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SHARP LOWER AND UPPER BOUNDS FOR THE q–GAMMA FUNCTION

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Abstract. This paper is devoted to provide sharp bounds for the q-gamma function from below and above for all $q > 0$ by means of investigating the monotonicity property to analytical functions involving logarithm q-gamma function. It turns out that these results refine and improve lower and upper bounds for the q-gamma function which have been given by Salem [13].

1. Introduction

The q-gamma function is defined as [11]

$$\Gamma_q(x) = |1-q|^{1-x} q^{\frac{x(x-1)}{2}H(q-1)} \prod_{n=0}^{\infty} \frac{1-\hat{q}^{n+1}}{1-\hat{q}^{n+x}}, \qquad 0 < q \neq 1 \tag{1}$$

where $|\cdot|$ is the absolute value, $H(\cdot)$ denotes the Heaviside step function and

$$\hat{q} = \begin{cases} q & \text{if} \quad 0 < q \leqslant 1 \\ q^{-1} & \text{if} \quad q \geqslant 1. \end{cases}$$

The close connection between two branches of the q-gamma function when $0 < q < 1$ and $q \geqslant 1$ is given by

$$\Gamma_q(x) = q^{\frac{(x-1)(x-2)}{2}} \Gamma_{q^{-1}}(x), \qquad q \geqslant 1. \tag{2}$$

In the recent past, numerous papers were published presenting remarkable inequalities involving the q-gamma function (see [13, 14, 15, 16, 2, 17, 18, 19, 20, 21, 22, 23, 24, 4, 5, 7, 3, 8, 6, 9] and the extensive list of references given therein). The author in [13] proved that the function

$$F_q(x) = \frac{\log\Gamma_q(x+1) - \frac{x(x-1)}{2}H(q-1)\log q}{x\log[x]_q - x(x-1)H(q-1)\log q}, \qquad 0 < x \neq 1, \quad q > 0 \tag{3}$$

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and $F_q(1) = 1 - \hat{q}^{-1}\gamma_{\hat{q}}$ where $[x]_q = (1-q^x)/(1-q)$, is increasing on $(0,\infty)$ and used this result to establish sharp bounds for the q-gamma function stated for $q > 0$ as

$$q^{\frac{x(1-x)}{2}H(q-1)}[x]_q^{\alpha x-1} \leqslant \Gamma_q(x) \leqslant q^{\frac{x(x-1)}{2}H(q-1)}[x]_q^{\beta x-1}, \qquad x \in (0,1] \tag{4}$$

with the best possible constants $\alpha = 1$ and $\beta = 0$ and

$$q^{\frac{x(1-x)}{2}(2\alpha-1)H(q-1)}[x]_q^{\alpha x-1} \leqslant \Gamma_q(x) \leqslant q^{\frac{x(1-x)}{2}H(q-1)}[x]_q^{\beta x-1}, \quad x \in [1,\infty) \tag{5}$$

with the best possible constants $\alpha = 1 - \hat{q}^{-1}\gamma_{\hat{q}}$ and $\beta = 1$, where γ_q is the q-analogue of the Euler-Mascheroni constant defined as

$$\gamma_q = \frac{1-q}{\log q}\psi_q(1) \tag{6}$$

and $\psi_q(x)$ denotes the q-digamma function which is defined as the logarithmic derivative of the q-gamma function

$$\psi_q(x) = \frac{d}{dx}(\log\Gamma_q(x)) = \frac{\Gamma_q'(x)}{\Gamma_q(x)} \tag{7}$$

The q-digamma function $\psi_q(x)$ appeared in the work of Krattenthaler and Srivastava [10] where they studied the summations for basic hypergeometric series. Some of its properties are presented and proved in their work and also in [25]. From (1), the q-digamma function, for $0 < q < 1$ and for all real variable $x > 0$, can be represented as

$$\psi_q(x) = -\log(1-q) + \log q \sum_{k=1}^{\infty} \frac{q^{xk}}{1-q^k} \tag{8}$$

and satisfies the identity (see [25])

$$\psi_q(x+1) = \psi_q(x) - \frac{q^x \log q}{1-q^x}. \tag{9}$$

The nth derivatives of the q-gamma function is the so-called the q-polygamma functions which can be represented as

$$\psi_q^{(n)}(x) = \log^{n+1} q \sum_{k=1}^{\infty} \frac{k^n q^{xk}}{1-q^k}, \qquad n \in \mathbb{N},\ 0 < q < 1. \tag{10}$$

The purpose of this paper is to establish inequalities companion to the inequalities (4) and (5). These improvements will be shown in Theorems 1 and 5 and will be compared with the previous results (4) and (5) in the final section. It will turn out that the new results are superior.

We need the following inequalities. In [23] (Corollaries 3.5 and 3.6), it is proven that

$$\beta_1(1) < \psi_q(x) < \beta_1(2) \tag{11}$$

holds true for all positive reals x and q, where

$$\beta_1(1) = \log[x]_q + \frac{1}{2}\frac{q^x \log q}{1-q^x} - \frac{1}{12}\frac{q^x \log^2 q}{(1-q^x)^2},$$
$$\beta_1(2) = \beta_1(1) + \frac{1}{720}\frac{(q^{2x}+4q^x+1)q^x \log^4 q}{(1-q^x)^4}$$

and the inequalities

$$\psi_q(x) < \log[x]_q + \frac{1}{2}\frac{q^x \log q}{1-q^x}, \tag{12}$$
$$\psi_q'(x) > -\beta_2(0) = -\frac{q^x \log q}{1-q^x} + \frac{1}{2}\frac{q^x \log^2 q}{(1-q^x)^2} \tag{13}$$

hold true for all positive reals x and q. Also, we need the inequalities obtained in [19] (Corollary 3.7):

$$\log[x]_q + \frac{q^x \log q}{1-q^x} < \psi_q(x) < \log[x]_q + \frac{1-q+\log q}{(1-q)\log q}\frac{q^x \log q}{1-q^x} \tag{14}$$
$$\psi_q'(x) > -\frac{q^x \log q}{1-q^x} + \frac{1-q+\log q}{(1-q)\log q}\frac{q^x \log^2 q}{(1-q^x)^2} \tag{15}$$

hold for all $x > 0$ and $0 < q < 1$ and the inequality (3.3) obtained in [17]:

$$\psi_q(x) > \log[x]_q + \frac{1}{2}\frac{q^x \log q}{1-q^x} + \frac{2(1-q+\log q)-(1-q)\log q}{2(1-q)\log^2 q}\frac{q^x \log^2 q}{(1-q^x)^2} \tag{16}$$

holds for all $x > 0$ and $0 < q < 1$.

2. Useful lemmas

The following lemmas will be used in the proofs of the main results of this paper.

LEMMA 1. *Let $k \in \mathbb{N}$ and $0 < q < 1$. Then*

$$kq^k \log q < 1 - q^k \leqslant k(1-q) < -k\log q. \tag{17}$$

LEMMA 2. *Let γ_q be defined in (6). Let*

$$\alpha(q) = 1 - q^{-1}\gamma_q, \qquad \delta(q) = \frac{1}{2q}\left(\psi_q(1) - \frac{1-q}{\log q}\psi_q'(1)\right). \tag{18}$$

If $0 < q < 1$, then $q/2 < \gamma_q < q$, $0 < \alpha(q) < 1/2$, and $\alpha(q) < \delta(q)$.

Proof. By virtue of (14) at $x=1$, we get $\psi_q(1) > \frac{q\log q}{1-q}$ which, by using (6), gives $\gamma_q < q$. Moreover, by using the inequality (12) at $x=1$, we can see that $\gamma_q > \frac{1}{2}q$. Therefore, we have $\frac{1}{2}q < \gamma_q < q$ which yields $0 < \alpha(q) < \frac{1}{2}$.

From (14) at $x=1$, we get

$$\alpha(q) = 1 - \frac{1-q}{q\log q}\psi_q(1) < 1 - \frac{1-q+\log q}{(1-q)\log q}$$

and from (15) and (16) at $x=1$, we get

$$\delta(q) > \frac{1}{2} + \frac{2q(1-q+\log q) - q(1-q)\log q}{4(1-q)^3}.$$

Whence

$$\begin{aligned}\alpha(q)-\delta(q) &< -\frac{4(1-q)^3+2(1-q)(1+q-q^2)\log q+q(1+q)\log^2 q}{4(1-q)^3\log q}\\ &= -\frac{q(1+q)}{4(1-q)^3\log q}u_1(q)v_1(q)\end{aligned}$$

where

$$u_1(q) = \log q + \frac{2(1-q)}{1+q} \qquad \text{and} \qquad v_1(q) = \log q + \frac{2(1-q)^2}{q}.$$

By differentiating u_1 gives

$$u_1'(q) = \frac{(1-q)^2}{q(1+q)^2}.$$

It is clear that $u_1'(q) > 0$ for all $0<q<1$ which yields $u_1(q)$ is increasing on $(0,1)$. Since $u_1(1)=0$, then $u_1(q)<0$ for all $0<q<1$.

Regarding to the $v_1(q)$, using (17) gives

$$v_1(q) > -\frac{1-q}{q} + \frac{2(1-q)^2}{q} = \frac{(1-q)(1-2q)}{q}.$$

It is obvious that $v_1(q)<0$ if $0<q<1/2$. In view of the above, we get $\alpha(q)<\delta(q)$ for all $0<q<1/2$.

On the other hand, from (11) and (13) at $x=1$, we get

$$\begin{aligned}\alpha(q)-\delta(q) &< 1-\frac{1-q}{q\log q}\beta_1(2)-\frac{1}{2q}\left(\beta_1(1)+\frac{1-q}{\log q}\beta_2(0)\right)\\ &= -\frac{\log q}{720(1-q)^3}\left(-60(1-q)^2-30(1-q)\log q+(1+4q+q^2)\log^2 q\right)\\ &= -\frac{(1+4q+q^2)\log q}{720(1-q)^3}u_2(q)v_2(q)\end{aligned}$$

where

$$u_2(q) = \log q - \frac{15+\sqrt{15}\sqrt{4q^2+16q+19}}{1+4q+q^2}(1-q)$$

and

$$v_2(q) = \log q - \frac{15-\sqrt{15}\sqrt{4q^2+16q+19}}{1+4q+q^2}(1-q).$$

It is clear that $u_2(q) < 0$ for all $0 < q < 1$. Differentiation gives

$$v_2'(q) = \frac{-s(q)}{q(1+4q+q^2)^2}\left(\frac{\sqrt{15}}{\sqrt{19+16q+4q^2}} - \frac{\ell(q)}{s(q)}\right).$$

where

$$s(q) = 87q + 90q^2 + 45q^3 + 12q^4$$

and

$$\ell(q) = 1 + 83q + 48q^2 - 7q^3 + q^4.$$

Let the function

$$z(q) = \frac{15}{19+16q+4q^2} - \left(\frac{\ell(q)}{s(q)}\right)^2 = -\frac{(1+4q+q^2)^2 n(q)}{(19+16q+4q^2)s^2(q)}$$

where

$$n(q) = 19 + 3018q - 2646q^2 - 4658q^3 - 1281q^4 - 72q^5 + 4q^6.$$

Differentiations give

$$n'(q) = 3018 - 5292q - 13974q^2 - 5124q^3 - 360q^4 + 24q^5$$
$$n''(q) = -5292 - 27948q - 15372q^2 - 1320q^3 - 120q^3(1-q).$$

It is obvious that $n''(q) < 0$ for all $0 < q < 1$ which leads to $n'(q)$ is decreasing on $(0,1)$. Since $n'(0) = 3018$ and $n'(1) = -21708$, then there exists $q_0 \in (0,1)$ such that $n'(q) > 0$ for all $q < q_0$ and $n'(q) < 0$ for all $q > q_0$ which yield $n(q)$ is increasing on $(0, q_0)$ and decreasing on $(q_0, 1)$. Since $n(0) = 19$ and $n(1) = -5616$, then there exists $q_1 \in (q_0, 1)$ such that $n(q) > 0$ for all $q < q_1$ and $n(q) < 0$ for all $q > q_1$ which yield $z(q) > 0$ for all $q > q_1$ and $z(q) < 0$ for all $q < q_1$. This concludes that $v_2'(q) > 0$ for all $q < q_1$ and $z(q) < 0$ for all $q > q_1$ which leads to $v_2(q)$ is increasing on $(0, q_1)$ and decreasing on $(q_1, 1)$. Since $v_2(0) = -\infty$ and $v_2(1) = 0$, then there exists a unique root at $q = q_2$ ($q_2 \simeq 0.278909$ numerically by Mathematica) such that $v_2(q) < 0$ for all $q < q_2$ and $v_2(q) \geqslant 0$ for all $q > q_2$. In conclusion, the function $\alpha(q) - \delta(q) < 0$ for all $q > q_2 \simeq 0.278909$.

LEMMA 3. *The function*

$$\lambda(k) = \sum_{r=1}^{k} \frac{1-q^r + r\log q}{r(1-q^r)} - \frac{1-q^k + k\log q}{1-q^k} \tag{19}$$

is nonnegative for all $k \in \mathbb{N}$ *and* $q \in (0,1)$.

Proof. Let Δ be the forward shift operator and $\Delta^n = \Delta(\Delta^{n-1})$. Then

$$\Delta\lambda(k) = \frac{\lambda_1(k)}{(k+1)(1-q^k)(1-q^{k+1})}$$

where

$$\lambda_1(k) = (1-q^k)(1-q^{k+1}) + k(k+1)(1-q)q^k \log q.$$

Again, $\Delta\lambda_1(k) = (1-q)q^k\lambda_2(k)$ where

$$\lambda_2(k) = (1+q)(1-q^{k+1}) - (k+1)(k(1-q)-2q)\log q$$

and then, we get

$$\Delta\lambda_2(k) = (1-q^2)q^{k+1} - 2(k+1)(1-q)\log q + 2q\log q.$$

In view of (17), we get $-2\log q > 1-q^2$ and $(k+1)(1-q) > 1-q^{k+1}$, which yield

$$\Delta\lambda_2(k) > (1-q^2)q^{k+1} + (1-q^2)(1-q^{k+1}) + 2q\log q = 1-q^2+2q\log q.$$

It is easy to see that $\Delta\lambda_2(k) > 0$ for all positive integer k and $0<q<1$, which yields that the function $\lambda_2(k)$ is increasing for all $k \geqslant 1$. To show the positivity of $\lambda_2(k)$, we have

$$\lambda_2(1) = (1+q)(1-q^2) - 2(1-q)\log q + 4q\log q > 2(1-q^2+2q\log q) > 0.$$

Also, we have

$$\lambda_1(1) = (1-q)(1-q^2+2q\log q) > 0.$$

Hence, $\lambda_1(k) > 0$ for all $k \in \mathbb{N}$ which lead to the function $\lambda(k)$ is increasing for all $k \in \mathbb{N}$. Since $\lambda(1) = 0$, then $\lambda(k) \geqslant 0$ for all $k \in \mathbb{N}$.

LEMMA 4. *The function*

$$\mu(k) = \frac{k(k-1)^2(k-2)}{1-q^k} + 2k\sum_{r=1}^{k-1}\frac{r(3r-1)}{1-q^r} - 2\sum_{r=1}^{k-1}\frac{r(6r^2-3r+1)}{1-q^r} \tag{20}$$

is nonnegative for all $k \in \mathbb{N}$ *and* $q \in (0,1)$.

Proof. Forward shift operator gives

$$\Delta\mu(k) = 2\sum_{r=1}^{k}\frac{r(3r-1)}{1-q^r} + \frac{k^2(k^2-1)}{1-q^{k+1}} - \frac{k^2(k+1)^2}{1-q^k}.$$

The second forward shift operator gives

$$\begin{aligned}\Delta^2\mu(k) &= \frac{k^2(k+1)^2}{1-q^k} - \frac{2k(k+1)^3}{1-q^{k+1}} + \frac{k(k+1)^2(k+2)}{1-q^{k+2}}\\ &= \frac{k(k+1)^2(1-q)q^k t(k)}{(1-q^k)(1-q^{k+1})(1-q^{k+2})}\end{aligned}$$

where

$$t(k) = k(1-q)(1+q^{k+1}) - 2q(1-q^k). \tag{21}$$

Then, we have

$$\Delta t(k) = (1-q)(1-q^{k+1}(1+(1-q)(k+1)))$$

$$\Delta^2 t(k) = (1-q)^3 q^{k+1}(k+2) \geqslant 0, \qquad k \geqslant -2.$$

Since $\Delta t(-1) = 0$, then $\Delta t(k) \geqslant 0$ for all $k \geqslant -1$. Also, since $t(0) = 0$, then $t(k) \geqslant 0$ for all $k \geqslant 0$ which reveals that $\Delta^2 \mu(k) \geqslant 0$ for all $k \geqslant 0$. It is not difficult to see that $\Delta \mu(1) = 0$ which leads to $\Delta \mu(k) \geqslant 0$ for all $k \in \mathbb{N}$. Since $\mu(1) = 0$, then $\mu(k) \geqslant 0$ for all $k \in \mathbb{N}$.

LEMMA 5. *The function*

$$\eta(k) = \frac{k(k-1)^2}{1-q^k} + 2\sum_{r=1}^{k-1} \frac{r(k-3r)}{1-q^r} - \frac{2(k-1)}{\log q} \tag{22}$$

is nonnegative for all $k \in \mathbb{N}$ *and* $q \in (0,1)$.

Proof. Forward shift operator gives

$$\Delta \eta(k) = \frac{k^2(k+1)}{1-q^{k+1}} - \frac{k(k+1)^2}{1-q^k} + 2\sum_{r=1}^{k} \frac{r}{1-q^r} - \frac{2}{\log q}$$

and

$$\Delta^2 \eta(k) = \frac{(k+1)^2(1-q)q^k t(k)}{(1-q^k)(1-q^{k+1})(1-q^{k+2})}$$

where $t(k)$ is defined in (21), which yields that the function $\Delta^2 \eta(k) > 0$. Computing $\Delta \eta(1)$ gets

$$\Delta \eta(1) = \frac{-2(1-q^2+q\log q)}{(1-q^2)\log q} > 0$$

which means that $\Delta \eta(k) > 0$ for all $k \in \mathbb{N}$. Since $\eta(1) = 0$, then $\eta(k) > 0$ for all $k \in \mathbb{N}$.

LEMMA 6. *Let* a *and* q *be positive reals with* $0 < q < 1$ *and*

$$\varphi_a(x) = \frac{(1-q^x)^2}{q^x \log^2 q} \psi_q'(x+1) + \frac{2a(1-q^x)}{\log q} + ax. \tag{23}$$

Then there exists $\bar{x} > 0$ *such that* $\varphi_a(x) \leqslant 0$ *for all* $x \leqslant \bar{x}$ *and* $\varphi_a(x) > 0$ *for all* $x > \bar{x}$.

Proof. Differentiations give

$$\varphi_a'(x) = \frac{(1-q^x)^2}{q^x\log^2 q}\psi_q''(x+1) - \frac{1-q^{2x}}{q^x\log q}\psi_q'(x+1) - 2aq^x + a, \tag{24}$$

$$\varphi_a''(x) = \frac{(1-q^x)^2}{q^x\log^2 q}\psi_q'''(x+1) - \frac{2(1-q^{2x})}{q^x\log q}\psi_q''(x+1) + \frac{1+q^{2x}}{q^x}\psi_q'(x+1) - 2aq^x\log q. \tag{25}$$

Inserting the identity (9) and its derivatives into the last equation with supposing $\phi(x) = q^{-x}\varphi_a''(x)$ yields

$$\phi(x) = \frac{(1-q^x)^2}{q^{2x}\log^2 q}\psi_q'''(x) - \frac{2(1-q^{2x})}{q^{2x}\log q}\psi_q''(x) + \frac{1+q^{2x}}{q^{2x}}\psi_q'(x) - 2a\log q \tag{26}$$

which has the derivative

$$\frac{q^{2x}\log^2 q}{(1-q^x)^2}\phi'(x) = \psi_q^{(4)}(x) - \frac{2(2+q^x)\log q}{1-q^x}\psi_q'''(x) + \frac{(5+q^{2x})\log^2 q}{(1-q^x)^2}\psi_q''(x) - \frac{2\log^3 q}{(1-q^x)^2}\psi_q'(x). \tag{27}$$

With using (8), the binomial theorem and the Cauchy product rule lead to

$$(1-q^x)^{-1}\psi_q'''(x) = \log^4 q\sum_{k=1}^{\infty} q^{xk}\sum_{r=1}^{k}\frac{r^3}{1-q^r},$$

$$(1-q^x)^{-2}\psi_q''(x) = \log^3\sum_{k=1}^{\infty} q^{xk}\sum_{r=1}^{k}\frac{r^2(k-r+1)}{1-q^r},$$

$$(1-q^x)^{-2}\psi_q'(x) = \log^2 q\sum_{k=1}^{\infty} q^{xk}\sum_{r=1}^{k}\frac{r(k-r+1)}{1-q^r}$$

which can be inserted into (27) to obtain

$$\begin{aligned}
\frac{q^{2x}}{(1-q^x)^2\log^3 q}\phi'(x) &= \sum_{k=2}^{\infty} q^{xk}\left\{\frac{k^4}{1-q^k} + \sum_{r=1}^{k}\frac{r(k(5r-2)-9r^2+7r-2)}{1-q^r}\right\} \\
&\quad - 2\sum_{k=2}^{\infty} q^{xk}\sum_{r=1}^{k-1}\frac{r^3}{1-q^r} + \sum_{k=3}^{\infty} q^{xk}\sum_{r=1}^{k-2}\frac{r^2(k-r-1)}{1-q^r} \\
&= \sum_{k=3}^{\infty} q^{xk}\left\{\frac{k(k-1)^2(k-2)}{1-q^k} + \sum_{r=1}^{k-1}\frac{r(k(5r-2)-11r^2+7r-2)}{1-q^r}\right\} \\
&\quad + \sum_{k=3}^{\infty} q^{xk}\sum_{r=1}^{k-1}\frac{r^2(k-r-1)}{1-q^r} = \sum_{k=3}^{\infty} q^{xk}\mu(k)
\end{aligned}$$

where $\mu(k)$ defined in (20), which means that $\phi'(x) < 0$ for all $x \geqslant 0$ and so the function $\phi(x)$ is decreasing on $(0,\infty)$. From (25), we have

$$\phi(0) = \varphi_a''(0) = 2\psi_q'(1) - 2a\log q > 0$$

and from (26) with aiding (10), we get

$$\lim_{x\to\infty}\phi(x)=\frac{2\log^2 q}{1-q^2}-\frac{2\log^2 q}{1-q}-2a\log q=\frac{-2\log q}{1-q^2}[q\log q+a(1-q^2)].$$

Consequently, we have two cases:

The first case: When $a\geqslant-\frac{q\log q}{1-q^2}$, then $\phi(x)>0$ for all $x\geqslant 0$ and so does the function $\varphi_a''(x)$ which leads to $\varphi_a'(x)$ is increasing on $(0,\infty)$. It is not difficult, from (24), to show that

$$\varphi_a'(0)=-a<0 \qquad \text{and} \qquad \lim_{x\to\infty}\varphi_a'(x)=a>0 \tag{28}$$

and so there exists $x_0>0$ such that $\varphi_a'(x)<0$ for all $x<x_0$ and $\varphi_a'(x)>0$ for all $x>x_0$ which yields $\varphi_a(x)$ is decreasing on $(0,x_0)$ and increasing on (x_0,∞).

The second case: When $a<-\frac{q\log q}{1-q^2}$, then there exists $0<x_1<\infty$ such that $\phi(x)>0$ for all $x<x_1$ and $\phi(x)<0$ for all $x>x_1$ and so does the function $\varphi_a''(x)$ which reveals that $\varphi_a'(x)$ is increasing on $(0,x_1)$ and decreasing on (x_1,∞). By virtue of (28), we see that $\varphi_a'(x_1)>0$ and so there exists $0<x_2<x_1$ such that $\varphi_a'(x)<0$ for all $x<x_2$ and $\varphi_a'(x)>0$ for all $x>x_2$.

In view of both cases for all $a>0$, we conclude that there exists $0<\tilde{x}<\infty$ such that $\varphi_a(x)$ is decreasing on $(0,\tilde{x})$ and increasing on $(\tilde{x},\infty)$. From (23), we get $\varphi_a(0)=0$ and $\lim_{x\to\infty}\varphi_a(x)=\infty$ which mean that $\varphi_a(\tilde{x})<0$ and there exists $\bar{x}>\tilde{x}>0$ such that $\varphi_a(x)\leqslant 0$ for all $x\leqslant\bar{x}$ and $\varphi_a(x)>0$ for all $x>\bar{x}$.

3. The main results

In this section, we prove the following refinements of (4) and (5).

THEOREM 1. *Let q be positive real. Then*

$$\begin{aligned}&[x]_q^{x-\hat{q}^{-1}\gamma_{\hat{q}}}q^{(x-1)(q\gamma_{q^{-1}}-\frac{1}{2})H(q-1)}\exp\left(\frac{Li_2(1-q^x)-Li_2(1-q)}{\log q}\right)\\ \leqslant&\Gamma_q(x)\leqslant[x]_q^{x-\frac{1}{2}}\exp\left(\frac{Li_2(1-q^x)-Li_2(1-q)}{\log q}\right),\end{aligned} \tag{29}$$

for every $x\geqslant 1$. The constants $\hat{q}^{-1}\gamma_{\hat{q}}$ and $\frac{1}{2}$ are the best possible for all $q>0$. Moreover, the left-hand side inequality holds for all $x>0$ and the right-hand side inequality is reversed for all $x\leqslant 1$.

Proof. Let the function

$$S_q(x)=\log\Gamma_q(x)-\left(x-\frac{1}{2}\right)\log[x]_q-\frac{\mathrm{Li}_2(1-q^x)-\mathrm{Li}_2(1-q)}{\log q} \tag{30}$$

be defined for all $x,q \in \mathbb{R}^+$, where $\mathrm{Li}_2(z)$ is the dilogarithm function defined for all complex argument z as [1]

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t}dt, \qquad z \notin (1,\infty)$$

which satisfies the identity

$$\mathrm{Li}_2\left(\frac{z-1}{z}\right) = -\mathrm{Li}_2(1-z) - \frac{1}{2}\log^2 z. \tag{31}$$

Let $0 < q < 1$. Differentiating (30) gives

$$S_q'(x) = \psi_q(x) - \log[x]_q - \frac{1}{2}\frac{q^x \log q}{1-q^x} = -\frac{1}{2}\sum_{k=1}^{\infty}\frac{q^{xk}}{k(1-q^k)}s(y), \quad y = q^k$$

where

$$s(y) = (1-y)\log y - 2\log y - 2(1-y) = y\sum_{n=3}^{\infty}\frac{\log^n(1/y)}{n!}(n-2) > 0$$

which yields that $S_q'(x) < 0$ for all $x > 0$ and so the function $S_q(x)$ is decreasing on $(0,\infty)$. Since $S_q(1) = 0$, then $S_q(x) > 0$ for all $x \in (0,1)$ and $S_q(x) < 0$ for all $x \in (1,\infty)$. When $q \geqslant 1$, in view of (2), (30) and (31), we get $S_{q^{-1}}(x) = S_q(x)$. Hence, for all $q \in \mathbb{R}^+$, we get $S_q(x) > 0$ for all $x \in (0,1)$ and $S_q(x) < 0$ for all $x \in (1,\infty)$. Also, let the function

$$\begin{aligned} T_q(x) = &\log\Gamma_q(x) - (x - \hat{q}^{-1}\gamma_{\hat{q}})\log[x]_q - \frac{\mathrm{Li}_2(1-q^x) - \mathrm{Li}_2(1-q)}{\log q} \\ &- \frac{(x-1)(2q\gamma_{q^{-1}}-1)}{2}H(q-1)\log q \end{aligned} \tag{32}$$

be defined for all $x,q \in \mathbb{R}^+$. Differentiation gives

$$T_q'(x) = \psi_q(x) - \log[x]_q - \frac{\gamma_q q^{x-1}\log q}{1-q^x} = -\frac{q^x \log q}{1-q^x}\tau(x), \qquad 0 < q < 1$$

where

$$\tau(x) = -\frac{1-q^x}{q^x \log q}(\psi_q(x) - \log[x]_q) + q^{-1}\gamma_q. \tag{33}$$

Differentiation gives

$$\frac{q^x \log q}{q^x - 1}\tau'(x) = -\frac{\log q}{1-q^x}(\psi_q(x) - \log[x]_q) + \psi_q'(x) + \frac{q^x \log q}{1-q^x}.$$

Using (7) and Cauchy product rule yield

$$\frac{q^x}{1-q^x}\tau'(x) = \sum_{k=1}^{\infty} q^{xk}\lambda(k).$$

Lemma 3 tells that $\tau'(x) > 0$ for all $x > 0$ which yields that $\tau(x)$ is increasing on $(0,\infty)$. It is clear from (6) and (33) that the function $\tau(1) = 0$ which gives $\tau(x) < 0$ if $x \in (0,1)$ and $\tau(x) > 0$ if $x \in (1,\infty)$ and so does the function $T_q'(x)$. Therefore, the function $T_q(x)$ is decreasing on $(0,1)$ and increasing on $(1,\infty)$. In conclusion, since $T_q(1) = 0$, then we get $T_q(x) \geqslant 0$ for all $x > 0$. When $q \geqslant 1$, in view of (2), (31) and (32), we get $T_q(x) = T_{q^{-1}}(x)$. Hence $T_q(x) > 0$ for all $x > 0$ and $q > 0$. The inequality (29) can be rewritten as

$$\frac{1}{2} < U(x;q) < \hat{q}^{-1}\gamma_{\hat{q}} - \frac{(x-1)(2q\gamma_{q^{-1}} - 1)}{2\log[x]_q} H(q-1)\log q$$

where

$$U(x;q) = \frac{\mathrm{Li}_2(1-q^x) - \mathrm{Li}_2(1-q)}{\log q \log[x]_q} + \frac{x\log[x]_q - \log\Gamma_q(x)}{\log[x]_q}.$$

With the Euler-Maclaurin formula, Moak [12] obtained the following q-analogue of Stirling formula (see also [24])

$$\log\Gamma_q(x) \sim \left(x - \frac{1}{2}\right)\log[x]_q + \frac{\mathrm{Li}_2(1-q^x)}{\log q} + \frac{1}{2}H(q-1)\log q + C_{\hat{q}}, \quad x \to \infty \tag{34}$$

where $C_{\hat{q}}$ is appropriate constant depending on the value of q. This asymptotic expansion can be exploited to compute

$$\lim_{x\to\infty} U(x;q) = \frac{1}{2}, \qquad q > 0$$

Also, using l'Hôpital's rule yields

$$\lim_{x\to 1} U(x;q) = q^{-1}\gamma_q, \qquad 0 < q < 1.$$

When $q \geqslant 1$, we find

$$\lim_{x\to 1}\left(\frac{(x-1)(2q\gamma_{q^{-1}} - 1)}{2\log[x]_q}\log q\right) = \frac{1-q}{2q}(1 - 2q\gamma_{q^{-1}})$$

and thus

$$\lim_{x\to 1}\left(U(x;q) + \frac{(x-1)(2q\gamma_{q^{-1}} - 1)}{2\log[x]_q}H(q-1)\log q\right) = q^{-1}\gamma_q + \frac{1-q}{2q}(1 - 2q\gamma_{q^{-1}}).$$

When $q \geqslant 1$, the logarithmic derivative of (2) at $x = 1$ gives

$$q^{-1}\gamma_q + \frac{1-q}{2q}(1 - 2q\gamma_{q^{-1}}) = q\gamma_{q^{-1}}.$$

Therefore, the constants $\frac{1}{2}$ and $\hat{q}^{-1}\gamma_{\hat{q}}$ are the best possible for all $q > 0$.

THEOREM 2. *Let $q>0$. Then*

$$f_q(x)=\log\Gamma_q(x+1)-\alpha(\hat{q})x\log[x]_q-\frac{x(x-1)(1-2\alpha(q^{-1}))}{2}H(q-1)\log q \quad (35)$$

is non-negative for all $x\geqslant 0$, where $\alpha(q)$ is defined in (18).

Proof. Let $0<q<1$. Differentiation gives

$$f_q'(x)=\psi_q(x+1)-\alpha(q)\log[x]_q+\frac{\alpha(q)xq^x\log q}{1-q^x} \quad (36)$$

and

$$\frac{(1-q^x)^2}{q^x\log^2 q}f_q''(x)=\varphi_{\alpha(q)}(x)$$

where $\varphi_{\alpha(q)}(x)$ defined as in (23). By virtue of the results obtained in Lemma 6, there exists $\bar{x}>0$ such that the function $f_q'(x)$ is decreasing on $(0,\bar{x})$ and increasing on $(\bar{x},\infty)$. Let $\bar{x}\geqslant 1$ and since $f_q'(1)=0$, then $f_q'(x)>0$ for all $x\in(0,1)$ and so $f_q(x)$ is increasing on $(0,1)$ which contradicts with $f_q(0)=f_q(1)=0$. Hence, we have to take $\bar{x}<1$. Since $f_q'(x)$ is increasing on $(\bar{x},1)\subset(\bar{x},\infty)$, then $f_q'(\bar{x})<0$. From (36), we get $\lim_{x\to 0}f_q'(x)=\infty$ and $\lim_{x\to\infty}f_q'(x)=-(1-\alpha(q))\log(1-q)>0$, where $\alpha(q)<1/2$ by Lemma 2, which yields that there exists $0<\bar{\bar{x}}<\bar{x}<1$ such that $f_q'(x)>0$ for all $x\in(0,\bar{\bar{x}})\cup(1,\infty)$ and $f_q'(x)<0$ for all $x\in(\bar{\bar{x}},1)$ and so $f_q(x)$ is increasing on $(0,\bar{\bar{x}})\cup(1,\infty)$ and decreasing on $(\bar{\bar{x}},1)$. In conclusion, since $f_q(0)=f_q(1)=0$, then $f_q(x)\geqslant 0$ for all $x\geqslant 0$. In view of (2) and (35), we get $f_q(x)=f_{q^{-1}}(x)$ for all $q\geqslant 1$ which concludes that $f_q(x)\geqslant 0$ for all $x\geqslant 0$ and $q>0$.

THEOREM 3. *Let $q>0$. Then*

$$g_q(x)=(x-\hat{q}^{-1}\gamma_{\hat{q}})\log[x]_q-\log\Gamma_q(x+1)-\frac{(x-1)(x-2q\gamma_{q^{-1}})}{2}H(q-1)\log q \quad (37)$$

is non-negative for all $x>0$.

Proof. Let $0<q<1$. Differentiation gives

$$g_q'(x)=\log[x]_q-\frac{(x-q^{-1}\gamma_q)q^x\log q}{1-q^x}-\psi_q(x+1) \quad (38)$$

and

$$\frac{(1-q^x)^2}{q^x\log^2 q}g_q''(x)=-\frac{2(1-q^x)}{\log q}-x+q^{-1}\gamma_q-\frac{(1-q^x)^2}{q^x\log^2 q}\psi_q'(x+1)\triangleq\theta(x). \quad (39)$$

The function $\theta(x)$ can be represented by using (9) as

$$\theta(x)=-\frac{2(1-q^x)}{\log q}-x+q^{-1}\gamma_q-\frac{(1-q^x)^2}{q^x\log^2 q}\psi_q'(x)+1.$$

Differentiation gives

$$\theta'(x) = 2q^x - 1 - \frac{(1-q^x)^2}{q^x \log^2 q}\psi_q''(x) + \frac{1-q^{2x}}{q^x \log q}\psi_q'(x) \tag{40}$$

and

$$\frac{q^x \log^2 q}{(1-q^x)^2}\theta''(x) = \frac{2q^{2x}\log^3 q}{(1-q^x)^2} - \psi_q'''(x) + \frac{2(1+q^x)\log q}{1-q^x}\psi_q''(x) - \frac{(1+q^{2x})\log^2 q}{(1-q^x)^2}\psi_q'(x)$$

which can be represented as

$$\frac{q^x \log^2 q}{(1-q^x)^2}\theta''(x) = -\log^4 q \sum_{k=2}^{\infty} q^{xk}\eta(k)$$

where $\eta(k)$ defined in (22), and so $\theta''(x) < 0$ for all $x > 0$ which reveals that $\theta'(x)$ is decreasing on $(0,\infty)$. According to the equation (40), $\theta'(0) = 1 > 0$ and $\lim_{x\to\infty}\theta'(x) = -1 < 0$, then there exists $x_0 > 0$ such that $\theta'(x) > 0$ for all $x < x_0$ and $\theta'(x) < 0$ for all $x > x_0$ which yields that $\theta(x)$ is increasing on $(0,x_0)$ and decreasing on (x_0,∞). Since, from (39), $\theta(0) = q^{-1}\Gamma_q > 0$ and $\lim_{x\to\infty}\theta(x) = -\infty$, then there exists $x_1 > x_0 > 0$ such that $\theta(x) > 0$ for all $x < x_1$ and $\theta(x) < 0$ for all $x > x_0$ and does the function $g_q''(x)$ which reveals that $g_q'(x)$ is increasing on $(0,x_1)$ and decreasing on (x_1,∞). By using the relations (6), (9) and (38), we get $g_q'(1) = 0$ and $\lim_{x\to\infty} g_q'(x) = 0$ which mean that $x_1 > 1$ and $g_q'(x) < 0$ for all $x < 1$ and $g_q'(x) > 0$ for all $x > 1$ and so $g_q(x)$ is decreasing on $(0,1)$ and increasing on $(1,\infty)$. Since $g_q(1) = 0$, then $g_q(x) \geqslant 0$ for all $x > 0$. In view of (2) and (37), we get $g_q(x) = g_{q^{-1}}(x)$ for all $q \geqslant 1$ which concludes that $g_q(x) \geqslant 0$ for all $x > 0$ and $q > 0$.

THEOREM 4. *Let $q > 0$. Then*

$$\begin{aligned} h_q(x) =& \log\Gamma_q(x+1) - [\delta(\hat{q})(x-1) + \alpha(\hat{q})]\log[x]_q \\ & - \frac{(x-1)[(1-2\delta(q^{-1}))x + 2(\delta(q^{-1}) - \alpha(q^{-1})]}{2}H(q-1)\log q \end{aligned} \tag{41}$$

is negative for all $x \in (0,1)$ and positive for all $x \in (1,\infty)$, where $\alpha(q)$ and $\delta(q)$ are defined in (18).

Proof. Let $0 < q < 1$. Differentiation gives

$$h_q'(x) = \psi_q(x+1) - \delta(q)\log[x]_q + [\delta(q)(x-1) + \alpha(q)]\frac{q^x \log q}{1-q^x} \tag{42}$$

and

$$\frac{(1-q^x)^2}{q^x \log^2 q}h_q''(x) = \frac{(1-q^x)^2}{q^x \log^2 q}\psi_q'(x+1) + \frac{2\delta(q)(1-q^x)}{\log q} + \delta(q)(x-1) + \alpha(q) \triangleq \omega(x). \tag{43}$$

Note that $\omega(x) = \varphi_{\delta(q)}(x) + \alpha(q) - \delta(q)$ where φ defined as in (2.8). According to results obtained in Lemma 2 and in the proof of Lemma 6, there exists $\bar{x} > 0$ such that $\omega(x)$ is decreasing on $(0, \bar{x})$ and increasing on $(\bar{x}, \infty)$. It is easy from (43) to see that $\omega(0) = \alpha(q) - \delta(q) < 0$ (see Lemma 2), $\omega(1) = 0$ and $\lim_{x\to\infty} \omega(x) = \infty$ which mean that $\omega(x) < 0$ for all $x < 1$ and $\omega(x) > 0$ for all $x > 1$ and so the function $h_q'(x)$ is decreasing on $(0,1)$ and increasing on $(1,\infty)$. Since $h_q'(1) = 0$, then $h_q'(x) \geqslant 0$ for all $x > 0$ and so $h_q(x)$ is increasing on $(0,\infty)$. In conclusion, since $h_q(1) = 0$ then, $h_q(x) < 0$ for all $x \in (0,1)$ and $h_q(x) > 0$ for all $x \in (1,\infty)$. When $q \geqslant 1$, (2) gives $h_q(x) = h_{q^{-1}}(x)$. Therefore, $h_q(x) < 0$ if $x \in (0,1)$ and $h_q(x) > 0$ if $x \in (1,\infty)$ for all $q > 0$.

THEOREM 5. *Let q be a positive real and let $\alpha(q)$ and $\delta(q)$ be defined in (18). Then*

$$[x]_q^{a(x-1)-\hat{q}^{-1}\gamma_{\hat{q}}} q^{c_1} < \Gamma_q(x) < [x]_q^{b(x-1)-\hat{q}^{-1}\gamma_{\hat{q}}} q^{c_3} \tag{44}$$

holds for $x \in (0,1)$ with the best possible constants $a = \alpha(\hat{q})$ and $b = \delta(\hat{q})$. Also,

$$[x]_q^{a(x-1)-\hat{q}^{-1}\gamma_{\hat{q}}} q^{c_3} < \Gamma_q(x) < [x]_q^{b(x-1)-\hat{q}^{-1}\gamma_{\hat{q}}} q^{-c_2} \tag{45}$$

holds for $x \in (1,\infty)$ with the best possible constants $a = \delta(\hat{q})$ and $b = 1$ where

$$\begin{aligned}
c_1 &= \frac{x(x-1)(1-2\alpha(q^{-1}))}{2} H(q-1),\\
c_2 &= \frac{(x-1)(x-2q\gamma_{q^{-1}})}{2} H(q-1),\\
c_3 &= \frac{(x-1)((1-2\delta(q^{-1}))x+2(\delta(q^{-1})-\alpha(q^{-1}))}{2} H(q-1).
\end{aligned}$$

Proof. The proof comes immediately from the previous three theorems. It suffices to prove the constants are the best possible. In order to do this, we note from (44) and (45), respectively, that

$$a + \frac{c_1 \log q}{(x-1)\log[x]_q} < w(x) < b + \frac{c_3 \log q}{(x-1)\log[x]_q}, \qquad x \in (0,1)$$

and

$$a + \frac{c_3 \log q}{(x-1)\log[x]_q} < w(x) < b - \frac{c_2 \log q}{(x-1)\log[x]_q}, \qquad x \in (1,\infty)$$

where

$$w(x) = \frac{1}{x-1}\left(\hat{q}^{-1}\gamma_{\hat{q}} + \frac{\log\Gamma_q(x)}{\log[x]_q}\right).$$

Using l'Hôpital's rule gets

$$\lim_{x\to 0}\left(w(x)-\frac{c_1\log q}{(x-1)\log[x]_q}\right)=\alpha(\hat{q}),$$
$$\lim_{x\to 1}\left(w(x)-\frac{c_3\log q}{(x-1)\log[x]_q}\right)=\delta(\hat{q}),$$
$$\lim_{x\to \infty}\left(w(x)+\frac{c_2\log q}{(x-1)\log[x]_q}\right)=1.$$

The proof is done.

4. Conclusion

In this paper, the inequalities (29), (44) and (45) for the q-gamma function are established and proven for all $q>0$. The bounds in these inequalities are sharper than the bounds in inequalities (4) and (5) obtained in [13]. It will be confined in the following to show the improvements in the case when $0<q<1$.

4.1. Comparing (29) and (5):

The following theorem shows the bounds in inequality (29) are better than in (5).

THEOREM 6. *Let* $0<q<1$. *Then:*

- *The left hand side of inequality (29) is greater than left hand side of inequality (5) for all* $x\geqslant 1$.
- *The right hand side of inequality (29) is smaller than right hand side of inequality (5) for all* $x\geqslant 1$.

Proof. Define the function

$$f(x)=\frac{\mathrm{Li}_2(1-q^x)-\mathrm{Li}_2(1-q)}{\log q}+(x(1-\alpha(q))+\alpha(q))\log[x]_q.$$

Differentiation gives

$$f'(x)=\frac{\alpha(q)(x-1)q^x\log q}{1-q^x}+(1-\alpha(q))\log[x]_q$$

and

$$f''(x)=\frac{(2\alpha(q)-1)q^x\log q}{1-q^x}+\frac{\alpha(q)(x-1)q^x\log^2 q}{(1-q^x)^2}.$$

It is known from Lemma 2 that $0<\alpha(q)<1/2$ and so $2\alpha(q)-1<0$ which yields that $f''(x)>0$ for all $x\geqslant 1$. Thus $f'(x)$ is increasing on $(1,\infty)$. Since $f'(1)=0$, then $f'(x)>0$ for all $x>1$ which leads to $f(x)$ is increasing on $(0,\infty)$. Since $f(1)=0$, then $f(x) > 0$ for all $x > 1$ which yields $\exp(f(x)) > 1$. Notice that $\exp(f(x))$ is the ratio of the two lift sides of (29) and (5). This proves the first statement. To prove the second statement, define the function

$$g(x) = \frac{\mathrm{Li}_2(1-q^x) - \mathrm{Li}_2(1-q)}{\log q} + \frac{1}{2}\log[x]_q.$$

Differentiation gives

$$g'(x) = \frac{1}{2}\frac{(2x-1)q^x \log q}{1-q^x} < 0, \qquad x > 1.$$

Since $g(1) = 0$, then $g(x) < 0$ for all $x > 1$ which yields $\exp(g(x)) < 1$. Notice that $\exp(g(x))$ is the ratio of the two right sides of (29) and (5).

4.2. Comparing (44) and (4):

The following theorem shows the intervals in which the bounds in inequality (44) are better than in (4).

THEOREM 7. *Let* $0 < q < 1$. *Then:*

- *The left hand side of inequality (44) is greater than left hand side of inequality (4) for all* $x \in (0,1)$.
- *The right hand side of inequality (44) is smaller than right hand side of inequality (4) for all* $x \in (1-\alpha(q)/\delta(q), 1)$ *whereas the reverse is true for all* $x \in (0, 1-\alpha(q)/\delta(q))$.

Proof. It is easy to see that the ratio of two left sides of (44) and (4) is greater than one and so the left bound in (44) is greater (better) than the bound in (4) for all $x \in (0,1)$. The ratio of the two right bounds is $[x]_q^{\delta(q)(x-1)+\alpha(q)}$. The exponent is positive on $(1-\alpha(q)/\delta(q), 1)$ and negative in $(0, 1-\alpha(q)/\delta(q))$. Indeed, if $x < 1$ then $[x]_q < 1$ and thus the ratio is greater than one if $x \in (0, 1-\alpha(q)/\delta(q))$ and less than one if $x \in (1-\alpha(q)/\delta(q), 1)$. Therefore, the right bound of (44) is better than the right bound of (4) if $x \in (1-\alpha(q)/\delta(q), 1)$, whereas the reverse is true if $x \in (0, 1-\alpha(q)/\delta(q))$.

4.3. Comparing (45) and (5):

The following theorem shows the bounds in inequality (45) are better than in (5).

THEOREM 8. *Let* $0 < q < 1$. *Then:*

- *The left hand side of inequality (45) is greater than left hand side of inequality (5) for all* $x > 1$.

- *The right hand side of inequality (45) is smaller than right hand side of inequality (5) for all $x > 1$.*

Proof. The ratio of the left pound is $[x]_q^{[\delta(q)-\alpha(q)](x-1)} > 1$ due to $x > 1$ and $\delta(q) - \alpha(q) > 0$ (Lemma 2). Also the ratio of the two right bounds is less than one. Therefore the bounds of (45) are better than the bounds of (5).

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