Mathematical
Inequalities
& Applications
Volume 23, Number 3 (2020), 843–853 doi:10.7153/mia-2020-23-68

SHARP RATIONAL BOUNDS FOR THE GAMMA FUNCTION

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(*Communicated by N. Elezović*)

Abstract. In the article, we prove that the inequality

$$\Gamma(x+1) \leqslant \frac{x^2+p}{x+p}$$

holds for all $x \in (0,1)$ if and only if $p \geqslant p_0$, where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ is the gamma function, $p_0 = \left[x_0\Gamma(x_0+1) - x_0^2\right]/[1-\Gamma(x_0+1)] = 1.755\cdots$, $x_0 = 0.192\cdots$ is the unique solution of the equation $\psi(x+1) = [1-\Gamma(x+1)][2-\Gamma(x)]/[(1-x)\Gamma(x+1)]$ on the interval $(0,1)$ and $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the psi function. As applications, we present the best possible parameters λ_0 and μ_0 on the interval $(0,\infty)$ such that the double inequality

$$\frac{x^2+\lambda_0}{x+\lambda_0} < \Gamma(x+1) < \frac{x^2+\mu_0}{x+\mu_0}$$

holds for all $x \in (1/2,1)$, and the two-sided inequality

$$\frac{\pi x(1-x)(1-x+\mu_0)}{\sin(\pi x)[(1-x)^2+\mu_0]} < \Gamma(x+1) < \frac{\pi x(1-x)(1-x+\lambda_0)}{\sin(\pi x)[(1-x)^2+\lambda_0]}$$

takes place for all $x \in (0,1/2)$.

1. Introduction

Let $x > 0$. Then the classical Euler gamma function $\Gamma(x)$ [71] and its logarithmic derivative, the so-called psi function $\psi(x)$ [45] are given by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

respectively. They have wide applications in pure and applied mathematics [8, 9, 10, 11, 12, 13, 14, 16, 19, 20, 21, 23, 24, 25, 26, 28, 29, 30, 32, 33, 35, 36, 37, 38, 39, 48, 49, 50, 51, 52, 53, 54, 55, 59, 63, 70, 72, 73, 74, 75]. In particular, many special functions can be expressed by use of the gamma function [1, 2, 3, 4, 5, 6, 7, 18, 27, 40, 41, 42, 43, 44, 46, 47, 56, 57, 58, 60, 61, 62, 64, 67, 68, 69]. Recently, the bounds for the gamma function have attracted the attention of many researchers. It is well known that

Mathematics subject classification (2010): 41A60, 33B15, 26D07.
Keywords and phrases: Gamma function, psi function, rational bound, Euler-Mascheroni constant.
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$\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(n+1) = n!$. Therefore, we only need to focus our attention on $\Gamma(x+1)$ with $x \in (0,1)$.

Gautschi [17] proved that the double inequality

$$n^{1-s} < \frac{\Gamma(n+1)}{\Gamma(n+s)} < e^{(1-s)\psi(n+1)} \tag{1.1}$$

holds for all $s \in (0,1)$ and $n \in \mathbb{N}$.

Inequality (1.1) was generalized and improved by Kershaw [34] as follows:

$$\left(x+\frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < e^{(1-s)\psi[x+(1+s)/2]}$$

for all $x > 0$ and $s \in (0,1)$.

Elezović et al. [15] established the double inequality

$$\frac{x}{2} < \Gamma(x)^{-\frac{1}{1-x}} < -\frac{1}{2} + \sqrt{\frac{1}{4}+x}$$

for the gamma function being valid for all $x \in (0,1)$, and asked for "other bounds for the gamma function in terms of elementary functions".

In [31], Ivády provided the bounds for gamma function in terms of very simple rational functions as follows:

$$\frac{x^2+1}{x+1} < \Gamma(x+1) < \frac{x^2+2}{x+2} \tag{1.2}$$

for all $x \in (0,1)$. Inequality (1.2) can be regarded as a simple estimation of the value of the gamma function.

In 2017, Yang et al. [66] proved that the inequality

$$\Gamma(x+1) > \frac{x^2+q}{x+q} \tag{1.3}$$

holds for all $x \in (0,1)$ if and only if $q \leqslant \gamma/(1-\gamma) = 1.365\cdots$, where

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) = 0.577\cdots$$

is Euler-Mascheroni constant [22].

The aim of this paper is to prove that the inequality

$$\Gamma(x+1) \leqslant \frac{x^2+p}{x+p}$$

is valid for all $x \in (0,1)$ if and only if $p \geqslant p_0$, and the double inequality

$$\frac{x^2+\lambda}{x+\lambda} < \Gamma(x+1) < \frac{x^2+\mu}{x+\mu}$$

holds for all $x \in (1/2,1)$ and the two-sided inequality

$$\frac{\pi x(1-x)(1-x+\mu)}{\sin(\pi x)[(1-x)^2+\mu]} < \Gamma(x+1) < \frac{\pi x(1-x)(1-x+\lambda)}{\sin(\pi x)[(1-x)^2+\lambda]}$$

takes place for all $x \in (0,1/2)$ if and only if $\lambda \leqslant \lambda_0 = \gamma/(1-\gamma) = 1.365\cdots$ and $\mu \geqslant \mu_0 = (\pi+\sqrt{\pi}-2)/(8-2\pi) = 1.697\cdots$, where $p_0 = \frac{x_0\Gamma(x_0+1)-x_0^2}{1-\Gamma(x_0+1)} = 1.755\cdots$ and $x_0 = 0.192\cdots$ is the unique solution of the equation

$$\psi(x+1) = \frac{1-\Gamma(x+1)][2-\Gamma(x)]}{(1-x)\Gamma(x+1)}$$

on the interval $(0,1)$.

2. Lemmas

In order to establish our main results we need several lemmas, which we present in this section.

LEMMA 2.1. *(See* [65, Corollary 3]*) The double inequality*

$$\frac{1}{24(x+1/2)^2}-\frac{7}{960(x+1/2)^4}+\log\left(x+\frac{1}{2}\right)<\psi(x+1)<\frac{1}{24(x+1/2)^2}+\log\left(x+\frac{1}{2}\right)$$

holds for all $x \in (-1/2,\infty)$.

LEMMA 2.2. *(See* [66, Lemma 2.11]*) Let* $p \in [8/5,9/5]$, $x \in (0,1)$ *and the function* $h(p,x)$ *be defined by*

$$h(p,x) = \psi'(x+1)+\frac{4x^2}{(x^2+p)^2}-\frac{2}{x^2+p}-\frac{1}{(x+p)^2}. \tag{2.1}$$

Then there exist $\eta_1(p),\eta_2(p) \in (0,1)$ *with* $\eta_1(p) < \eta_2(p)$ *such that* $h(p,x) > 0$ *for* $x \in (0,\eta_1(p)) \cup (\eta_2(p),1)$ *and* $h(p,x) < 0$ *for* $x \in (\eta_1(p),\eta_2(p))$.

LEMMA 2.3. *Let* $p \in (7/4,44/25)$, $x \in (0,1)$ *and the function* $g(p,x)$ *be defined by*

$$g(p,x) = \psi(x+1)-\frac{2x}{x^2+p}+\frac{1}{x+p}. \tag{2.2}$$

Then $g(p,1/10) > 0$ *and* $g(p,1/2) < 0$.

Proof. It follows from Lemma 2.1 and the well known identity $\psi(x+1) = \psi(x)+1/x$ that

$$\psi(x+1) = \psi(x+2)-\frac{1}{x+1} > \frac{1}{24(x+3/2)^2}-\frac{7}{960(x+3/2)^4}+\log\left(x+\frac{3}{2}\right)-\frac{1}{x+1}, \tag{2.3}$$

$$\psi(x+1) < \frac{1}{24(x+3/2)^2} + \log\left(x+\frac{3}{2}\right) - \frac{1}{x+1}. \tag{2.4}$$

Inequalities (2.3) and (2.4) lead to

$$\psi\left(\frac{1}{10}+1\right) > \frac{1}{24(1/10+3/2)^2} - \frac{7}{960(1/10+3/2)^4} + \log\left(\frac{1}{10}+\frac{3}{2}\right) - \frac{1}{\frac{1}{10}+1} = \log\frac{8}{5} - \frac{2577715}{2883584}, \tag{2.5}$$

$$\psi\left(\frac{1}{2}+1\right) < \frac{1}{24(1/2+3/2)^2} + \log\left(\frac{1}{2}+\frac{3}{2}\right) - \frac{1}{\frac{1}{2}+1} = \log 2 - \frac{21}{32}. \tag{2.6}$$

From (2.2) we get

$$\frac{dg(p,1/10)}{dp} = \left[\frac{(2x-1)p^2+2x^2p+(2-x)x^3}{(x^2+p)^2(x+p)^2}\right]_{x=1/10} = -\frac{100(8000p^2-200p-19)}{(1000p^2+110p+1)^2} < 0, \tag{2.7}$$

$$\frac{dg(p,1/2)}{dp} = \left[\frac{(2x-1)p^2+2x^2p+(2-x)x^3}{(x^2+p)^2(x+p)^2}\right]_{x=1/2} = \frac{4(8p+3)}{(2p+1)^2(4p+1)^2} > 0 \tag{2.8}$$

for $p \in (7/4, 44/25)$.

It follows from (2.2) and (2.5)-(2.8) that

$$\begin{aligned} g(p,1/10) > g(44/25,1/10) &= \psi\left(\frac{1}{10}+1\right) - \frac{2/10}{1/100+44/25} + \frac{1}{1/10+44/25} \\ &> \log\frac{8}{5} - \frac{2577715}{2883584} - \frac{2/10}{1/100+44/25} + \frac{1}{1/10+44/25} = 0.000716\cdots > 0, \\ g(p,1/2) < g(44/25,1/2) &= \psi\left(\frac{1}{2}+1\right) - \frac{1}{1/4+44/25} + \frac{1}{1/2+44/25} \\ &< \log 2 - \frac{21}{32} - \frac{1}{1/4+44/25} + \frac{1}{1/2+44/25} = -0.01813\cdots < 0 \end{aligned}$$

for $p \in (7/4, 44/25)$. $\square$

LEMMA 2.4. *Let $p \in (7/4, 44/25)$, $x \in (0,1)$, $g(p,x)$ be defined by (2.2), and $\eta_1(p)$ and $\eta_2(p)$ be defined by Lemma 2.2. Then $g(p,\eta_1(p)) > 0$ and $g(p,\eta_2(p)) < 0$.*

Proof. Let $h(p,x)$ be defined by (2.1). Then from (2.1) and (2.2) we clearly see that

$$\frac{\partial g(p,x)}{\partial x} = h(p,x) \tag{2.9}$$

and

$$g(p,0^+) = \psi(1) + \frac{1}{p} < -\gamma + \frac{4}{7} < 0, \quad g(p,1^-) = \psi(2) - \frac{1}{1+p} > 1 - \gamma - \frac{4}{11} > 0 \tag{2.10}$$

for $p \in (7/4, 44/25)$.

We use the proof by contradiction to prove the desired results. We first prove that $g(p, \eta_1(p)) > 0$. Indeed, if $g(p, \eta_1(p)) \leqslant 0$, then from (2.9) and (2.10) together with $(7/4, 44/25) \subset [8/5, 9/5]$ and Lemma 2.2 we clearly see that there exists $\omega_1(p) \in (\eta_2(p), 1)$ such that $g(p,x) \leqslant 0$ for $x \in (0, \omega_1(p))$ and $g(p,x) > 0$ for $x \in (\omega_1(p), 1)$, which contradicts with Lemma 2.3.

Next, we prove that $g(p, \eta_2(p)) < 0$. In fact, if $g(p, \eta_2(p)) \geqslant 0$, then (2.9) and (2.10) together with $(7/4, 44/25) \subset [8/5, 9/5]$ and Lemma 2.2 lead to the conclusion that there exists $\omega_2(p) \in (0, \eta_1(p))$ such that $g(p,x) < 0$ for $x \in (0, \omega_2(p))$ and $g(p,x) \geqslant 0$ for $x \in (\omega_2(p), 1)$, which also contradicts with Lemma 2.3. □

3. Main results

THEOREM 3.1. *Let $p > 0$. Then the inequality*

$$\Gamma(x+1) \leqslant \frac{x^2+p}{x+p}$$

holds for all $x \in (0,1)$ if and only if $p \geqslant p_0$, where

$$p_0 = \frac{x_0\Gamma(x_0+1) - x_0^2}{1 - \Gamma(x_0+1)} = 1.755\cdots$$

and $x_0 = 0.192\cdots$ is the unique solution of the equation

$$\psi(x+1) = \frac{1 - \Gamma(x+1)][2 - \Gamma(x)]}{(1-x)\Gamma(x+1)}$$

on the interval $(0,1)$.

Proof. Let $p \in (7/4, 44/25)$, $x \in (0,1)$, $\eta_1(p)$ and $\eta_2(p)$ be defined by Lemma 2.2, $h(p,x)$ and $g(p,x)$ be respectively defined by (2.1) and (2.2), and $f(p,x)$ be defined by

$$f(p,x) = \log\Gamma(x+1) - \log\frac{x^2+p}{x+p}. \tag{3.1}$$

Then from (2.2) and (3.1) we clearly see that

$$\frac{\partial f(p,x)}{\partial x} = g(p,x), \tag{3.2}$$

$$f(p, 0^+) = f(p, 1^-) = 0. \tag{3.3}$$

It follows from Lemma 2.2, Lemma 2.4, (2.9) and (2.10) that there exist $\tau_1(p) \in (0, \eta_1(p))$, $\tau_0(p) \in (\eta_1(p), \eta_2(p))$ and $\tau_2(p) \in (\eta_2(p), 1)$ such that $g(p,x) < 0$ for $x \in (0, \tau_1(p)) \cup (\tau_0(p), \tau_2(p))$ and $g(p,x) > 0$ for $x \in (\tau_1(p), \tau_0(p)) \cup (\tau_2(p), 1)$. Then (3.2) leads to the conclusion that $f(p,x)$ is strictly decreasing on $(0, \tau_1(p)) \cup (\tau_0(p), \tau_2(p))$ and strictly increasing on $(\tau_1(p), \tau_0(p)) \cup (\tau_2(p), 1)$.

Let $x_0 = 0.192\cdots$ be the unique solution of the equation

$$\psi(x+1) = \frac{[1-\Gamma(x+1)][2-\Gamma(x)]}{(1-x)\Gamma(x+1)}$$

on the interval $(0,1)$ and $x_0 = \tau_0(p_0)$, where

$$p_0 = \frac{x_0\Gamma(x_0+1) - x_0^2}{1-\Gamma(x_0+1)} = 1.755\cdots \in (7/4, 44/25).$$

Then we clearly see that $(p_0, x_0) \in (7/4, 44/25) \times (0,1)$ is the the unique solution of the simultaneous equations

$$\log\Gamma(x+1) = \log\frac{x^2+p}{x+p}, \quad \psi(x+1) = \frac{2x}{x^2+p} - \frac{1}{x+p}$$

and

$$f(p_0, x_0) = f(p_0, \tau_0(p_0)) = 0. \tag{3.4}$$

From (3.1), (3.3), (3.4) and the piecewise monotonicity of the function $f(p_0, x)$ on the interval $(0,1)$ we get

$$\Gamma(x+1) \leqslant \frac{x^2+p_0}{x+p_0} \tag{3.5}$$

for all $x \in (0,1)$, and inequality (3.5) becomes equality if and only if $x = x_0$.

It is easy to verify that the function $p \to (x^2+p)/(x+p)$ is strictly increasing on $(0,\infty)$ for all $x \in (0,1)$. Therefore,

$$\Gamma(x+1) \leqslant \frac{x^2+p}{x+p} \tag{3.6}$$

for all $x \in (0,1)$ and $p \geqslant p_0$ follows from (3.5).

Next, we prove that $p \geqslant p_0$ if inequality (3.6) holds for all $x \in (0,1)$. Indeed, inequality (3.6) implies that

$$p \geqslant \frac{x\Gamma(x+1) - x^2}{1-\Gamma(x+1)} \tag{3.7}$$

for all $x \in (0,1)$. In particular, taking $x = x_0$, then (3.7) leads to the conclusion that

$$p \geqslant \frac{x_0\Gamma(x_0+1) - x_0^2}{1-\Gamma(x_0+1)} = p_0. \quad \square$$

THEOREM 3.2. *The double inequality*

$$\frac{x^2+\lambda}{x+\lambda} < \Gamma(x+1) < \frac{x^2+\mu}{x+\mu}$$

holds for all $x \in (1/2, 1)$ *and the two-sided inequality*

$$\frac{\pi x(1-x)(1-x+\mu)}{\sin(\pi x)[(1-x)^2+\mu]} < \Gamma(x+1) < \frac{\pi x(1-x)(1-x+\lambda)}{\sin(\pi x)[(1-x)^2+\lambda]}$$

takes place for all $x \in (0, 1/2)$ *if and only if* $\lambda \leqslant \lambda_0 = \gamma/(1-\gamma) = 1.365\cdots$ *and* $\mu \geqslant \mu_0 = (\pi+\sqrt{\pi}-2)/(8-2\pi) = 1.697\cdots$.

Proof. Let $x \in (0,1)$, $x_0 = 0.192\cdots$ be defined by Theorem 3.1, and $H(x)$ and $P(x)$ be respectively defined by

$$H(x) = \psi(x+1) + \frac{[1-\Gamma(x+1)][\Gamma(x)-2]}{(1-x)\Gamma(x+1)}, \tag{3.8}$$

$$P(x) = \frac{x\Gamma(x+1)-x^2}{1-\Gamma(x+1)}. \tag{3.9}$$

Then from the proof of Theorem 3.1 we know that x_0 is the unique solution of the equation $H(x) = 0$ on the interval $(0,1)$.

It follows from (3.8) and (3.9) that

$$P\left(\frac{1}{2}\right) = \mu_0, \quad P(1^-) = \lambda_0, \tag{3.10}$$

$$\lim_{x\to 0^+} \frac{H(x)}{x} = \frac{\pi^2}{12} - \frac{\gamma^2}{2} - \gamma > 0, \quad \lim_{x\to 1^-} \frac{H(x)}{1-x} = -\frac{\pi^2}{12} - \frac{3\gamma^2}{2} + 2\gamma < 0, \tag{3.11}$$

$$P'(x) = \frac{x(1-x)\Gamma(x+1)}{[1-\Gamma(x+1)]^2} H(x). \tag{3.12}$$

From (3.11) and (3.12) together with x_0 is the unique solution of the equation $H(x) = 0$ on the interval $(0,1)$ we clearly see that $P(x)$ is strictly increasing on $(0,x_0)$ and strictly decreasing on $(x_0,1)$, which implies that $P(x)$ is strictly decreasing on $(1/2,1)$. Therefore, λ_0 and μ_0 are the best possible constants such that the double inequality

$$\frac{x^2+\lambda_0}{x+\lambda_0} < \Gamma(x+1) < \frac{x^2+\mu_0}{x+\mu_0} \tag{3.13}$$

holds for all $x \in (1/2,1)$ follow from (3.10) and the monotonicity of the function $P(x)$ on the interval $(1/2,1)$.

It is well known that $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$ for all $x \in (0,1)$, which leads to the conclusion that

$$\Gamma(2-x) = \frac{\pi x(1-x)}{\sin(\pi x)\Gamma(x+1)} \tag{3.14}$$

for all $x \in (0,1)$. Therefore, λ_0 and μ_0 are the best possible constants such that the two-sided inequality

$$\frac{\pi x(1-x)(1-x+\mu_0)}{\sin(\pi x)[(1-x)^2+\mu_0]} < \Gamma(x+1) < \frac{\pi x(1-x)(1-x+\lambda_0)}{\sin(\pi x)[(1-x)^2+\lambda_0]}$$

takes place for all $x \in (0,1/2)$ follow easily from (3.13) and (3.14) together with $1-x \in (1/2,1)$. □

Let $\lambda_0 = \gamma/(1-\gamma)$ and $x_{\lambda_0} = \sqrt{\lambda_0(\lambda_0+1)} - \lambda_0$. Then simple computations show that $\left(\frac{x^2+\lambda_0}{x+\lambda_0}\right)' = \frac{x+\lambda_0+\sqrt{\lambda_0(\lambda_0+1)}}{(x+\lambda_0)^2}(x-x_{\lambda_0})$, which implies that

$$\min_{x\in(0,1)} \frac{x^2+\lambda_0}{x+\lambda_0} = \frac{x_{\lambda_0}^2+\lambda_0}{x_{\lambda_0}+\lambda_0} = \frac{2\sqrt{\gamma}}{1+\sqrt{\gamma}}. \tag{3.15}$$

REMARK 3.3. From (1.3) and (3.15) we clearly see that the inequality

$$\Gamma(x+1) > \frac{2\sqrt{\gamma}}{1+\sqrt{\gamma}}$$

holds for all $x \in (0,1)$.

Acknowledgements. The work was supported by the Natural Science Foundation of China (Grant Nos. 61673169, 11301127, 11701176, 11626101, 11601485) and the Natural Science Foundation of Huzhou City (Grant No. 2018YZ07).

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(Received May 24, 2017)

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