}} W^{\frac{q}{p'}}(\varphi(x_k)) \int_{x_k}^{x_{k+1}} W^{q(\alpha-1)}(x)v(x)dx \qquad (14) \\
&\ll A^q \sum_k \left(\int_{t_{k-1}}^{t_{k+1}} |f(s)|^p w(s)ds \right)^{\frac{q}{p}} \ll A^q \left(\sum_k \int_{t_{k-1}}^{t_{k+1}} |f(s)|^p w(s)ds \right)^{\frac{q}{p}} \\
&\ll A^q \|f\|_{p,w}^q. \qquad (15)
\end{aligned}$$

In order to estimate F_2 we use (9), (10) and the estimate $W(x) \geqslant W(\varphi(x)),\ \ x \in I$, to deduce that

$$\begin{aligned}
F_2 &:= \sum_k \int_{x_k}^{x_{k+1}} v(x) \left(\int_a^{\varphi(x_{k-1})} \frac{f(s)w(s)ds}{(W(x)-W(s))^{1-\alpha}} \right)^q dx \\
&\leqslant \sum_k \int_{x_k}^{x_{k+1}} v(x) \left(\int_a^{\varphi(x_{k-1})} \frac{f(s)w(s)ds}{(W(x)-W(\varphi(x_{k-1})))^{1-\alpha}} \right)^q dx \\
&\leqslant \sum_k \int_{x_k}^{x_{k+1}} \frac{v(x)dx}{(W(x)-W(\varphi(x_{k-1})))^{q(1-\alpha)}} \left(\int_a^{\varphi(x_{k-1})} f(s)w(s)ds \right)^q .
\end{aligned}$$

Taking the following estimates

$$\begin{aligned}
W(x)-W(\varphi(x_{k-1})) &= W(x) - \frac{1}{2}\cdot 2^k = W(x) - \frac{1}{2}W(\varphi(x_k)) \\
&\geqslant W(x) - \frac{1}{2}W(x_k) \geqslant W(x) - \frac{1}{2}W(x) = \frac{1}{2}W(x),
\end{aligned}$$

for $x_k \leqslant x \leqslant x_{k+1}$, into account, we obtain that

$$\begin{aligned}
F_2 &\leqslant 2^{q(1-\alpha)} \sum_k \int_{x_k}^{x_{k+1}} \frac{v(x)}{W^{q(1-\alpha)}(x)} \left(\int_0^{\varphi(x_{k-1})} f(s)w(s)ds \right)^q dx \\
&\ll \sum_k \int_{x_k}^{x_{k+1}} v(x) \left(\frac{1}{W^{1-\alpha}(x)} \int_a^{\varphi(x)} f(s)w(s)ds \right)^q dx \leqslant \|H_\varphi f\|_{q,v}^q. \qquad (16)
\end{aligned}$$

Hence, on the basis of Lemma 1,

$$F_2 \ll A^q \|f\|_{p,w}^q. \tag{17}$$

From (12), (15) and (17) it follows that the operator (1) is bounded from $L_{p,w}$ to $L_{q,v}$, Moreover, $\|K_{\alpha,\varphi}\| \ll A$, which together with (12) gives (6). The proof is complete. $\square$

Proof of Theorem 2.

Necessity. Let the operator (1) be bounded from $L_{p,w}$ to $L_{q,v}$. Then, as in Theorem 1, from (5) and from Lemma 2, we have

$$\|K_{\alpha,\varphi}\| \gg B. \tag{18}$$

Sufficiency. Let $B < \infty$. To estimate the norm of the operator (1), we proceed from the relation (12). By virtue of (16) and Lemma 2, we have

$$F_2 \ll B^q \|f\|_{p,w}^q. \tag{19}$$

Estimating F_1 in a similar way as in Theorem 1, we obtain the relation (14) and applying Hölder's inequality with exponents $\frac{p}{q}$ and $\frac{p}{p-q}$, we have

$$\begin{aligned}
F_1 \ll& \sum_k \left(\int_{t_{k-1}}^{t_{k+1}} |f(s)|^p w(s) ds \right)^{\frac{q}{p}} W^{\frac{q}{p'}}(\varphi(x_k)) \int_{x_k}^{x_{k+1}} W^{q(\alpha-1)}(x) v(x) dx \\
\leqslant& \left(\sum_k \int_{t_{k-1}}^{t_{k+1}} |f(s)|^p w(s) ds \right)^{\frac{q}{p}} \\
&\times \left(\sum_k W^{\frac{q(p-1)}{p-q}}(\varphi(x_k)) \left(\int_{x_k}^{x_{k+1}} W^{q(\alpha-1)}(x) v(x) dx \right)^{\frac{p}{p-q}} \right)^{\frac{p-q}{p}} \\
\leqslant& 2^{\frac{q}{p}} \|f\|_{p,w}^q \left(\frac{p}{p-q} \sum_k W^{\frac{q(p-1)}{p-q}}(\varphi(x_k)) \right. \\
&\times \left. \int_{x_k}^{x_{k+1}} \left(\int_t^{x_{k+1}} W^{q(\alpha-1)}(x) v(x) dx \right)^{\frac{q}{p-q}} W^{q(\alpha-1)}(t) v(t) dt \right)^{\frac{p-q}{p}} \\
\ll& \left(\sum_k \int_{x_k}^{x_{k+1}} \left(\int_t^b W^{q(\alpha-1)}(x) v(x) dx \right)^{\frac{q}{p-q}} W^{\frac{q(p-1)}{p-q}}(\varphi(t)) \frac{v(t) dt}{W^{q(1-\alpha)}(t)} \right)^{\frac{p-q}{p}} \|f\|_{p,w}^q \\
\leqslant& B^q \|f\|_{p,w}^q.
\end{aligned} \tag{20}$$

From (12), (19) and (20) it follows that the operator (1) is bounded from $L_{p,w}$ to $L_{q,v}$ and, moreover, $\|K_{\alpha,\varphi}\| \ll B$, which together with (18) gives (7). The proof is complete. □

Proofs of Theorems 3 and 4. The proof are similar to those of Theorems 1 and 2, respectively, so we omit the details. □

Proof of Theorem 5.

Necessity. Suppose that the operator (1) is compact from $L_{p,w}(I)$ to $L_{q,v}(I)$. We show that (ii) is true.

Since the operator $K_{\alpha,\varphi}$ is compact we get that the operator (1) is bounded. Then, from Theorem 1 its follows that $A<\infty$.

To prove $\lim\limits_{t\to a^+} A(t) = \lim\limits_{t\to b^-} A(t) = 0$ we use the well known fact that a compact operator maps a weakly convergent sequence into a strongly convergent one. For $a < s < b$ consider the family of functions

$$f_s(x) = \chi_{(a,\varphi(s)]}(x) W^{-\frac{1}{p}}(\varphi(s)),\ x \in I. \tag{21}$$

It is easy to see that $\{f_s\}_{s\in(a,b)} \in L_{p,w}$.

Indeed,

$$\|f_s\|_{p,w} = \left(\int_a^b |f_s(x)|^p w(x)dx\right)^{\frac{1}{p}} = W^{-\frac{1}{p}}(\varphi(s)) \left(\int_a^{\varphi(s)} w(x)dx\right)^{\frac{1}{p}} = 1. \tag{22}$$

We show that the family of functions (21) converges weakly to zero in $L_{p,w}$.

By using properties of $\varphi(x)$ and the Hölder inequality together with (22) we find that

$$\begin{aligned}\int_a^b f_s(x)g(x)dx &= \int_a^{\varphi(s)} f_s(x)g(x)dx \\ &\leqslant \left(\int_a^b |f_s(x)|^p w(x)dx\right)^{\frac{1}{p}} \left(\int_a^s |g(x)|^{p'} w^{1-p'}(x)dx\right)^{\frac{1}{p'}} \\ &= \left(\int_a^s |g(x)|^{p'} w^{1-p'}(x)dx\right)^{\frac{1}{p'}}\end{aligned} \tag{23}$$

for all $g \in L_{p',w^{1-p'}}$.

Since $g \in L_{p',w^{1-p'}}$, then last integral in (23) tends to zero when $s\to a^+$, which means weak convergence $f_s \to 0$ at $s \to a^+$. Since a compact operator in a Banach space every weakly convergent sequence translates into a strongly convergent one, then we get that

$$\lim_{s\to a^+} \|K_{\alpha,\varphi} f_s\|_{q,v} = 0. \tag{24}$$

On the other hand, by using properties of functions $W(x)$ and $\varphi(x)$ we have

$$\begin{aligned}
\|K_{\alpha,\varphi}f_s\|_{q,v} &= \left(\int_a^b v(x)\left|\int_a^{\varphi(x)} \frac{f_s(t)w(t)dt}{(W(x)-W(t))^{1-\alpha}}\right|^q dx\right)^{\frac{1}{q}} \\
&\geqslant \left(\int_s^b v(x)\left|\int_a^{\varphi(s)} \frac{W^{-\frac{1}{p}}(\varphi(s))w(t)dt}{(W(x)-W(t))^{1-\alpha}}\right|^q dx\right)^{\frac{1}{q}} \\
&\geqslant W^{-\frac{1}{p}}(\varphi(s))\left(\int_s^b v(x)W^{q(\alpha-1)}(x)dx\right)^{\frac{1}{q}} \int_a^{\varphi(s)} w(t)dt \\
&= W^{\frac{1}{p'}}(\varphi(s))\left(\int_s^b v(x)W^{q(\alpha-1)}(x)dx\right)^{\frac{1}{q}} = A(s). \qquad (25)
\end{aligned}$$

By combining (24) and (25) we find that $\lim\limits_{s\to a^+} A(s)=0$.

Next we show that $\lim\limits_{t\to b^-} A(t)=0$. The compactness of the operator $K_{\alpha,\varphi}$ implies compactness of the dual operator

$$K^*_{\alpha,\varphi}g(t) = w(t)\int_{\varphi^{-1}(t)}^b \frac{g(x)dx}{(W(x)-W(t))^{1-\alpha}},\ t\in I, \qquad (26)$$

from $L_{q',v^{1-q'}}$ to $L_{p',w^{1-p'}}$.

For $a<s<b$ we consider the family of functions

$$g_s(x) = \chi_{[s,b)}(x)\left(\int_s^b v(t)W^{q(\alpha-1)}(t)dt\right)^{-\frac{1}{q'}} W^{(q-1)(\alpha-1)}(x)v(x),\ x\in I. \qquad (27)$$

These functions are properly defined, since the integrals in the definition of the functions $g_s(x)$, are finite because $A<\infty$.

In addition, $g_s\in L_{q',v^{1-q'}}$, for any $s\in(a,b)$. Indeed,

$$\begin{aligned}
\|g_s\|_{q',v^{1-q'}} &= \left(\int_a^b |g_s(x)|^{q'}v^{1-q'}(x)dx\right)^{\frac{1}{q'}} \\
&= \left(\int_s^b W^{q(\alpha-1)}(t)v(t)dt\right)^{-\frac{1}{q'}}\left(\int_s^b |W^{(q-1)(\alpha-1)}(x)v(x)|^{q'}v^{1-q'}(x)dx\right)^{\frac{1}{q'}} \\
&= \left(\int_s^b W^{q(\alpha-1)}(t)v(t)dt\right)^{-\frac{1}{q'}}\left(\int_s^b W^{q(\alpha-1)}(t)v(t)dt\right)^{\frac{1}{q'}} = 1. \qquad (28)
\end{aligned}$$

From (28) it follows that

$$\int_a^b g_s(x)f(x)dx = \int_s^b g_s(x)f(x)dx \leqslant \left(\int_s^b |g_s(x)|^q v^{-\frac{q'}{q}}(x)dx\right)^{\frac{1}{q'}} \left(\int_s^b |f(x)|^q v(x)dx\right)^{\frac{1}{q}}$$
$$\leqslant \left(\int_s^b |f(x)|^q v(x)dx\right)^{\frac{1}{q}} \|g_s\|_{q',v^{1-q'}} = \left(\int_s^b |f(x)|^q v(x)dx\right)^{\frac{1}{q}}$$

for all $f \in L_{q,v}$.

Since $f \in L_{q,v}$, the last integral tends to zero at $s \to b^-$. Hence, the family of functions $\{g_s\}_{s\in(a,b)}$ converge weakly to zero in $L_{q',v^{1-q'}}$ when $s \to b^-$.

The dual operator $K^*_{\alpha,\varphi}$ is compact from $L_{q',v^{1-q'}}$ to $L_{p',w^{1-p'}}$. Therefore,

$$\lim_{s\to b^-} \|K^*_{\alpha,\varphi} g_s\|_{p',w^{1-p'}} = 0. \tag{29}$$

However, the following estimate holds:

$$\|K^*_{\alpha,\varphi} g_s\|_{p',w^{1-p'}}$$
$$= \left(\int_a^b w(t)\left|\int_{\varphi^{-1}(t)}^b \frac{g_s(x)dx}{(W(x)-W(t))^{1-\alpha}}\right|^{p'} dt\right)^{\frac{1}{p'}}$$
$$\geqslant \left(\int_a^{\varphi(s)} w(t)\left|\int_{\varphi^{-1}(t)}^b \frac{g_s(x)dx}{(W(x)-W(t))^{1-\alpha}}\right|^{p'} dt\right)^{\frac{1}{p'}}$$
$$\geqslant \left(\int_a^{\varphi(s)} w(t)\left|\int_s^b \frac{W^{(q-1)(\alpha-1)}(x)v(x)dx}{(W(x))^{1-\alpha}}\right|^{p'} dt\right)^{\frac{1}{p'}} \left(\int_s^b W^{q(\alpha-1)}(t)v(t)dt\right)^{-\frac{1}{q'}}$$
$$= \left(\int_s^b W^{q(\alpha-1)}(t)v(t)dt\right)^{-\frac{1}{q'}} \int_s^b W^{q(\alpha-1)}(t)v(t)dt \left(\int_a^{\varphi(s)} w(t)dt\right)^{\frac{1}{p'}} = A(s).$$

Consequently, by using (29) we have that $\lim\limits_{s\to b^-} A(s) = 0.$ Thus, the implication (i) $\Rightarrow$ (ii) holds.

Sufficiency. Now we will prove (ii)$\Rightarrow$ (i).

Let $a < c < d < b$. We take d such that $\varphi(d) > c$ and put $P_c f = \chi_{(a,c]} f,\ P_{cd} f = \chi_{(c,d]} f,\ Q_d f = \chi_{(d,b)} f$.

Then $f = \chi_{(a,c]} f + \chi_{(c,d]} f + \chi_{(d,b)} f = P_c f + P_{cd} f + Q_d f$.

We find that

$$\begin{aligned}K_{\alpha,\varphi}f =&(P_c+P_{cd}+Q_d)K_{\alpha,\varphi}f = (P_c+P_{cd})K_{\alpha,\varphi}(P_c+P_{cd}+Q_d)f+Q_dK_{\alpha,\varphi}f\\ =&P_cK_{\alpha,\varphi}P_cf+P_cK_{\alpha,\varphi}P_{cd}f+P_cK_{\alpha,\varphi}Q_df+P_{cd}K_{\alpha,\varphi}P_cf\\ &+P_{cd}K_{\alpha,\varphi}P_{cd}f+P_{cd}K_{\alpha,\varphi}Q_df+Q_dK_{\alpha,\varphi}f.\end{aligned}$$

Thus, since $P_cK_{\alpha,\varphi}P_{cd}\equiv 0$, $P_cK_{\alpha,\varphi}Q_d\equiv 0$, $P_{cd}K_{\alpha,\varphi}Q_d\equiv 0$ we can conclude that

$$K_{\alpha,\varphi}f = P_cK_{\alpha,\varphi}P_cf+P_{cd}K_{\alpha,\varphi}P_cf+P_{cd}K_{\alpha,\varphi}P_{cd}f+Q_dK_{\alpha,\varphi}f. \tag{30}$$

We show that the operator $P_{cd}K_{\alpha,\varphi}P_{cd}$ is compact from $L_{p,w}(I)$ to $L_{q,v}(I)$. Since $P_{cd}K_{\alpha,\varphi}P_{cd}f(x)=0$ when $x\in I\backslash(c,d]$, then it suffices to show that the operator $P_{cd}K_{\alpha,\varphi}P_{cd}$ is compact from $L_{p,w}(c,d)$ to $L_{q,v}(c,d)$ and this is equivalent to the compactness from $L_{p,w}(c,d)$ to $L_{q,v}(c,d)$ of the operator $Kf(x)=\int\limits_c^d K(x,s)f(s)ds$ with the kernel

$$K(x,t)=\frac{v^{\frac{1}{q}}(x)\chi_{(c,d]}(t)\theta(\varphi(x)-t)w^{\frac{1}{p'}}(t)}{(W(x)-W(t))^{(1-\alpha)}},$$

where $\theta(z)$ is Heaviside's unit step function, (that is, $\theta(z)=1$ for $z\geqslant 0$ and $\theta(z)=0$ for $z<0$).

From the proof of the Theorem 1 there are points x_k,x_i such that $k-i=m\geqslant 1$, $x_k\geqslant d$ and $c\geqslant x_i$. Therefore, making the change of the variable $W(s)=W(x)z$ in the integral below and applying Lemma 3, we have that

$$\begin{aligned}\int\limits_c^d\left(\int\limits_c^d|K(x,t)|^{p'}dt\right)^{\frac{q}{p'}}dx &= \int\limits_c^d v(x)\left(\int\limits_c^{\varphi(x)}\frac{\chi_{(c,d]}(t)w(t)dt}{(W(x)-W(t))^{p'(1-\alpha)}}\right)^{\frac{q}{p'}}dx\\ &\leqslant \int\limits_c^d v(x)\left(\int\limits_a^{\varphi(x)}\frac{w(t)dt}{(W(x)-W(t))^{p'(1-\alpha)}}\right)^{\frac{q}{p'}}dx\\ &\ll \int\limits_{x_i}^{x_k} v(x)W^{q(\alpha-1)}(x)v(x)W^{\frac{q}{p'}}(\varphi(x))dx\\ &\leqslant W^{\frac{q}{p'}}(\varphi(x_k))\int\limits_{x_i}^{x_k}v(x)W^{q(\alpha-1)}(x)dx\\ &\ll W^{\frac{q}{p'}}(\varphi(x_i))\int\limits_{x_i}^{b}v(x)W^{q(\alpha-1)}(x)dx\leqslant A^q<\infty.\end{aligned}$$

Therefore, on the basis of the theorem in Kantorovich and Akilov (see [5], page 420), the operator K is compact from $L_p(c,d)$ to $L_q(c,d)$, which is equivalent to the compactness of the operator $P_{cd}K_{\alpha,\varphi}P_{cd}$ from $L_{p,w}(I)$ to $L_{q,v}(I)$.

By using (30) we find that

$$\|K_{\alpha,\varphi} - P_{cd}K_{\alpha,\varphi}\| \leqslant \|P_cK_{\alpha,\varphi}\| + \|Q_dK_{\alpha,\varphi}\| + \|P_{cd}K_{\alpha,\varphi}P_c\|. \tag{31}$$

We will show that the right-hand side of (31) tends to zero as $c \to a^+$ and $d \to b^-$. This will imply that the operator $K_{\alpha,\varphi}$ being a uniform limit of compact operators, is compact from $L_{p,w}(I)$ to $L_{q,v}(I)$.

Consider each of the operators in (31) separately. By Theorem 1 we have

$$\|P_cK_{\alpha,\varphi}P_cf\|_{q,v} = \left(\int_a^c v(x)\left|\int_a^{\varphi(x)} \frac{f(t)w(t)dt}{(W(x)-W(t))^{(1-\alpha)}}\right|^q dx\right)^{\frac{1}{q}}$$

$$\ll \sup_{a<t<c} W^{\frac{1}{p'}}(\varphi(t))\left(\int_t^c W^{q(\alpha-1)}(x)v(x)dx\right)^{\frac{1}{q}} \|f\|_{p,w} \leqslant \sup_{a<t<c} A(t)\|f\|_{p,w}.$$

Hence, $\|P_cK_{\alpha,\varphi}P_c\| \ll \sup_{a<t<c} A(t)$. Then

$$\lim_{c\to a^+} \|P_cK_{\alpha,\varphi}P_c\| \ll \lim_{t\to a^+} A(t) = 0. \tag{32}$$

Let $v_d = Q_dv$. Then, by Theorem 1 we obtain that

$$\|Q_bK_{\alpha,\varphi}f\|_{q,v} = \|K_{\alpha,\varphi}f\|_{q,v_d} \ll \sup_{a<t<b} W^{\frac{1}{p'}}(\varphi(t))\left(\int_t^b W^{q(\alpha-1)}(x)v_d(x)dx\right)^{\frac{1}{q}} \|f\|_{p,w}$$

$$= \sup_{d<t<b} W^{\frac{1}{p'}}(\varphi(t))\left(\int_t^b W^{q(\alpha-1)}(x)v(x)dx\right)^{\frac{1}{q}} \|f\|_{p,w} = \sup_{d<t<b} A(t)\|f\|_{p,w}.$$

Consequently,

$$\lim_{d\to b^-} \|Q_dK_{\alpha,\varphi}\| \ll \lim_{t\to b^-} A(t) = 0. \tag{33}$$

Now we will prove that

$$\lim_{c\to a^+} \|P_{cd}K_{\alpha,\varphi}P_c\| = 0. \tag{34}$$

Since $\varphi(d) > c$ and the function $\varphi(x)$ is continuous then there exists a point $z \in (c,d)$ such that $\varphi(z) = c$. Since $\varphi(x)$ is a strictly increasing function, then $z = \varphi^{-1}(c)$.

We have that

$$\|P_{cd}K_{\alpha,\varphi}P_cf\|_{q,v}^q = \int_c^{\varphi^{-1}(c)} v(x)\left|\int_a^{\varphi(x)} \frac{\chi_{(a,c]}(t)f(t)w(t)dt}{(W(x)-W(t))^{(1-\alpha)}}\right|^q dx$$

$$+ \int_{\varphi^{-1}(c)}^d v(x)\left|\int_a^{\varphi(x)} \frac{\chi_{(a,c]}(t)f(t)w(t)dt}{(W(x)-W(t))^{(1-\alpha)}}\right|^q dx := J_1 + J_2. \tag{35}$$

By Theorem 1, we get that

$$J_1 \leqslant \int_a^{\varphi^{-1}(c)} v(x) \left| \int_a^{\varphi(x)} \frac{f(t)w(t)dt}{(W(x)-W(t))^{(1-\alpha)}} \right|^q dx \ll \sup_{a<t<\varphi^{-1}(c)} A^q(t)\|f\|^q_{p,w}. \tag{36}$$

Making the change of the variable $W(t) = W(x)s$ in the integral below and applying Hölder's inequality and Lemma 1 we obtain that

$$\begin{aligned}
J_2 &= \int_{\varphi^{-1}(c)}^{d} v(x) \left(\int_a^c \frac{f(t)w(t)dt}{(W(x)-W(t))^{(1-\alpha)}} \right)^q dx \\
&\leqslant \int_{\varphi^{-1}(c)}^{d} v(x) \left(\int_a^c \frac{w(t)dt}{(W(x)-W(t))^{p'(1-\alpha)}} \right)^{\frac{q}{p'}} dx \|f\|^q_{p,w} \\
&= \int_{\varphi^{-1}(c)}^{d} v(x) \frac{(W(x))^{\frac{q}{p'}}}{(W(x))^{q(1-\alpha)}} \left(\int_a^{\frac{W(c)}{W(x)}} \frac{ds}{(1-s)^{p'(1-\alpha)}} \right)^{\frac{q}{p'}} dx \|f\|^q_{p,w} \\
&\ll \int_{\varphi^{-1}(c)}^{d} v(x) \frac{(W(x))^{\frac{q}{p'}}}{(W(x))^{q(1-\alpha)}} \left(\frac{W(c)}{W(x)} \right)^{\frac{q}{p'}} dx \|f\|^q_{p,w} \\
&= W^{\frac{q}{p'}}(c) \int_{\varphi^{-1}(c)}^{d} v(x)(W(x))^{q(1-\alpha)} dx \|f\|^q_{p,w} = A^q(\varphi^{-1}(c)) \|f\|^q_{p,w}.
\end{aligned} \tag{37}$$

Since $\varphi^{-1}(c) \to a^+$ at $c \to a^+$, then from (36), (37) and (35) we have (34).

From (32), (33) and (34) it follows that the right side of (31) tends to zero with $c \to a^+$ and $d \to b^-$. The proof is complete. □

Proof of Theorem 6. The statement of Theorem 6 follows by Ando Theorem and its generalizations [6]. □

REFERENCES

[1] A.M. ABYLAYEVA, *Criterion of the boundedness of* $0 < \alpha < 1$ *- order fractional integration type operator*, Bulletin of L.N. Gumilyov ENU, **46**, 6 2005, 130–136 (in Russian).

[2] A.M. ABYLAYEVA, *Inequalities for Some Classes of Hardy Type Operators and Compactness in Weighted Lebesgue Spaces*, Doctoral Thesis, Luleå University of Technology, Sweden, ISSN 1402-1544, December 2016.

[3] A.M. ABYLAYEVA AND R. OINAROV, *Criterion of the boundedness of a class of fractional integration operators*, Math. J. Almaty, **4**, 12 (2004), No.2, 5–14 (in Russian).

[4] D.E. EDMUNDS, V. KOKILASHVILI AND A. MESKHI, *Bounded and compact integral operators*, Kluwer Academic Published, Boston / Dordrecht / London, 2002.

[5] L.V. KANTAROVICH AND G.R. AKILOV, *Functional analysis*, Publishing House "Nauka", Moscow, 1977, 741 (in Russian).

[6] M.A. KRASNOSELSKII, P.P. ZABREIKO, E.I. PUSTYLNIK AND P.E. SOBOLEVSKII, *Integral operators in space of summable functions*, Publishing House "Nauka", Moscow, 1966, 500 (in Russian).

[7] A. KUFNER, L. MALIGRANDA AND L.-E. PERSSON, *The Hardy Inequality. About its History and some Related Results*, Vydavatelsky Servis Publishing House, Pilsen, 2007.

[8] A. KUFNER, L.-E. PERSSON AND N.SAMKO, *Weighted Inequalities of Hardy Type*, Second Edition, World Scientific, New Jersey-London-Singapore-Beijing-ShanghaiHong Kong-Taipei-Chennai-Tokyo, 2017.

[9] M.A. LORENTE, *A characterization of two weight norm inequalities for one-side operators of fractional type*, Can.Math.J. **49**, 5 (1997), 1010–1033.

[10] A. MESKHI, *Solution of some weight problems for the Reimann -Liouville and Weyl operators*, Georgian Math.J. **5**, 6 (1998), 565–574.

[11] D.V. PROKHOROV, *On the boundedness and compactness of a class of integral operators*, J.London Math. Soc. **61**, 2 (2000), 617–628.

[12] D.V. PROKHOROV AND V.D. STEPANOV, *Riemann-Liouville operators*, Reports of the Russian Academy of Sciencesh. **382**, 4 (2002), 452–455 (in Russian).

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