ASYMPTOTIC EXPANSION AND BOUNDS
FOR COMPLETE ELLIPTIC INTEGRALS

MIAO-KUN WANG, YU-MING CHU*, YONG-MIN LI AND WEN ZHANG

(*Communicated by J. Pečarić*)

Abstract. In the article, we present several new bounds for the the complete elliptic integrals $\mathcal{K}(r) = \int_0^{\pi/2}(1-r^2\sin^2\theta)^{-1/2}d\theta$ and $\mathcal{E}(r) = \int_0^{\pi/2}(1-r^2\sin^2\theta)^{1/2}d\theta$, and find an asymptotic expansion for $\mathcal{K}(r)$ as $r \to 1$, which are the refinements and improvements of the previously well-known results.

1. Introduction

For $r \in [0,1]$, Lengedre's complete elliptic integrals of the first and second kind [18, 20, 21, 22, 37, 60, 72, 73, 74, 82, 83, 87] are defined by

$$\mathcal{K} = \mathcal{K}(r) = \int_0^{\pi/2}(1-r^2\sin^2\theta)^{-1/2}d\theta, \quad \mathcal{K}' = \mathcal{K}'(r) = \mathcal{K}(r'),$$

$$\mathcal{K}(0) = \pi/2, \qquad \mathcal{K}(1) = \infty,$$

and

$$\mathcal{E} = \mathcal{E}(r) = \int_0^{\pi/2}(1-r^2\sin^2\theta)^{1/2}d\theta, \quad \mathcal{E}' = \mathcal{E}'(r) = \mathcal{E}(r'),$$

$$\mathcal{E}(0) = \pi/2, \qquad \mathcal{E}(1) = 1,$$

respectively. Here and in what follows we set $r' = \sqrt{1-r^2}$. These integrals are special cases of Guassian hypergeometric function [43, 55, 56, 57, 58, 62, 63, 66, 79, 91, 92, 94, 95]

$$F(a,b;c;x) = {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!} \quad (-1 < x < 1),$$

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where $(a)_0 = 1$, $(a)_n = \prod_{k=0}^{n-1}(a+k) = \Gamma(n+a)/\Gamma(a)$ for $n \geqslant 1$ and $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ is the Euler gamma function [29, 30, 46, 49, 50, 65, 78, 84, 85, 86, 89, 93]. Indeed, we have

$$\mathscr{K}(r) = \frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;r^2\right) \quad \text{and} \quad \mathscr{E}(r) = \frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2};1;r^2\right).$$

It is well known that the complete elliptic integrals have many important applications in physics, engineering, geometric function theory, theory of mean values, number theory and other related fields [1, 4, 7, 10, 31, 34, 35, 48, 52, 75]. Particularly, in the past two decades, a great deal of mathematical effort in the complete elliptic integrals has been devoted to the study of distortion estimates in quasiconformal mappings [24, 25, 26, 27, 28, 32, 69, 70]. For more properties and recent development of $\mathscr{K}$ and $\mathscr{E}$ see the book [13], and the research articles [23, 36, 61, 64, 68, 81, 88].

1.1. The complete elliptic integrals and their related means

An important method to study the complete elliptic integrals is to consider their convexity and monotonicity properties, and their related bivariate means [2, 3, 5, 6, 8, 9, 16, 33, 39, 47, 76, 77, 90]. In 1875, Lagrange firstly defined the arithmetic-geometric mean $AG(a,b)$ [19, 38, 42, 45, 71, 80] of two positive real numbers a and b as the common limit of the following sequences $\{a_n\}$ and $\{b_n\}$:

$$a_0 = a, \quad b_0 = b,$$

$$a_{n+1} = A(a_n,b_n) = \frac{a_n+b_n}{2}, \quad b_{n+1} = G(a_n,b_n) = \sqrt{a_nb_n}.$$

Later, Gauss proved that

$$AG(a,b) = \frac{\pi/2}{\displaystyle\int_0^{\pi/2}\frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}} = \begin{cases} \pi a \Big/ \left[2\mathscr{K}\left(\sqrt{1-(b/a)^2}\right)\right], & a \geqslant b, \\ \pi b \Big/ \left[2\mathscr{K}\left(\sqrt{1-(a/b)^2}\right)\right], & a < b. \end{cases}$$

From then on, $AG(a,b)$ and related iterations have been used for efficient numerical evaluation of the complete elliptic integral $\mathscr{K}(r)$.

In 1998, a class of quasi-arithmetic mean related to the arithmetic-geometric mean was introduced by Toader in [53], which is defined by

$$M_{p,n}(a,b) = p^{-1}\left(\frac{1}{\pi}\int_0^\pi p(r_n(\theta)d\theta)\right) = p^{-1}\left(\frac{2}{\pi}\int_0^{\pi/2}p(r_n(\theta)d\theta)\right),$$

where $r_n(\theta) = (a^n\cos^2\theta + b^n\sin^2\theta)^{1/n}$ for $n \neq 0$, and $r_0(\theta) = a^{\cos^2\theta}b^{\sin^2\theta}$, and p is a strictly monotonic function. In the particular case $p = x^{-1}$ and $n = 2$, $M_{p,n}(a,b)$ reduces to $AG(a,b)$. Moreover, taking $p = \sqrt{x}$ and $n = 1$, a new symmetric mean $E(a,b)$ [17, 40, 41, 67, 96] involving the complete elliptic integral of the second kind

was obtained

$$E(a,b) = \begin{cases} 4a \left[\mathscr{E} \left(\sqrt{1-b/a} \right) \right]^2 / \pi^2, \ a \geqslant b, \\ 4b \left[\mathscr{E} \left(\sqrt{1-a/b} \right) \right]^2 / \pi^2, \ a < b, \end{cases}$$

which will be studied and derived several asymptotic bounds for $\mathscr{E}$ in this paper.

1.2. Some well known bounds for $\mathscr{K}$ and $\mathscr{E}$

Ramanujan's work on the asymptotic behavior of the hypergeometric functions shows that $\mathscr{K}(r)$ satisfies (see [13, 1.48])

$$\mathscr{K}(r) + \log r' = \log 4 + O((1-r^2)\log(1-r^2)), \quad r \to 1. \tag{1.1}$$

Anderson, Vamanamurthy and Vuorinen [12] approximated $\mathscr{K}(r)$ by the inverse hyperbolic tangent function arth, obtaining the inequalities

$$\frac{\pi}{2} \left(\frac{\mathrm{arth} r}{r} \right)^{1/2} < \mathscr{K}(r) < \frac{\pi}{2} \frac{\mathrm{arth} r}{r} \tag{1.2}$$

for all $r \in (0,1)$. In 1995, Sándor [51] refined the elegant inequalities for $AG(a,b)$ in terms of arithmetic and the logarithmic means

$$\frac{a-b}{\log a - \log b} = L(a,b) < AG(a,b) < A(a,b) = \frac{a+b}{2},$$

or equivalently

$$\frac{\pi}{2} \frac{1}{A(1,r')} < \mathscr{K}(r) < \frac{\pi}{2} \frac{1}{L(1,r')}$$

and proved that for $r \in (0,1)$

$$\frac{\pi}{2} \left(\frac{2/\pi}{A(1,r')} + \frac{1-2/\pi}{L(1,r')} \right) < \mathscr{K}(r) < \frac{\pi}{2} \left(\frac{12/(5\pi)}{A(1,r')} + \frac{1-12/(5\pi)}{L(1,r')} \right). \tag{1.3}$$

In 2004, Alzer and Qiu [11] refined (1.2) and (1.3) as

$$\frac{\pi}{2} \left(\frac{\mathrm{arth} r}{r} \right)^{3/4} < \mathscr{K}(r) < \frac{\pi}{2} \frac{\mathrm{arth} r}{r} \tag{1.4}$$

with the best possible exponents $3/4$ and 1, and

$$\frac{\pi}{2} \left(\frac{\alpha}{A(1,r')} + \frac{1-\alpha}{L(1,r')} \right) < \mathscr{K}(r) < \frac{\pi}{2} \left(\frac{\beta}{A(1,r')} + \frac{1-\beta}{L(1,r')} \right) \tag{1.5}$$

with the optimal parameters $\alpha = 2/\pi$ and $\beta = 3/4$.

In 1997, Vuorinen [54] conjectured that the inequality

$$\mathscr{E}(r) > \frac{\pi}{2} \left(\frac{1 + r'^{3/2}}{2} \right)^{2/3} \tag{1.6}$$

holds for all $r \in (0,1)$. Later, Barnard et al. [14, 15] verified the conjecture. And in [11] the authors also provided an optimal upper bound for $\mathscr{E}(r)$ in terms of power mean of 1 and r', namely, for $0 < r < 1$,

$$\mathscr{E}(r) < \frac{\pi}{2} \left(\frac{1 + r'^{\log 2/\log(\pi/2)}}{2} \right)^{\log(\pi/2)/\log 2}. \tag{1.7}$$

In this paper, we shall give an asymptotic expansion for $\mathscr{K}(r)$ as $r \to 1$ in Section 2, which is a refinement of (1.1). In Section 3, several sharp symmetrical bounds for $AG(a,b)$ and $E(a,b)$ are found. These results lead to several new bounds for $\mathscr{K}(r)$ and $\mathscr{E}(r)$ in Section 4. Finally, in Section 5 we shall compare our results with (1.4)-(1.7).

In order to prove our main results we need some formulas [13, Appendix E, pp. 474-475] and a technical lemma, which we present in this section.

$$\frac{d\mathscr{K}}{dr} = \frac{\mathscr{E} - r'^2 \mathscr{K}}{rr'^2}, \quad \frac{d\mathscr{E}}{dr} = \frac{\mathscr{E} - \mathscr{K}}{r}, \quad \frac{d(\mathscr{E} - r'^2 \mathscr{K})}{dr} = r\mathscr{K},$$

$$\mathscr{K}\left(\frac{2\sqrt{r}}{1+r} \right) = (1+r)\mathscr{K}(r), \quad \mathscr{E}\left(\frac{2\sqrt{r}}{1+r} \right) = \frac{2\mathscr{E} - r'^2 \mathscr{K}}{1+r}.$$

LEMMA 1.1. [13, Theorem 1.25]. *For* $-\infty < a < b < \infty$, *let* $f, g : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$, *and be differentiable on* (a,b), *let* $g'(x) \neq 0$ *on* (a,b). *If* $f'(x)/g'(x)$ *is increasing (decreasing) on* (a,b), *then so are*

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad and \quad \frac{f(x) - f(b)}{g(x) - g(b)}.$$

If $f'(x)/g'(x)$ *is strictly monotone, then the monotonicity in the conclusion is also strict.*

2. An asymptotic expansion for $\mathscr{K}(r)$ as $r \to 1$

In this section, we find an infinite series for the function $r \to \mathscr{K}(r) + \log r'$ defined on $(0,1]$. This result yields an asymptotic expansion for $\mathscr{K}(r)$ as $r \to 1$, which is a refinement of (1.1).

THEOREM 2.1. *Let* $r \in (0,1]$, $r_0 = r$, $r_1 = 2\sqrt{r}/(1+r) = \varphi_2(r_0)$, $r_2 = 2\sqrt{r_1}/(1+r_1) = \varphi_2(r_1) = \varphi_4(r_0), \cdots, r_n = 2\sqrt{r_{n-1}}/(1+r_{n-1}) = \varphi_2(r_{n-1}) = \varphi_{2^n}(r_0)$. *Then*

$$\mathscr{K}(r) + \log r' = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(3 + r_k)\log(1 + r_k) - (1 - r_k)\log(1 - r_k)}{\prod_{i=0}^{k}(1 + r_i)}.$$

Proof. By simple computation we get

$$\mathscr{K}(r_1) + \log r'_1 = \mathscr{K}\left(\frac{2\sqrt{r}}{1+r} \right) + \log\left(\frac{1-r}{1+r} \right) = (1+r)\mathscr{K}(r) + \log\left(\frac{1-r}{1+r} \right)$$

$$=(1+r)[\mathscr{K}(r)+\log r']-\frac{1+r}{2}\log(1-r^2)+\log\left(\frac{1-r}{1+r}\right)$$

$$=(1+r)[\mathscr{K}(r)+\log r']-\frac{(3+r)\log(1+r)-(1-r)\log(1-r)}{2},$$

that is,

$$\frac{\mathscr{K}(r_1)+\log r_1'}{1+r_0}=\mathscr{K}(r_0)+\log r_0'-\frac{(3+r_0)\log(1+r_0)-(1-r_0)\log(1-r_0)}{2(1+r_0)}. \quad (2.1)$$

Putting r_{n-1} into (2.1) instead of r_0, one has

$$\frac{\mathscr{K}(r_n)+\log r_n'}{1+r_{n-1}}$$
$$=\mathscr{K}(r_{n-1})+\log r_{n-1}'-\frac{(3+r_{n-1})\log(1+r_{n-1})-(1-r_{n-1})\log(1-r_{n-1})}{2(1+r_{n-1})}. \quad (2.2)$$

It follows from (2.1) and (2.2) that

$$\mathscr{K}(r)+\log r'-\frac{(3+r_0)\log(1+r_0)-(1-r_0)\log(1-r_0)}{2(1+r_0)}$$
$$-\frac{(3+r_1)\log(1+r_1)-(1-r_1)\log(1-r_1)}{2(1+r_0)(1+r_1)}$$
$$=\frac{\mathscr{K}(r_2)+\log r_2'}{(1+r_0)(1+r_1)},$$

and

$$\mathscr{K}(r)+\log r'-\frac{(3+r_0)\log(1+r_0)-(1-r_0)\log(1-r_0)}{2(1+r_0)}$$
$$-\frac{(3+r_1)\log(1+r_1)-(1-r_1)\log(1-r_1)}{2(1+r_0)(1+r_1)}$$
$$-\frac{(3+r_2)\log(1+r_2)-(1-r_2)\log(1-r_2)}{2(1+r_0)(1+r_1)(1+r_2)}$$
$$=\frac{\mathscr{K}(r_3)+\log r_3'}{(1+r_0)(1+r_1)(1+r_2)}.$$

Generally, by recursion we can obtain

$$\mathscr{K}(r)+\log r'-\frac{1}{2}\sum_{k=0}^{n}\frac{(3+r_k)\log(1+r_k)-(1-r_k)\log(1-r_k)}{\prod_{i=0}^{k}(1+r_i)}=\frac{\mathscr{K}(r_{n+1})+\log r_{n+1}'}{\prod_{i=0}^{n}(1+r_i)}.$$

Since the function $r\to\mathscr{K}(r)+\log r'$ is strictly decreasing from $(0,1]$ onto $[\log 4,\pi/2)$,

$$0<\mathscr{K}(r)+\log r'-\frac{1}{2}\sum_{k=0}^{n}\frac{(3+r_k)\log(1+r_k)-(1-r_k)\log(1-r_k)}{\prod_{i=0}^{k}(1+r_i)}<\frac{\pi/2}{\prod_{i=0}^{n}(1+r_i)}.$$

Letting $n \to \infty$, then $\prod_{i=0}^{\infty}(1+r_i) = +\infty$ for $r \in (0,1]$, and thereby

$$\mathcal{K}(r) + \log r' = \frac{1}{2}\sum_{k=0}^{\infty} \frac{(3+r_k)\log(1+r_k) - (1-r_k)\log(1-r_k)}{\prod_{i=0}^{k}(1+r_i)}.$$

COROLLARY 2.2. *Let r_n be defined as in Theorem 2.1. Then the complete elliptic integrals of the first kind $\mathcal{K}(r)$ has the following asymptotic formula*

$$\mathcal{K}(r) + \log r' = \log 4 - \sum_{k=0}^{\infty} \frac{(1-r_k)\log(1-r_k)}{2^{k+2}}, \quad r \to 1.$$

3. Bounds for $AG(a,b)$ and $E(a,b)$

In this section, we shall prove several sharp symmetrical bounds for $AG(a,b)$ and $E(a,b)$ in terms of some classical mean values, such as arithmetic mean $A(a,b)$, geometric mean $G(a,b)$, harmonic mean $H(a,b)$ and logarithmic mean $L(a,b)$. We first establish three monotonicity lemmas involving the complete elliptic integrals $\mathcal{K}(r)$ and $\mathcal{E}(r)$.

LEMMA 3.1. (1) *The function $r \mapsto (\mathcal{E} - r'^2\mathcal{K})/r^2$ is strictly increasing from $(0,1)$ onto $(\pi/4,1)$;*

(2) *the function $r \mapsto 2\mathcal{E} - r'^2\mathcal{K}$ is strictly increasing from $(0,1)$ onto $(\pi/2,2)$;*

(3) *the function $r \mapsto [\mathcal{K} - \mathcal{E} - (\mathcal{E} - r'^2\mathcal{K})]/r^4$ is strictly increasing from $(0,1)$ onto $(\pi/16,\infty)$;*

(4) *the function $r \mapsto r'^{3/4}(\mathcal{K} - \mathcal{E})/r^2$ is strictly decreasing from $(0,1)$ onto $(\pi/4,\infty)$;*

(5) *the function $r \mapsto \mathcal{E} + r^2\mathcal{K}$ is strictly increasing from $(0,1)$ onto $(\pi/2,\infty)$;*

(6) *the function $r \mapsto [4(2\mathcal{E} - r'^2\mathcal{K})^2/\pi^2 - (1-r^2)]/r^2$ is strictly increasing from $(0,1)$ onto $(3/2,16/\pi^2)$.*

Proof. Parts (1)-(4) can be found in [13, Theorem 3.21(1), Exercise 3.43(13)], [59, Lemma 2.2(4)] and [11, Theorem 15].

For part (5), let $g(r) = \mathcal{E} + r^2\mathcal{K}$, then by differentiation we have

$$g'(r) = \frac{\mathcal{E} - \mathcal{K}}{r} + 2r\mathcal{K} + r^2\frac{\mathcal{E} - r'^2\mathcal{K}}{rr'^2} = \frac{\mathcal{E} - r'^2\mathcal{K} + r^2r'^2\mathcal{K}}{rr'^2} > 0$$

for all $r \in (0,1)$. Thus $g(r)$ is strictly increasing on $(0,1)$, and the limiting values are clear.

For part (6), let $h_1(r) = 4(2\mathscr{E} - r'^2\mathscr{K})^2/\pi^2 - (1 - r^2)$, $h_2(r) = r^2$ and $h(r) = h_1(r)/h_2(r)$. Then $h(1^-) = 16/\pi^2$, $h_1(0) = h_2(0) = 0$ and

$$\frac{h_1'(r)}{h_2'(r)} = \frac{4}{\pi^2}\left(2\mathscr{E} - r'^2\mathscr{K}\right)\left(\frac{\mathscr{E} - r'^2\mathscr{K}}{r^2}\right) + 1. \tag{3.1}$$

It follows from (3.1) and parts (1) and (2) together with Lemma 1.1 that $h(r)$ is strictly increasing on $(0,1)$ and $h(0^+) = 3/2$.

LEMMA 3.2. *The function*

$$f(r) = \frac{4(1+r^2)(2\mathscr{E} - r'^2\mathscr{K})^2/\pi^2 - (1-r^2)^2}{r^2}$$

is strictly decreasing from $(0,1)$ *onto* $(32/\pi^2, 7/2)$.

Proof. Clearly $f(1^-) = 32/\pi^2$, and by l'Hôpital's rule and Lemma 3.1(1), (2) we have

$$\lim_{r\to 0^+} f(r) = \lim_{r\to 0}\left[\frac{4}{\pi^2}(2\mathscr{E} - r'^2\mathscr{K})^2 + \frac{4}{\pi^2}(1+r^2)(2\mathscr{E} - r'^2\mathscr{K})\frac{\mathscr{E} - r'^2\mathscr{K}}{r^2} + 2(1-r^2)\right]$$

$$= \frac{7}{2}.$$

Differentiating f yields

$$f'(r) = \frac{[8r(2\mathscr{E} - r'^2\mathscr{K})^2/\pi^2 + 8(1+r^2)(2\mathscr{E} - r'^2\mathscr{K})(\mathscr{E} - r'^2\mathscr{K})/r/\pi^2 + 4r(1-r^2)]r^2}{r^4}$$

$$- \frac{[4(1+r^2)(\mathscr{E} - r'^2\mathscr{K})^2/\pi^2 - (1-r^2)^2]2r}{r^4}$$

$$= \frac{1}{r^3}\left[-\frac{8}{\pi^2}(2\mathscr{E} - r'^2\mathscr{K})^2 + \frac{8}{\pi^2}(1+r^2)(2\mathscr{E} - r'^2\mathscr{K})(\mathscr{E} - r'^2\mathscr{K}) + 2(1-r^4)\right]$$

$$= \frac{2r'^2}{r^3}\left[1 + r^2 - \frac{4}{\pi^2}(2\mathscr{E} - r'^2\mathscr{K})(\mathscr{E} + r^2\mathscr{K})\right]. \tag{3.2}$$

Let

$$f_1(r) = 1 + r^2 - \frac{4}{\pi^2}(2\mathscr{E} - r'^2\mathscr{K})(\mathscr{E} + r^2\mathscr{K}), \quad r \in (0,1). \tag{3.3}$$

Then

$$\lim_{r\to 0^+} f_1(r) = 0, \tag{3.4}$$

$$\frac{f_1'(r)}{r} = \frac{1}{r}\left[2r - \frac{4}{\pi^2}\frac{\mathscr{E} - r'^2\mathscr{K}}{r}(\mathscr{E} + r^2\mathscr{K}) - \frac{4}{\pi^2}(2\mathscr{E} - r'^2\mathscr{K})\frac{\mathscr{E} - r'^2\mathscr{K} + r^2r'^2\mathscr{K}}{rr'^2}\right]$$

$$=2 - \frac{4}{\pi^2} \frac{\mathscr{E} - r'^2 \mathscr{K}}{r^2} (\mathscr{E} + r^2 \mathscr{K}) - \frac{4}{\pi^2} (2\mathscr{E} - r'^2 \mathscr{K}) \left(\frac{\mathscr{E} - r'^2 \mathscr{K}}{r^2 r'^2} + \mathscr{K} \right) = f_2(r).$$

$$(3.5)$$

The equation (3.5) and Lemma 3.1(1), (2) and (5) imply that $f_2(r)$ is strictly decreasing on $(0,1)$. Note that $f_2(0) = 0$. Thus $f_2(r) < 0$ for $r \in (0,1)$. By (3.4) and (3.5) we conclude that $f_1(r)$ is strictly decreasing on $(0,1)$, and $f_1(r) < 0$ for $r \in (0,1)$. Therefore, using (3.2) and (3.3), the monotonicity of f follows.

LEMMA 3.3. *Let $c \in \mathbb{R}$. Then the function*

$$f(r) = r'^c \left[\frac{\mathscr{E} - r'^2 \mathscr{K} - r'^2 (\mathscr{K} - \mathscr{E})}{r^4} \right]$$

is strictly decreasing from $(0,1)$ onto $(0, 3\pi/16)$ if and only if $c \geqslant 1/2$.

Proof. By power series expansion, one has

$$\frac{2}{\pi} \left[\mathscr{E} - r'^2 \mathscr{K} - r'^2 (\mathscr{K} - \mathscr{E}) \right]$$

$$= \frac{2}{\pi} \left[(1 + r'^2) \mathscr{E} - 2r'^2 \mathscr{K} \right]$$

$$= (2 - r^2) \sum_{n=0}^{\infty} \frac{(-1/2)_n (1/2)_n}{(n!)^2} r^{2n} - 2(1 - r^2) \sum_{n=0}^{\infty} \frac{(1/2)_n (1/2)_n}{(n!)^2} r^{2n}$$

$$= r^2 + (2 - r^2) \sum_{n=1}^{\infty} \frac{(-1/2)_n (1/2)_n}{(n!)^2} r^{2n} - 2(1 - r^2) \sum_{n=1}^{\infty} \frac{(1/2)_n (1/2)_n}{(n!)^2} r^{2n}$$

$$= r^2 - \frac{(2 - r^2)}{2} \sum_{n=0}^{\infty} \frac{(1/2)_n (1/2)_{n+1}}{[(n+1)!]^2} r^{2n+2} - 2(1 - r^2) \sum_{n=0}^{\infty} \frac{(1/2)_{n+1} (1/2)_{n+1}}{[(n+1)!]^2} r^{2n+2}$$

$$= r^2 - \sum_{n=0}^{\infty} \frac{(1/2)_n (1/2)_{n+1}}{[(n+1)!]^2} r^{2n+2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(1/2)_n (1/2)_{n+1}}{[(n+1)!]^2} r^{2n+4}$$

$$- 2 \sum_{n=0}^{\infty} \frac{(1/2)_{n+1} (1/2)_{n+1}}{[(n+1)!]^2} r^{2n+2} + 2 \sum_{n=0}^{\infty} \frac{(1/2)_{n+1} (1/2)_{n+1}}{[(n+1)!]^2} r^{2n+4}$$

$$= - \sum_{n=0}^{\infty} \frac{(1/2)_{n+1} (1/2)_{n+2}}{[(n+2)!]^2} r^{2n+4} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(1/2)_n (1/2)_{n+1}}{[(n+1)!]^2} r^{2n+4}$$

$$- 2 \sum_{n=0}^{\infty} \frac{(1/2)_{n+2} (1/2)_{n+2}}{[(n+2)!]^2} r^{2n+4} + 2 \sum_{n=0}^{\infty} \frac{(1/2)_{n+1} (1/2)_{n+1}}{[(n+1)!]^2} r^{2n+4}$$

$$= \frac{3}{8} \sum_{n=0}^{\infty} \frac{(1/2)_n (3/2)_n}{(3)_n} \frac{r^{2n+4}}{n!} = \frac{3}{8} r^4 F\left(\frac{1}{2}, \frac{3}{2}; 3; r^2 \right).$$

$$(3.6)$$

According to (3.6), $f(r)$ can be rewritten as

$$f(r) = \frac{3\pi}{16} r'^c F\left(\frac{1}{2}, \frac{3}{2}; 3; r^2 \right).$$

$$(3.7)$$

Lemma 2.15(1) in [44] shows that $x \to (1-x)^d F(a,b;a+b+1,x)$ $(a,b>0)$ is strictly decreasing on $(0,1)$ if and only if $d \geqslant ab/(a+b+1)$. Taking $a = 1/2$ and $b = 3/2$, it follows that $f(r)$ is strictly decreasing on $(0,1)$ if and only if $c \geqslant 1/2$. Moreover, when $c \geqslant 1/2$, by (3.7) we get $f(0^+) = 3\pi/16$, and $f(1^+) = 0$.

THEOREM 3.4. *The double inequality*

$$\frac{\alpha}{L(a,b)} + \frac{1-\alpha}{He(a,b)} < \frac{1}{AG(a,b)} < \frac{\beta}{L(a,b)} + \frac{1-\beta}{He(a,b)}$$

holds for all $a,b > 0$ *with* $a \neq b$ *if and only if* $\alpha \leqslant 1/2$ *and* $\beta \geqslant 2/\pi$, *where* $He(a,b) = 2A(a,b)/3 + G(a,b)/3$.

Proof. According to the fact that $He(a,b) > L(a,b)$ for all $a,b > 0$ with $a \neq b$, it follows that Theorem 3.4 is equivalent to

$$\alpha < \frac{\dfrac{1}{AG(a,b)} - \dfrac{1}{He(a,b)}}{\dfrac{1}{L(a,b)} - \dfrac{1}{He(a,b)}} < \beta \tag{3.8}$$

for all $a,b > 0$ with $a \neq b$ if and only if $\alpha \leqslant 1/2$ and $\beta \geqslant 2/\pi$.

Since $L(a,b)$, $He(a,b)$ and $AG(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a = 1 > b$. Let $b = (1-r)/(1+r)(r \in (0,1))$. Then

$$\frac{A(a,b)}{L(a,b)} = \frac{1/(1+r)}{2r/\log[(1+r)/(1-r)]/(1+r)} = \frac{\mathrm{arth}r}{r}, \tag{3.9}$$

$$\frac{A(a,b)}{AG(a,b)} = \frac{1/(1+r)}{(\pi/2)/\mathscr{K}[2\sqrt{r}/(1+r)]} = \frac{2\mathscr{K}(r)}{\pi}, \tag{3.10}$$

$$\frac{A(a,b)}{He(a,b)} = \frac{1/(1+r)}{2/[3(1+r)] + (1/3)\sqrt{(1-r)/(1+r)}} = \frac{3}{2+r'}, \tag{3.11}$$

$$\frac{\dfrac{1}{AG(a,b)} - \dfrac{1}{He(a,b)}}{\dfrac{1}{L(a,b)} - \dfrac{1}{He(a,b)}} = \frac{2\mathscr{K}(r)/\pi - 3/(2+r')}{\mathrm{arth}r/r - 3/(2+r')} = \frac{2r\mathscr{K}(r)/\pi - 3r/(2+r')}{\mathrm{arth}r - 3r/(2+r')}. \tag{3.12}$$

Let

$$G(r) = \frac{2r\mathscr{K}(r)/\pi - 3r/(2+r')}{\mathrm{arth}r - 3r/(2+r')}, \quad r \in (0,1), \tag{3.13}$$

$G_1(r) = 2r\mathscr{K}(r)/\pi - 3r/(2+r')$, $G_2(r) = \mathrm{arth}r - 3r/(2+r')$, $G_3(r) = 2\mathscr{E}(r)/\pi - [3r'(1+2r')]/(2+r')^2$, $G_4(r) = 1 - [3r'(1+2r')]/(2+r')^2$. Then $G(r) = G_1(r)/G_2(r)$, $G_1(0^+) = G_2(0^+) = G_3(0^+) = G_4(0^+) = 0$ and

$$\frac{G_1'(r)}{G_2'(r)} = \frac{2\mathscr{E}(r)/(\pi r'^2) - [3(1+2r')]/[r'(2+r')^2]}{1/r'^2 - [3(1+2r')]/[r'(2+r')^2]} = \frac{G_3(r)}{G_4(r)},$$

$$\frac{G_3'(r)}{G_4'(r)} = \frac{2(\mathscr{E} - \mathscr{K})/(\pi r) + [3r(2 + 7r')]/[r'(2 + r')^3]}{[3r(2 + 7r')]/[r'(2 + r')^3]}$$

$$= 1 - \frac{2}{3\pi} \frac{r'^{3/4}(\mathscr{K} - \mathscr{E})}{r^2} \cdot \frac{r'^{1/4}(2 + r')^3}{7r' + 2}. \tag{3.14}$$

It is easy to check that $r \to r(2 + r^4)^3/(7r^4 + 2)$ is strictly increasing from $(0,1)$ onto $(0,3)$. Then from (3.14) and Lemma 3.1(4) we know that $G_3'(r)/G_4'(r)$ is strictly increasing on $(0,1)$, and $\lim_{r \to 0^+} [G_3'(r)/G_4'(r)] = 1/2$. Applying Lemma 1.1 two times, we obtain that $G(r)$ is strictly increasing on $(0,1)$, and by l'Hôpital's rule, $G(0^+) = \lim_{r \to 0^+} [G_3'(r)/G_4'(r)] = 1/2$ and $G(1^-) = \lim_{r \to 1^-} [G_3(r)/G_4(r)] = 2/\pi$. Therefore, Theorem 3.4 directly follows from (3.8)-(3.13).

Combining (1.5) and (3.8), we see that

$$\frac{1}{2L(a,b)} + \frac{1}{2He(a,b)} < \frac{1}{AG(a,b)} < \frac{3}{4L(a,b)} + \frac{1}{4A(a,b)} \tag{3.15}$$

for all $a,b > 0$ with $a \neq b$. The following Theorem 3.5 will be a refinement of (3.15).

THEOREM 3.5. *The double inequality*

$$\alpha \left(\frac{3}{4L(a,b)} + \frac{1}{4A(a,b)} \right) + (1 - \alpha) \left(\frac{1}{2L(a,b)} + \frac{1}{2He(a,b)} \right)$$

$$< \frac{1}{AG(a,b)} \beta \left(\frac{3}{4L(a,b)} + \frac{1}{4A(a,b)} \right) + (1 - \beta) \left(\frac{1}{2L(a,b)} + \frac{1}{2He(a,b)} \right) \tag{3.16}$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $\alpha \leqslant 17/44$ and $\beta \geqslant 8/\pi - 2$, where $He(a,b) = 2A(a,b)/3 + G(a,b)/3$.

Proof. It suffices to establish the double inequality

$$\alpha < \frac{\dfrac{1}{AG(a,b)} - \left(\dfrac{1}{2L(a,b)} + \dfrac{1}{2He(a,b)} \right)}{\dfrac{3}{4L(a,b)} + \dfrac{1}{4A(a,b)} - \left(\dfrac{1}{2L(a,b)} + \dfrac{1}{2He(a,b)} \right)} < \beta \tag{3.17}$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $\alpha \leqslant 17/44$ and $\beta \geqslant 8/\pi - 2$.

Since $L(a,b)$, $A(a,b)$, $He(a,b)$ and $AG(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a = 1 > b$. Let $b = (1 - r)/(1 + r)(r \in (0,1))$. Then from (3.9)-(3.12) we have

$$\frac{\dfrac{1}{AG(a,b)} - \left(\dfrac{1}{2L(a,b)} + \dfrac{1}{2He(a,b)} \right)}{\dfrac{3}{4L(a,b)} + \dfrac{1}{4A(a,b)} - \left(\dfrac{1}{2L(a,b)} + \dfrac{1}{2He(a,b)} \right)}$$

$$=\frac{2\mathscr{K}(r)/\pi-\mathrm{arth}r/(2r)-3/[2(2+r')]}{\mathrm{arth}r/(4r)+1/4-3/[2(2+r')]}$$

$$=\frac{4r\mathscr{K}(r)/\pi-\mathrm{arth}r-3r/(2+r')}{\mathrm{arth}r/2+r/2-3r/(2+r')}. \qquad (3.18)$$

Let

$$F(r)=\frac{4r\mathscr{K}(r)/\pi-\mathrm{arth}r-3r/(2+r')}{\mathrm{arth}r/2+r/2-3r/(2+r')}, \quad r\in(0,1), \qquad (3.19)$$

$F_1(r)=4r\mathscr{K}(r)/\pi-\mathrm{arth}r-3r/(2+r')$, $F_2(r)=\mathrm{arth}r/2+r/2-3r/(2+r')$, $F_3(r)=4\mathscr{E}(r)/\pi-1-[3r'(1+2r')]/(2+r')^2$, $F_4(r)=1/2+r'^2/2-[3r'(1+2r')]/(2+r')^2$, $F_5(r)=-4(\mathscr{K}-\mathscr{E})/(\pi r^2)+[3(7r'+2)]/[r'(2+r')^3]$, $F_6(r)=-1+[3(7r'+2)]/[r'(2+r')^3]$. Then $F(r)=F_1(r)/F_2(r)$, $F_1(0^+)=F_2(0^+)=F_3(0^+)=F_4(0^+)=F_5(0^+)=F_6(0^+)=0$ and

$$\frac{F_1'(r)}{F_2'(r)}=\frac{4\mathscr{E}(r)/(\pi r'^2)-1/r'^2-[3(1+2r')]/[r'(2+r')^2]}{1/(2r'^2)+1/2-[3(1+2r')]/[r'(2+r')^2]}=\frac{F_3(r)}{F_4(r)}$$

$$\frac{F_3'(r)}{F_4'(r)}=\frac{4(\mathscr{E}-\mathscr{K})/(\pi r)+[3r(2+7r')]/[r'(2+r')^3]}{-r+[3r(2+7r')]/[r'(2+r')^3]}=\frac{F_5(r)}{F_6(r)}$$

$$\frac{F_5'(r)}{F_6'(r)}=\frac{-4[\mathscr{E}-r'^2\mathscr{K}-r'^2(\mathscr{K}-\mathscr{E})]/(\pi r^3 r'^2)+3r(21r'^2+8r'+4)/[r'^3(2+r')^4]}{3r(21r'^2+8r'+4)/[r'^3(2+r')^4]}$$

$$=1-\frac{4}{3\pi}\frac{r'^{1/2}[\mathscr{E}-r'^2\mathscr{K}-r'^2(\mathscr{K}-\mathscr{E})]}{r^4}\cdot\frac{r'^{1/2}(2+r')^4}{21r'^2+8r'+4}. \qquad (3.20)$$

It is not difficult to verify that $r\to r(2+r^2)^4/(21r^4+8r^2+4)$ is strictly increasing from $(0,1)$ onto $(0,27/11)$. Then from (3.20) and Lemma 3.3 we know that $F_5'(r)/F_6'(r)$ is strictly increasing on $(0,1)$, and $\lim_{r\to0^+}[F_5'(r)/F_6'(r)]=17/44$. Applying Lemma 1.1 three times, we obtain that $F(r)$ is strictly increasing on $(0,1)$, and by l'Hôpital's rule, $F(0^+)=\lim_{r\to0^+}[F_5'(r)/F_6'(r)]=17/44$ and $F(1^-)=\lim_{r\to1^-}[F_3(r)/F_4(r)]=8/\pi-2$.

Therefore, Theorem 3.5 directly follows from (3.17)-(3.19) together with the monotonicity and range of F on $(0,1)$.

THEOREM 3.6. *Let* $\alpha_1,\alpha_2,\beta_1,\beta_2\in(0,1)$. *Then:*

(1) *the double inequality*

$$\alpha_1A(a,b)+(1-\alpha_1)G(a,b)<E(a,b)<\beta_1A(a,b)+(1-\beta_1)G(a,b)$$

holds for all $a,b>0$ *with* $a\neq b$ *if and only if* $\alpha_1\leqslant3/4$ *and* $\beta_1\geqslant8/\pi^2$;

(2) *the double inequality*

$$\alpha_2A(a,b)+(1-\alpha_2)H(a,b)<E(a,b)<\beta_2A(a,b)+(1-\beta_2)H(a,b)$$

holds for all $a,b>0$ *with* $a\neq b$ *if and only if* $\alpha_2\leqslant8/\pi^2$ *and* $\beta_2\geqslant7/8$, *where* $H(a,b)=2ab/(a+b)$ *is classical harmonic mean of* a *and* b.

Proof. It is clear to see that parts (1) and (2) are respectively equivalent to

$$\alpha_1 < \frac{E(a,b) - G(a,b)}{A(a,b) - G(a,b)} < \beta_1,$$

$$\alpha_2 < \frac{E(a,b) - H(a,b)}{A(a,b) - H(a,b)} < \beta_2$$

hold for all $a,b > 0$ with $a \neq b$ if and only if $\alpha_1 \leqslant 3/4$, $\beta_1 \geqslant 8/\pi^2$, $\alpha_2 \leqslant 8/\pi^2$ and $\beta_2 \geqslant 7/8$.

Without loss of generality, we assume that $a = 1 > b$. Let $b = [(1-r)/(1+r)]^2 (r \in (0,1))$. Then simple computations leads to

$$\frac{E(a,b)}{A(a,b)} = \frac{4\left[\mathscr{E}\left(\frac{2\sqrt{r}}{1+r}\right)\right]^2 / \pi^2}{(1+r^2)/(1+r)^2} = \frac{4}{\pi^2} \frac{(2\mathscr{E} - r'^2 \mathscr{K})^2}{1+r^2},$$

$$\frac{G(a,b)}{A(a,b)} = \frac{1-r^2}{1+r^2}, \quad \frac{H(a,b)}{A(a,b)} = \left(\frac{1-r^2}{1+r^2}\right)^2.$$

Consequently,

$$\frac{E(a,b) - G(a,b)}{A(a,b) - G(a,b)} = \frac{4(2\mathscr{E} - r'^2 \mathscr{K})^2 / [\pi^2(1+r^2)] - (1-r^2)/(1+r^2)}{1 - (1-r^2)/(1+r^2)}$$

$$= \frac{4(2\mathscr{E} - r'^2 \mathscr{K})^2 / \pi^2 - (1-r^2)}{2r^2}, \tag{3.21}$$

$$\frac{E(a,b) - H(a,b)}{A(a,b) - H(a,b)} = \frac{4(2\mathscr{E} - r'^2 \mathscr{K})^2 / [\pi^2(1+r^2)] - (1-r^2)^2/(1+r^2)^2}{1 - (1-r^2)^2/(1+r^2)^2}$$

$$= \frac{4(1+r^2)(2\mathscr{E} - r'^2 \mathscr{K})^2 / \pi^2 - (1-r^2)^2}{4r^2}, \tag{3.22}$$

Therefore, Theorem 3.6 directly follows from (3.21), (3.22), Lemma 3.1(6) and Lemma 3.2.

THEOREM 3.7. *The double inequality*

$$\alpha\left[\frac{7}{8}A(a,b) + \frac{1}{8}H(a,b)\right] + (1-\alpha)\left[\frac{3}{4}A(a,b) + \frac{1}{4}G(a,b)\right]$$

$$< E(a,b) < \beta\left[\frac{7}{8}A(a,b) + \frac{1}{8}H(a,b)\right] + (1-\beta)\left[\frac{3}{4}A(a,b) + \frac{1}{4}G(a,b)\right]$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $\alpha \leqslant 3/16$ and $\beta \geqslant 4(16/\pi^2 - 3/2)$.

Proof. With the similar argument in Theorem 3.6, we assume that $a = 1$ and $b = [(1-r)/(1+r)]^2 (r \in (0,1))$. Thus we only need to prove that

$$J(r) = \frac{E(a,b) - [3A(a,b)/4 + G(a,b)/4]}{7A(a,b)/8 + H(a,b)/8 - [3A(a,b)/4 + G(a,b)/4]}$$

$$
\begin{aligned}
&= \frac{E(a,b)/A(a,b) - [3/4 + G(a,b)/(4A(a,b))]}{1/8 + H(a,b)/(8A(a,b)) - [G(a,b)/(4A(a,b))]}\\
&= \frac{4(2\mathcal{E} - r'^2\mathcal{K})^2/\pi^2/(1+r^2) - 3/4 - (1-r^2)/(1+r^2)/4}{1/8 + (1-r^2)^2/(1+r^2)^2/8 - (1-r^2)/(1+r^2)/4}\\
&= \frac{2(1+r^2)\left[4(2\mathcal{E} - r'^2\mathcal{K})^2/\pi^2 - 1 - r^2/2\right]}{r^4}
\end{aligned}
\tag{3.23}
$$

is strictly increasing from $(0,1)$ onto $(3/16, 4(16/\pi^2 - 3/2))$.

Let $J_1(r) = 2(1+r^2)\left[4(2\mathcal{E} - r'^2\mathcal{K})^2/\pi^2 - 1 - r^2/2\right]$, $J_2(r) = r^4$, $J_3(r) = 4(2\mathcal{E} - r'^2\mathcal{K})^2/\pi^2 - 1 - r^2/2 + 1/2(1+r^2)[8(2\mathcal{E} - r'^2\mathcal{K})(\mathcal{E} - r'^2\mathcal{K})/(\pi^2 r^2) - 1]$ and $J_4(r) = r^2$. Then $J(r) = J_1(r)/J_2(r)$, $J_1(0^+) = J_2(0^+) = J_3(0^+) = J_4(0^+) = 0$ and

$$
\begin{aligned}
\frac{J_1'(r)}{J_2'(r)} &= \frac{4(2\mathcal{E} - r'^2\mathcal{K})^2/\pi^2 - 1 - r^2/2}{r^2} + \frac{1+r^2}{2r^2}\left[\frac{8}{\pi^2}(2\mathcal{E} - r'^2\mathcal{K})\frac{\mathcal{E} - r'^2\mathcal{K}}{r^2} - 1\right]\\
&= \frac{J_3(r)}{J_4(r)},
\end{aligned}
\tag{3.24}
$$

$$
\begin{aligned}
\frac{J_3'(r)}{J_4'(r)} &= \frac{8}{\pi^2}(2\mathcal{E} - r'^2\mathcal{K})\frac{\mathcal{E} - r'^2\mathcal{K}}{r^2} - 1 + \frac{2}{\pi^2}(1+r^2)\\
&\quad \times \left[\left(\frac{\mathcal{E} - r'^2\mathcal{K}}{r^2}\right)^2 + (2\mathcal{E} - r'^2\mathcal{K})\frac{\mathcal{K} - \mathcal{E} - (\mathcal{E} - r'^2\mathcal{K})}{r^4}\right].
\end{aligned}
\tag{3.25}
$$

The equation (3.25) and Lemma 3.1(1)-(3) show that $J_3'(r)/J_4'(r)$ is strictly increasing from $(0,1)$ onto $(3/16, +\infty)$. Therefore, the monotonicity of $J(r)$ follows from (3.23) and (3.24) together with Lemma 1.1. Moreover, by l'Hopital's rule, $J(0^+) = \lim_{r\to 0^+}[J_3'(r)/J_4'(r)] = 3/16$. And $J(1^-) = 4(16/\pi^2 - 3/2)$ is clear.

4. Bounds for $\mathcal{K}(r)$ and $\mathcal{E}(r)$

THEOREM 4.1. *For all $r \in (0,1)$ we have*

$$
\frac{\pi}{2}\left[\alpha\frac{\text{arth}r}{r} + (1-\alpha)\frac{3}{2+r'}\right] < \mathcal{K}(r) < \frac{\pi}{2}\left[\beta\frac{\text{arth}r}{r} + (1-\beta)\frac{3}{2+r'}\right]
\tag{4.1}
$$

with the best possible constants

$$
\alpha = 1/2, \quad \beta = 2/\pi.
$$

Proof. Taking $a = 1+r$ and $b = 1-r$ in Theorem 3.4, we obtain the result.

THEOREM 4.2. *For all $r \in (0,1)$ we have*

$$
\frac{\pi}{2}\left[\alpha\left(\frac{3}{4L(1,r')} + \frac{1}{4A(1,r')}\right) + (1-\alpha)\left(\frac{1}{2L(1,r')} + \frac{1}{2He(1,r')}\right)\right]
$$

$$
< \mathscr{K}(r) < \frac{\pi}{2} \left[\beta \left(\frac{3}{4L(1,r')} + \frac{1}{4A(1,r')} \right) + (1-\beta) \left(\frac{1}{2L(1,r')} + \frac{1}{2He(1,r')} \right) \right],
$$

(4.2)

$$
\frac{\pi}{2} \left[\alpha \left(\frac{3}{4} \frac{\text{arth} r}{r} + \frac{1}{4} \right) + (1-\alpha) \left(\frac{\text{arth} r}{2r} + \frac{3}{2(2+r')} \right) \right]
$$
$$
< \mathscr{K}(r) < \frac{\pi}{2} \left[\beta \left(\frac{3}{4} \frac{\text{arth} r}{r} + \frac{1}{4} \right) + (1-\beta) \left(\frac{\text{arth} r}{2r} + \frac{3}{2(2+r')} \right) \right]
$$

(4.3)

with the best possible constants

$$
\alpha = 17/44, \quad \beta = 8/\pi - 2.
$$

Proof. Two inequalities in Theorem 4.2 follow easily from Theorem 3.5 with $a = 1$, $b = r'$ and $a = 1+r$, $b = 1-r$, respectively.

THEOREM 4.3. *For all $r \in (0,1)$ we have*

$$
\frac{\pi}{2} \sqrt{\alpha_1 \frac{1+r'^2}{2} + (1-\alpha_1) r'} < \mathscr{E}(r) < \frac{\pi}{2} \sqrt{\beta_1 \frac{1+r'^2}{2} + (1-\beta_1) r'}, \qquad (4.4)
$$

$$
\frac{\pi}{2} \sqrt{\alpha_2 \frac{1+r'^2}{2} + (1-\alpha_2) \frac{2r'^2}{1+r'^2}} < \mathscr{E}(r) < \frac{\pi}{2} \sqrt{\beta_2 \frac{1+r'^2}{2} + (1-\beta_2) \frac{2r'^2}{1+r'^2}} \quad (4.5)
$$

with the best possible constants

$$
\alpha_1 = 3/4, \quad \beta_1 = 8/\pi^2, \quad \alpha_2 = 8/\pi^2, \quad \beta_2 = 7/8.
$$

Proof. Theorem 4.3 follows from Theorem 3.6 with $a = 1$ and $b = r'^2$. Similarly, letting $a = 1$ and $b = r'^2$ in Theorem 3.7, one has

THEOREM 4.4. *For all $r \in (0,1)$ we have*

$$
\frac{\pi}{2} \sqrt{\alpha \left[\frac{7+18r'^2+7r'^4}{16(1+r'^2)} \right] + (1-\alpha) \left(\frac{3r'^2+2r'+3}{8} \right)}
$$
$$
< \mathscr{E}(r) < \frac{\pi}{2} \sqrt{\beta \left[\frac{7+18r'^2+7r'^4}{16(1+r'^2)} \right] + (1-\beta) \left(\frac{3r'^2+2r'+3}{8} \right)} \qquad (4.6)
$$

with the best possible constants

$$
\alpha = 3/16, \quad \beta = 4(16/\pi^2 - 3/2).
$$

5. Comparisons

REMARK 5.1. Since $1/AG(1+r,1-r) = 2\mathcal{K}(r)/\pi$, $1/L(1+r,1-r) = \text{arth}r/r$ and $1/A(1+r,1-r) = 1$, the right-hand side of inequality (1.4) can be expressed as

$$\frac{1}{AG(a,b)} < \frac{1}{L(a,b)}$$

with $a = 1+r$ and $b = 1-r$. Thus Theorems 3.4 and 3.5 imply that our upper bounds in (4.1) and (4.3) are better than that in (1.4). On the other hand, it follows from (3.15) and (3.16) that the upper bound in (4.2) is better than that in (1.5).

Next, we shall compare the lower bound in (4.3) with the lower bound in (1.4). For this purpose we need the following two propositions.

PROPOSITION 5.2. *For all $r \in (0,1)$, we have*

$$\frac{r^2 \text{arth}r}{r - r'^2 \text{arth}r} > \frac{3}{2} + \frac{1}{5}r^2 + \frac{23}{175}r^4, \tag{5.1}$$

$$35\text{arth}r + \frac{216r'}{(2+r')^2}\left(\frac{3}{2}r + \frac{1}{5}r^3 + \frac{23}{175}r^5\right) - \frac{162r}{2+r'} - 17r > 0. \tag{5.2}$$

Proof. Clearly, inequality (5.1) can be rewritten as

$$\frac{\text{arth}r}{r} > \frac{r - r'^2 \text{arth}r}{r^3}\left(\frac{3}{2} + \frac{1}{5}r^2 + \frac{23}{175}r^4\right).$$

Since $\text{arth}r/r = \sum_{n=0}^{\infty}\left[r^{2n}/(2n+1)\right]$ and $(r - r'^2\text{arth}r)/r^3 = [1 - (1 - r^2)\text{arth}r/r]/r^2 = \sum_{n=0}^{\infty}(2r^{2n})/[(2n+1)(2n+3)]$, one has

$$\frac{\text{arth}r}{r} - \frac{r - r'^2\text{arth}r}{r^3}\left(\frac{3}{2} + \frac{1}{5}r^2 + \frac{23}{175}r^4\right)$$

$$= \sum_{n=0}^{\infty}\frac{r^{2n}}{2n+1} - \left(\frac{3}{2} + \frac{1}{5}r^2 + \frac{23}{175}r^4\right)\sum_{n=0}^{\infty}\frac{2r^{2n}}{(2n+1)(2n+3)}$$

$$= \sum_{n=2}^{\infty}\frac{r^{2n}}{2n+1} - \sum_{n=2}^{\infty}\frac{3r^{2n}}{(2n+1)(2n+3)} - \frac{2}{5}\sum_{n=1}^{\infty}\frac{r^{2n+2}}{(2n+1)(2n+3)} - \frac{46}{175}\sum_{n=0}^{\infty}\frac{r^{2n+4}}{(2n+1)(2n+3)}$$

$$= \sum_{n=0}^{\infty}\frac{r^{2n+4}}{2n+5} - \sum_{n=0}^{\infty}\frac{3r^{2n+4}}{(2n+5)(2n+7)} - \frac{2}{5}\sum_{n=0}^{\infty}\frac{r^{2n+4}}{(2n+3)(2n+5)} - \frac{46}{175}\sum_{n=0}^{\infty}\frac{r^{2n+4}}{(2n+1)(2n+3)}$$

$$= \sum_{n=0}^{\infty}\frac{2n(700n^2 + 2568n + 2213)}{175(2n+1)(2n+3)(2n+5)(2n+7)}r^{2n+4} > 0$$

for all $r \in (0,1)$.

For the inequality (5.2), set

$$f(r) = 35\text{arth}r + \frac{216r'}{(2+r')^2}\left(\frac{3}{2}r + \frac{1}{5}r^3 + \frac{23}{175}r^5\right) - \frac{162r}{2+r'} - 17r, \quad r \in (0,1).$$

Then $f(0^+) = 0$, and by tedious computations we have

$$f'(r) = \frac{19872r^8 - 29592r^6 - 30810r^4 + 130480r^2 - 40950 - \sqrt{1-r^2}g(r)}{175(1-r^2)(2+r')^3}, \quad (5.3)$$

where $g(r) = 59616r^6 + 13775r^4 + 89215r^2 - 40950$ is strictly increasing on $(0,1)$, and there exists a unique zero point $r_0 = 0.62778\cdots$ such that $g(r) < 0$ for $r \in (0, r_0)$, and $g(r) > 0$ for $r \in (r_0, 1)$.

Finally, it is sufficient to prove that $f'(r) > 0$ for $r \in (0, r_0]$ and $r \in (r_0, 1)$. When $r \in (0, r_0]$, noting that the fact $1 - r^2 < \sqrt{1-r^2}$ for $r \in (0,1)$, we have

$$19872r^8 - 29592r^6 - 30810r^4 + 130480r^2 - 40950 - \sqrt{1-r^2}g(r)$$
$$> 19872r^8 - 29592r^6 - 30810r^4 + 130480r^2 - 40950 - (1-r^2)g(r)$$
$$= r^2[79488r^6 + r^2(44630 - 75433r^2) + 315] > 0$$

for all $r \in (0, r_0]$. Thus it follows from (5.3) that $f'(r) > 0$ for all $r \in (0, r_0]$.

When $r \in (r_0, 1)$, by the inequality $\sqrt{1-r^2} < 1 - r^2/2 - r^4/8$, we get

$$19872r^8 - 29592r^6 - 30810r^4 + 130480r^2 - 40950 - \sqrt{1-r^2}g(r)$$
$$> 19872r^8 - 29592r^6 - 30810r^4 + 130480r^2 - 40950 - \left(1 - r^2/2 - r^4/8\right)g(r)$$
$$= \frac{\left(59616r^8 + 411215r^6 - 569349r^4 - 40770r^2 + 166320\right)r^2}{8}.$$

It is not difficult to verify that inequality $59616r^8 + 411215r^6 - 569349r^4 - 40770r^2 + 166320 > 0$ takes place for all $r \in (0,1)$. Thus (5.3) leads to that $f'(r) > 0$ for all $r \in (r_0, 1)$.

PROPOSITION 5.3. *Let*

$$F(r) = \log\left(\frac{105}{176}\frac{\text{arth}r}{r} + \frac{81}{88(2+r')} + \frac{17}{176}\right) - \frac{3}{4}\log\left(\frac{\text{arth}r}{r}\right).$$

Then $F(r)$ is strictly increasing from $(0,1)$ onto $(0,\infty)$.

Proof. Differentiating F yields

$$F'(r) = \frac{105(r - r'^2\text{arth}r)/(r^2r'^2) + 162r/[r'(2+r')^2]}{105(\text{arth}r)/r + 162/(2+r') + 17} - \frac{3}{4}\frac{r - r'^2\text{arth}r}{rr'^2\text{arth}r}$$
$$= \frac{3(r - r'^2\text{arth}r)F_1(r)}{4rr'^2\text{arth}r\,[105(\text{arth}r)/r + 162/(2+r') + 17]} \quad (5.4)$$

where

$$F_1(r) = 35\frac{\text{arth}r}{r} + \frac{216r'}{(2+r')^2}\frac{r^2\text{arth}r}{r - r'^2\text{arth}r} - \frac{162}{2+r'} - 17. \quad (5.5)$$

It follows from (5.1) and (5.2) that

$$F_1(r) > 35\frac{\text{arth}r}{r} + \frac{216r'}{(2+r')^2}\left(\frac{3}{2} + \frac{1}{5}r^2 + \frac{23}{175}r^4\right) - \frac{162}{2+r'} - 17 > 0 \qquad (5.6)$$

for all $r \in (0,1)$.

Therefore, the monotonicity of $F(r)$ follows from (5.4)-(5.6). The limiting values are clear.

REMARK 5.4. From Proposition 5.3 we conclude that the lower bound in (4.3) is better than the lower bound in (1.4).

REMARK 5.5. The lower bound in (4.5) is better than the lower bound in (1.6) when $r \to 1$. Computational and numerical experiments show that the lower bound in (4.6) is tighter than that in (1.6) for $0 \leqslant r \leqslant 0.972$. Aslo, computational and numerical experiments show that the upper bound in (4.6) is tighter than that in (1.7) for $0 \leqslant r \leqslant 0.953$.

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Miao-Kun Wang
Department of Mathematics
Huzhou University
Huzhou 313000, P. R. China
e-mail: wmk000@126.com; wangmiaokun@zjhu.edu.cn

Yu-Ming Chu
College of Science
Hunan City University
Yiyang 413000, P. R. China
School of Mathematics and Statistics
Changsha University of Science & Technology
Changsha 410114, P. R. China
e-mail: chuyuming2005@126.com

Yong-Min Li
Department of Mathematics
Huzhou University
Huzhou 313000, P. R. China
e-mail: ymlwww@163.com

Wen Zhang
Friedman Brain Institute
Icahn School of Medicine at Mount Sinai
New York NY 10029, USA
e-mail: zhang.wen81@gmail.com

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