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CHARACTERIZATIONS OF WEIGHTED HARDY–RELLICH INEQUALITIES AND THEIR APPLICATIONS

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Abstract. Let Ω be a bounded open domain in $\mathbb{R}^n$. We establish characterizations of the weighted Hardy-Rellich inequalities that connect the integrals over Ω of the first and second derivatives of the considered functions, via some weighted vector-valued Hardy inequalities and weighted dual Hardy inequalities. These characterizations are then applied to derive some new weighted Rellich inequalities with homogenous weights that admit singularities on unit sphere $\mathbb{S}^{n-1}$.

1. Introduction

The classical Hardy inequality, which connects integrals of functions and their derivatives, was first proved by G. H. Hardy [20] motivated by finding an elementary proof of Hilbert inequality. The *classical one-dimensional Hardy inequality* states that for any $p \in (1,\infty)$ and $f \in C_c^\infty(0,\infty)$, it holds

$$\int_0^\infty \frac{|f(x)|^p}{x^p}\,dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty \left|f'(x)\right|^p dx, \tag{1.1}$$

where the constant $(\frac{p}{p-1})^p$ is sharp but not attained. Since then, the Hardy inequality has been extended to various settings, which are of fundamental importance in many branches of mathematical analysis and mathematical physics; see [21, 34, 23, 4] and their references.

The one-dimensional Hardy inequality (1.1) can be extended directly to the weighted case (see [34, Lemma 1.3]). To be precise, let $p \in (1,\infty)$ and $\varepsilon \in \mathbb{R}$ with $\varepsilon \neq p-1$. The *classical one-dimensional weighted Hardy inequality* says that for any $f \in C_c^\infty(0,\infty)$,

$$\int_0^\infty |f(x)|^p x^{\varepsilon-p}\,dx \leqslant \left(\frac{p}{|p-1-\varepsilon|}\right)^p \int_0^\infty \left|f'(x)\right|^p x^\varepsilon\,dx, \tag{1.2}$$

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where the constant $(\frac{p}{|p-1-\varepsilon|})^p$ is also best possible.

On the other side, the *one-dimensional Rellich inequality* was first established by F. Rellich [36], which shows that for any $p \in (1,\infty)$ and $f \in C_c^\infty(0,\infty)$, it holds that

$$\int_0^\infty \frac{|f(x)|^p}{x^{2p}}\,dx \leqslant \left[\frac{p^2}{(p-1)(2p-1)}\right]^p \int_0^\infty \left|f''(x)\right|^p dx, \tag{1.3}$$

where the constant $[\frac{p^2}{(p-1)(2p-1)}]^p$ is sharp (see also [10]). It is easy to see that (1.3) can also be deduced from a combination of (1.1) and (1.2) with $\varepsilon = -p$.

Things become much more complicated when dimension of the underlying space gets higher. In many cases, there are two parallel choices on the distance function $\delta(x)$ in the considered inequalities: i) $\delta(x) \equiv |x| := \operatorname{dist}(x,0)$ is the distance to the origin; ii) $\delta(x) \equiv d(x) := \operatorname{dist}(x,\partial\Omega)$ is the distance to the boundary of the considered domain Ω. In the following, we may review some known results on the higher dimensional Hardy and Rellich inequalities with distance functions belong to the aforementioned two choices.

If $\delta(x) \equiv |x| := \operatorname{dist}(x,0)$, then *the n-dimensional Hardy inequality* (see [4, Corollary 1.2.6]) says that, for any $p \in (1,\infty)$, $p \neq n$ and $f \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$ with $n > 1$,

$$\int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^p}\,dx \leqslant \left(\frac{p}{|p-n|}\right)^p \int_{\mathbb{R}^n} |\nabla f(x)|^p\,dx. \tag{1.4}$$

Moreover if $n > p$, then (1.4) holds even for all $f \in C_c^\infty(\mathbb{R}^n)$. The above Hardy inequality can be extended to a *weighted version* (see [4, Corollary 1.2.9]) that for any $p \in (1,\infty)$, $\varepsilon \in \mathbb{R}$ satisfying $p - \varepsilon \neq n$ and $f \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$ with $n > 1$,

$$\int_{\mathbb{R}^n} |f(x)|^p |x|^{\varepsilon-p}\,dx \leqslant \left(\frac{p}{|p-n-\varepsilon|}\right)^p \int_{\mathbb{R}^n} |\nabla f(x)|^p\,|x|^\varepsilon\,dx. \tag{1.5}$$

In the case of Rellich inequality, Davies and Hinz [10, Theorem 12] proved the following *weighted n-dimensional Rellich inequality* that for any $p \in (1,\infty)$, $\varepsilon \in (-\infty, 2(p-1))$, $n > 2p - \varepsilon$ and $f \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$,

$$\begin{aligned}\int_{\mathbb{R}^n} &|f(x)|^p |x|^{\varepsilon-2p}\,dx \\ &\leqslant \left(\frac{p^2}{(n+\varepsilon-2p)[(p-1)(n-2)+2(p-1)-\varepsilon]}\right)^p \int_{\mathbb{R}^n} |\Delta f(x)|^p\,|x|^\varepsilon\,dx;\end{aligned} \tag{1.6}$$

see also [8, 30] and their references for an extension of (1.6) to a larger range of ε and n.

If Ω is a convex domain in $\mathbb{R}^n$ with C^1-boundary and $\delta(x) \equiv d(x) := \operatorname{dist}(x,\partial\Omega)$, then *the n-dimensional Hardy inequality* (see [26, 29]) says that for any $p \in (1,\infty)$ and $f \in C_c^\infty(\Omega)$,

$$\int_\Omega \frac{|f(x)|^p}{d(x)^p}\,dx \leqslant \left(\frac{p}{p-1}\right)^p \int_\Omega |\nabla f(x)|^p\,dx. \tag{1.7}$$

Recall that if Ω is a Lipschitz domain, then the associated n-dimensional Hardy inequality still holds true. Unfortunately, the sharp constant in this case is not clear (see [34, 28]).

The inequality (1.7) has a *weighted version* which says that for any $p \in (1,\infty)$, $\varepsilon \in (-\infty, p-1)$ and $f \in C_c^\infty(\Omega)$,

$$\int_\Omega |f(x)|^p d(x)^{\varepsilon-p}\,dx \leqslant \left(\frac{p}{p-1-\varepsilon}\right)^p \int_\Omega |\nabla f(x)|^p\, d(x)^\varepsilon\,dx; \tag{1.8}$$

see [1, Theorem 9] for more details. Recall that if Ω is in general a bounded Lipschitz domain, some variants of weighted Hardy inequalities without sharp constant were proved in [32]. We also point out that if $\varepsilon > p-1$, there are many problems related to the weighted Hardy inequality which are still open in this case (see [25, 14, 3]).

On the other hand, let Ω be a convex domain, Owen [35, Corollary 2.4] proved the following *n-dimensional Rellich inequality*, that for any $f \in C_c^\infty(\Omega)$,

$$\int_\Omega \frac{|f(x)|^2}{d(x)^4}\,dx \leqslant \frac{16}{9}\int_\Omega |\Delta f(x)|^2\,dx. \tag{1.9}$$

Recall that inequality (1.9) may still holds if Ω is not convex (see [2]). For general $p \in (1,\infty)$, it is not clear if a variant of inequality (1.9) with sharp constant still holds even when Ω is convex; see [5, p. 879] for further discussions. However, if we replace Δ by the p-Laplacian operator Δ_p, a version of (1.9) may holds for general $p \in (1,\infty)$ (see [13, (4.19)]). Another direction of extension is to consider the distance function $d(x) := \operatorname{dist}(x,K)$ to a closed piecewise surface K in $\mathbb{R}^n$. To be precise, let K be a closed piecewise smooth surface in $\mathbb{R}^n$ with codimension $k \in \{1,\dots,n\}$ and $\Omega \subset \mathbb{R}^n$ be a bounded open domain in $\mathbb{R}^n$ with $n \geqslant 2$ satisfying the condition

$$d\Delta d - k + 1 \geqslant 0 \ \text{ in } \Omega\setminus K \tag{1.10}$$

in the sense of distribution. Assume $\varepsilon \in (-\infty, 0]$, $p \in (1,\infty)$ satisfies $k+\varepsilon-2p > 0$ and

$$\begin{cases} -\varepsilon \neq \frac{3pk-8p^2-2k+6p}{4p-2} & \text{or} \\ p > \frac{13+\sqrt{105}}{4}. \end{cases}$$

Then Barbatis [5, Theorem 1] proved that for any $f \in C_c^\infty(\Omega\setminus K)$,

$$\int_\Omega |f(x)|^p d(x)^{\varepsilon-2p}\,dx \leqslant \left[\frac{p^2}{(k+\varepsilon-2p)(pk-\varepsilon-k)}\right]^p \int_\Omega |\Delta f(x)|^p\, d(x)^\varepsilon\,dx. \tag{1.11}$$

As a special case of (1.11), if $K \equiv \{x_0\}$ for some $x_0 \in \mathbb{R}^n$, then $k = n$ and (1.11) holds for any domain. In particular if $x_0 \equiv 0$, (1.11) reduces to (1.6). If $K \equiv \partial\Omega$, then $k = 1$. In this case, the condition (1.10) is satisfied if Ω is the complement of a convex domain. However, this violates the conditions $\varepsilon \in (-\infty, 0]$ and $k+\varepsilon-2p > 0$, which shows that (1.11) cannot be applied to the case $K \equiv \partial\Omega$ (see also [5, p. 879] for a detailed discussion).

As we have mentioned before, in the one-dimensional case, the sharp Rellich inequality (1.3) can be deduced directly from the Hardy inequality (1.1) with the help of its weighted version (1.2). Unfortunately, similar phenomenon fails in the higher dimensional case. To derive the sharp higher dimensional Rellich inequalities from the corresponding Hardy inequalities, one usually need a new kind of integral inequalities that connect the integrals of the first and second derivatives of the considered functions. These inequalities are called *Hardy-Rellich inequalities* throughout this article, as they build bridges between the Hardy and Rellich inequalities (in some literatures, they are also called the strong versions of Rellich inequality). The first L^2 *Hardy-Rellich inequality*, as far as we know, was proved by Tertikas and Zographopoulos [37, Theorem 1.7]. They proved that for any $n \geqslant 5$ and $f \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \frac{|\nabla f(x)|^2}{|x|^2}\,dx \leqslant \frac{4}{n^2}\int_{\mathbb{R}^n} |\Delta f(x)|^2\,dx, \tag{1.12}$$

where the constant $\frac{4}{n^2}$ is sharp. With the help of (1.12), one can deduce the sharp Rellich inequalities (1.6) with $\varepsilon = 0$ and $p = 2$ from the weighted Hardy inequalities (1.5) with $\varepsilon = -2$ and $p = 2$. Note that the L^2 Hardy-Rellich inequality was also used in [37] to derive some new improved Rellich inequalities.

The L^2 Hardy-Rellich inequality has a *weighted version* with $\delta(x) \equiv |x|$ (see [37, 9, 33]) that for any $\varepsilon \in \mathbb{R}$ satisfying $\varepsilon \geqslant 4-n$, $\varepsilon \neq n$ and $f \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |\nabla f(x)|^2 |x|^{\varepsilon-2}\,dx \leqslant \frac{4}{(n-\varepsilon)^2}\int_{\mathbb{R}^n} |\Delta f(x)|^2\,|x|^{\varepsilon}\,dx. \tag{1.13}$$

For general $p \in (1,n)$, Di et al. [11, Corollary 2(i)] proved a similar inequality that for any $f \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$, it holds

$$\int_{\mathbb{R}^n} |\nabla f(x)|^p\,dx \leqslant \left(\frac{p}{n-p}\right)^p \int_{\mathbb{R}^n} \left|\Delta_p f(x)\right|^p |x|^p\,dx,$$

where $\Delta_p f := \operatorname{div}\left(|\nabla f|^{p-2}\nabla f\right)$ denotes the p-Laplacian of f.

On the other hand, when the weight function $\delta(x) \equiv d(x) := d(x,\partial\Omega)$, Barbatis and Tertikas [6, Theorem 3] first proved that if Ω is a convex domain, d is bounded in Ω, then for any $f \in C_c^\infty(\Omega)$,

$$\int_{\Omega} \frac{|\nabla f(x)|^2}{d(x)^2}\,dx \leqslant 4\int_{\Omega} |\Delta f(x)|^2\,dx, \tag{1.14}$$

where the constant 4 is sharp. This combined (1.8) with $\varepsilon = -2$ and $p = 2$ immediately implies the sharp Rellich inequality (1.9). Note that (1.14) was also used in [6] to derive some new improved Rellich inequalities of the form similar to those in [37] but involving the distance from a hypersurface.

Motivated by the aforementioned results, the purpose of this article is to provide some characterizations of the weighted L^p $(1<p<\infty)$ versions of the Hardy-Rellich inequalities that include the inequalities (1.12), (1.13) and (1.14) as special cases. These characterizations are then applied to derive some new weighted Rellich inequalities with homogeneous weights, which admit singularities on the unit sphere $\mathbb{S}^{n-1}$ and relate to the potentials of some Schrödinger operators on $\mathbb{S}^{n-1}$.

To be precise, let $\alpha\in\mathbb{R}$, $p\in(1,\infty)$ and $c\in(0,\infty)$. Assume that Ω is a bounded open domain and δ a nonnegative measurable function on Ω. We first introduce the following three kinds of Hardy-type inequalities.

(i) (α,p,c)-**HR** (weighted Hardy-Rellich inequality): we say the domain Ω satisfies (α,p,c)-**HR** if for any $f\in C_c^\infty(\Omega)$,

$$\int_\Omega \frac{|\nabla f(x)|^p}{\delta(x)^{(\alpha+1)p}}\,dx \leqslant c^p\int_\Omega \frac{|\Delta f(x)|^p}{\delta(x)^{\alpha p}}\,dx. \tag{1.15}$$

(ii) (α,p,c)-**HV** (weighted vector-valued Hardy inequality): we say the domain Ω satisfies (α,p,c)-**HV** if for any $\vec{u}\in \mathrm{G}(\Omega;\mathbb{R}^n)$,

$$\int_\Omega \frac{|\vec{u}(x)|^p}{\delta(x)^{(\alpha+1)p}}\,dx \leqslant c^p\int_\Omega \frac{|\operatorname{div}\vec{u}(x)|^p}{\delta(x)^{\alpha p}}\,dx, \tag{1.16}$$

where $\mathrm{G}(\Omega;\mathbb{R}^n):=\nabla\left(C_c^\infty(\Omega)\right)$ with

$$\nabla\left(C_c^\infty(\Omega)\right):=\{\vec{u}\in C_c^\infty(\Omega;\mathbb{R}^n):\ \text{there exists } f\in C_c^\infty(\Omega),\ \text{s.t. } \vec{u}=\nabla f\}. \tag{1.17}$$

(iii) (α,p',c)-**HD** (weighted dual Hardy inequality): we say the domain Ω satisfies (α,p',c)-**HD** if for any $f\in C_c^\infty(\Omega)$,

$$\int_\Omega |f(x)|^{p'}\delta(x)^{\alpha p'}\,dx \leqslant c^{p'}\int_\Omega |\nabla f(x)|^{p'}\delta(x)^{(\alpha+1)p'}\,dx, \tag{1.18}$$

where $p':=p/(p-1)$ denotes the conjugate exponent of p.

To state the main result of this article, let Ω be a bounded open domain in $\mathbb{R}^n$. We need the following two assumptions.

Assumption (A1). Let δ be a positive continuous function in Ω.

Assumption (A2). Let $\delta(x)$ be as in Assumption **(A1)** and $w(x):=\delta(x)^{-(\alpha+1)p}$ a weight function on Ω with $\alpha\in\mathbb{R}$ and $p\in(1,\infty)$. The weighted vector-valued Lebesgue space $L^p(w,\Omega;\mathbb{R}^n)$ admits the *Helmholtz decomposition*

$$L^p(w,\Omega;\mathbb{R}^n)=\mathrm{V}^p(w,\Omega;\mathbb{R}^n)\bigoplus \mathrm{G}^p(w,\Omega;\mathbb{R}^n), \tag{1.19}$$

where $\mathrm{V}^p(w,\Omega;\mathbb{R}^n)$ and $\mathrm{G}^p(w,\Omega;\mathbb{R}^n)$ denotes the *solenoidal* (divergence-free) and *irrotational* (curl-free) vector-valued spaces defined respectively by

$$\mathrm{V}^p(w,\Omega;\mathbb{R}^n):=\left\{\vec{v}\in L^p(w,\Omega;\mathbb{R}^n):\ \text{for any } \phi\in \dot{W}^{1,p}(w,\Omega),\ \langle\vec{v},\nabla\phi\rangle=0\right\},$$

$$\mathrm{G}^p(w,\Omega;\mathbb{R}^n) := \nabla\left(\dot{W}^{1,p}(w,\Omega)\right) = \left\{\nabla f:\ f\in\dot{W}^{1,p}(w,\Omega)\right\}$$

and $\dot{W}^{1,p}(w,\Omega)$ is the *weighted homogeneous Sobolev space*

$$\dot{W}^{1,p}(w,\Omega) := \left\{f\in L^1_{\mathrm{loc}}(\overline{\Omega}):\ \text{for any } j\in\{1,\dots,n\},\ \frac{\partial f}{\partial x_j}\in L^p(w,\Omega)\right\}.$$

Note that the decomposition (1.19) means that for any $\vec{u}\in L^p(w,\Omega;\mathbb{R}^n)$, there exists a unique decomposition $\vec{u}=\vec{v}+\vec{h}$ satisfying $\vec{v}\in\mathrm{V}^p(w,\Omega;\mathbb{R}^n)$ and $\vec{h}\in\mathrm{G}^p(w,\Omega;\mathbb{R}^n)$ such that

$$\|\vec{v}\|_{L^p(w,\Omega;\mathbb{R}^n)}+\|\vec{h}\|_{L^p(w,\Omega;\mathbb{R}^n)}\leqslant c_p\|\vec{u}\|_{L^p(w,\Omega;\mathbb{R}^n)}, \tag{1.20}$$

where the constant $c_p\in(0,\infty)$ depends only on n, p, α and Ω.

REMARK 1. (i) Assumption **(A1)** is applied to show that the class $C_c^\infty(\Omega;\mathbb{R}^m)$ of all smooth compactly supported vector-valued functions on Ω is dense in the weighted vector-valued Lebesgue space $L^p(\delta^a,\Omega;\mathbb{R}^m)$ for any $a\in\mathbb{R}$ and $m\in\mathbb{N}$ (see Lemma 1 below). Note that similar but stronger conditions are used to show the density of smooth functions in the weighted Sobolev space (see [19] and their references). It is easy to see that if Ω is a bounded open domain with $0\notin\Omega$, then $\delta(x)\equiv|x|$ satisfies **(A1)**. Also, if Ω is a bounded open domain, then $\delta(x)\equiv d(x,\partial\Omega)$ satisfies **(A1)**.

(ii) Let $p\in(1,\infty)$ and $w\in A_p(\mathbb{R}^n)$ be a Muckenhoupt weight (see [31] for it's definition). If Ω is bounded with C^1-boundary, then by [17, Theorem 5], we know that $L^p(w,\Omega;\mathbb{R}^n)$ admits the Helmholtz decomposition, namely, Assumption **(A2)** holds true. Recall that if Ω is only Lipschitz, the Helmholtz decomposition holds only for partial $p\in(1,\infty)$ (see [15, Theorem 11.1]). Moreover, if $p=2$, then the space $L^2(w,\Omega;\mathbb{R}^n)$ always admit the Helmholtz decomposition for any domain Ω and weight w, as $L^2(w,\Omega;\mathbb{R}^n)$ is a Hilbert space. In this case, the constant c_2 in (1.20) equals to 1 and (1.19) is an orthogonal decomposition.

The following theorem is the main result of this article.

THEOREM 1. *Let $\alpha\in\mathbb{R}$, $p\in(1,\infty)$ and $c\in(0,\infty)$. Assume that Ω is a bounded open domain in $\mathbb{R}^n$ with $n\in\mathbb{N}$ and δ a nonnegative measurable function in Ω. Then*

(i) (α,p,c)-**HR** *is equivalent to* (α,p,c)-**HV**;

(ii) *under Assumption* **(A1)**, (α,p,c)-**HV** *implies* (α,p',c)-**HD**;

(iii) *under Assumption* **(A2)**, (α,p',c)-**HD** *implies* (α,p,cc_p)-**HV**, *where the constant c_p is the same as in the Helmholtz decomposition* (1.20).

Theorem 1 gives characterizations of the weighted L^p Hardy-Rellich inequalities by some weighted vector-valued Hardy inequalities and weighted dual Hardy inequalities. Based on these characterizations, one may derive some new weighted Hardy-Rellich inequalities from the latter two classes of inequalities. This further implies some new weighted Rellich inequalities. Moreover, as the weighted vector-valued Hardy inequalities (1.16) belong to more general div-curl inequalities for vector fields (see [7, 24]), these characterizations indicate a possible connection between Hardy-Rellich inequalities and some problems in fluid mechanics. Indeed, the Helmholtz decomposition **(A2)**, that used throughout in this article, is a fundamental tool in the study of Navier-Stokes equations (see [18]).

The main part of the proof for Theorem 1 is to show the equivalence between the vector-valued inequality (1.16) and the scalar-valued dual inequality (1.18). To show this equivalence, we find a correspondence between the scalar-valued and vector-valued functions. The correspondence from scalar-valued function to the vector-valued one is easy, we use only the gradient operator. However, for the converse direction, we first need to find a subclass of $\mathrm{G}^p(\Omega;\mathbb{R}^n)$ whose element corresponds a bounded linear functional on the weighted vector-valued Lebesgue space $L^{p'}(\delta^{(\alpha+1)p'},\Omega;\mathbb{R}^n)$. The vector-valued inequality (1.16) restricted to this subclass is proved via the boundedness of the functional. Then, we extend the subclass to the whole space $\mathrm{G}^p(\Omega;\mathbb{R}^n)$ by using the Helmholtz decomposition. It is in this last step that the constant c_p comes out. Note that if $p=2$, then $c_p=1$.

As a simple application of Theorem 1, we derive some new weighted Hardy-Rellich inequalities. The following corollary gives some weighted L^p Hardy-Rellich inequalities associated with the distance $\delta(x)\equiv|x|$ for any $p\in(1,\infty)$ and all dimension $n>1$, which are variants of the weighted L^2 Hardy-Rellich inequalities for $n>4$ in [37, Theorem 1.7], where the domain Ω is bounded and contains the origin.

COROLLARY 1. *Let $\alpha\in\mathbb{R}$, $p\in(1,\infty)$ and Ω be a bounded open domain in $\mathbb{R}^n$ with $0\notin\Omega$ and $n>1$. Assume that $\delta(x)\equiv|x|$. Then*

(i) *for any $\alpha\neq-n/2$, $(\alpha,2,c)$-**HR** holds with $c\equiv2/(n+2\alpha)$, namely, for any $f\in C_c^\infty(\Omega)$,*

$$\int_\Omega \frac{|\nabla f(x)|^2}{|x|^{2(\alpha+1)}}\,dx \leqslant \left(\frac{2}{|n+2\alpha|}\right)^2 \int_\Omega \frac{|\Delta f(x)|^2}{|x|^{2\alpha}}\,dx; \tag{1.21}$$

(ii) *for any $\alpha\in(-\frac{n}{p'}-1,\frac{n}{p}-1)\setminus\{-\frac{n}{p'}\}$, (α,p,c)-**HR** holds with $c\equiv c_p p'/(|\alpha p'+n|)$, namely, for any $f\in C_c^\infty(\Omega)$,*

$$\int_\Omega \frac{|\nabla f(x)|^p}{|x|^{(\alpha+1)p}}\,dx \leqslant \left(\frac{c_p p'}{|n+\alpha p'|}\right)^p \int_\Omega \frac{|\Delta f(x)|^p}{|x|^{\alpha p}}\,dx. \tag{1.22}$$

Note that the constant in (1.21) is sharp, as the corresponding equivalent weighted Hardy inequality (1.5) that we used in the proof is sharp (see Section 2 for a detailed proof).

If $\delta(x)\equiv d(x):=\operatorname{dist}(x,K)$ is the distance to $\partial\Omega$ or some closed piecewise smooth surface K in $\mathbb{R}^n$, we obtain the following weighted L^p Hardy-Rellich inequalities which are variants of [6, Theorem 3(i)] by neglecting the remainder terms.

COROLLARY 2. *Let* $\alpha \in \mathbb{R}$, $p \in (1,\infty)$ *and* Ω *be a bounded open domain in* $\mathbb{R}^n$. *Assume that* K *is a compact piecewise smooth surface in* $\mathbb{R}^n$ *with codimension* $k \in \{1,\dots,n\}$ *and* $\delta(x) \equiv d(x) := \operatorname{dist}(x,K)$. *Then,*

(i) *if* K *satisfies: a)* $(k+\alpha p')(d\Delta d - k+1) \geqslant 0$ *in* $\Omega \setminus K$ *in the sense of distribution; b)* $\alpha \neq -k/p'$. *Then for any* $\alpha \in (-\frac{k}{p'}-1, \frac{k}{p}-1) \setminus \{-\frac{k}{p'}\}$, *the* (α,p,c)-**HR** *holds with* $c \equiv c_p p'/(|k+\alpha p'|)$, *namely, for any* $f \in C_c^\infty(\Omega \setminus K)$,

$$\int_\Omega \frac{|\nabla f(x)|^p}{d(x)^{(\alpha+1)p}}\,dx \leqslant \left(\frac{c_p p'}{|k+\alpha p'|}\right)^p \int_\Omega \frac{|\Delta f(x)|^p}{d(x)^{\alpha p}}\,dx, \tag{1.23}$$

where c_p *is as in* (1.20)*;*

(ii) *if* $K = \partial\Omega$ *and* Ω *is convex, then for any* $\alpha \in (\frac{1}{p}-2, \frac{1}{p}-1)$, *the* (α,p,c)-**HR** *holds with* $c \equiv c_p p'/(|1+\alpha p'|)$, *namely, for any* $f \in C_c^\infty(\Omega)$,

$$\int_\Omega \frac{|\nabla f(x)|^p}{d(x)^{(\alpha+1)p}}\,dx \leqslant \left(\frac{c_p p'}{|1+\alpha p'|}\right)^p \int_\Omega \frac{|\Delta f(x)|^p}{d(x)^{\alpha p}}\,dx, \tag{1.24}$$

where c_p *is as in* (1.20).

REMARK 2. We point out that if $p=2$, then Corollary 2(i) and (ii) hold respectively for all $\alpha \neq -\frac{k}{2}$ and $\alpha \in (-\infty, -\frac{1}{2})$. Indeed, note that the proof of Corollary 2 depends on the weighted dual Hardy inequality (1.18) and the Helmholtz decomposition of the space $L^p(d(x)^{-(\alpha+1)p}, \Omega; \mathbb{R}^n)$, where the latter requires the condition that $-k < -(\alpha+1)p < k(p-1)$ and hence

$$-\frac{k}{p'} - 1 < \alpha < \frac{k}{p} - 1. \tag{1.25}$$

However, if $p=2$ then by Remark 1(ii), we know that the Helmholtz decomposition holds for any weight function with $c_p \equiv 1$. This shows that we can remove the restriction (1.25) in this case. By checking the conditions on the validity of (1.18) in [6, 1] (see the proof of Corollary 2 for more details), we deduce that Corollary 2(i) holds for all $\alpha \neq -\frac{k}{2}$ and Corollary 2(ii) holds for all $\alpha \in (-\infty, -\frac{1}{2})$. Note that the constant in (1.24) in this case is sharp as the equivalent inequality (1.18) in [1] is sharp.

Based on the weighted Hardy-Rellich inequalities in Corollary 2, we are able to obtain the following weighted Rellich inequalities associated with the distance $\delta(x) \equiv d(x) := \operatorname{dist}(x, \partial\Omega)$, which are variants of the L^2 Rellich inequalities [35, Corollary 2.4] with $m=2$.

COROLLARY 3. *Let* $\alpha \in \mathbb{R}$, $p \in (1,\infty)$ *and* Ω *be a bounded convex domain in* $\mathbb{R}^n$. *Assume that* $\delta(x) \equiv d(x) := \operatorname{dist}(x, \partial\Omega)$. *Then,*

(i) *for any* $\alpha \in (\max\{\frac{1}{p}-2, \frac{1}{p'}-2\}, \frac{1}{p}-1)$ *and* $f \in C_c^\infty(\Omega)$,

$$\int_\Omega \frac{|f(x)|^p}{d(x)^{(\alpha+2)p}}\,dx \leqslant \left[\frac{c_p p^2}{|(p-1)+\alpha p||1-(\alpha+2)p|}\right]^p \int_\Omega \frac{|\Delta f(x)|^p}{d(x)^{\alpha p}}\,dx; \tag{1.26}$$

(ii) *if* $p=2$, *then* (1.26) *holds for all* $\alpha\in(-\frac{3}{2},-\frac{1}{2})$ *with* $c_p=1$.

Theorem 1 and Corollaries 1 through 3 are proved in Section 2.

The second part of this paper is to apply the aforementioned characterizations to derive some new Rellich inequalities with homogeneous weights, which is actually our initial motivation to study the Hardy-Rellich inequalities. Unlike the usual weights, the homogeneous weights considered here may admit singularities on the unit sphere $\mathbb{S}^{n-1}$ and are related to the potentials of some Schrödinger operators defined on $\mathbb{S}^{n-1}$ (see [12]). In this setting, Hoffmann-Ostenhof and Laptev [22, Theorem 1.1] first establish the L^2 Hardy inequality with the homogeneous weight. The following theorem gives a corresponding L^2 Rellich inequality.

THEOREM 2. *Let* $n\geqslant 5$, $q\in[\frac{n-1}{2},\infty)$ *and* $0\leqslant\Phi\in L^q(\mathbb{S}^{n-1})$. *Then, for any* $f\in C_c^\infty(\mathbb{R}^n)$, *it holds that*

$$\int_{\mathbb{R}^n}|\Delta f(x)|^2\,dx\geqslant\sigma\int_{\mathbb{R}^n}\frac{\Phi(x/|x|)|f(x)|^2}{|x|^4}\,dx, \tag{1.27}$$

where the constant

$$\sigma:=\left[\frac{n(n-4)}{4}\right]^2\frac{|\mathbb{S}^{n-1}|^{\frac{1}{q}}}{\|\Phi\|_{L^q(\mathbb{S}^{n-1})}}$$

is sharp in the sense that if $\Phi\equiv 1$, *then* (1.27) *takes the sharp form of the classical Rellich inequality.*

For general $p\in(1,\infty)$, we also have the following L^p version of Rellich inequality with homogeneous weight.

THEOREM 3. *Let* $n\geqslant 5$, $p\in(1,\infty)$ *and* $0\leqslant\Phi\in L^{n/(2p)}(\mathbb{S}^{n-1})$. *Then, for any* $f\in C_c^\infty(\mathbb{R}^n)$, *it holds that*

$$\int_{\mathbb{R}^n}|\Delta f(x)|^p\,dx\geqslant\sigma\int_{\mathbb{R}^n}\frac{\Phi(x/|x|)|f(x)|^p}{|x|^{2p}}\,dx, \tag{1.28}$$

where the constant

$$\sigma:=\left[\frac{n(p-1)|n-2p|}{c_p p^2}\right]^p\frac{|\mathbb{S}^{n-1}|^{\frac{2p}{n}}}{\|\Phi\|_{L^{n/(2p)}(\mathbb{S}^{n-1})}}$$

with c_p *as in* (1.20).

Theorems 2 and 3 are proved in Section 3. To prove these two theorems, we establish two weighted Hardy inequalities with homogeneous weights which may also of independent interests (see Lemmas 3 and 4).

Finally, we make some conventions on notation. Let $\mathbb{N}:=\{1,2,\ldots\}$, $\mathbb{Z}_+:=\mathbb{N}\cup\{0\}$ and $\mathbb{Z}:=-\mathbb{N}\cup\{0\}\cup\mathbb{N}$. For any set $E\subset\mathbb{R}^n$, χ_E denotes its *characteristic function*. We use C to denote a *positive constant* that is independent of the main parameters involved, whose value may differ from line to line. Constants with subscripts, such as C_1, do not change in different occurrences. If $f \leqslant Cg$, we also write $f \lesssim g$ and, if $f \lesssim g \lesssim f$, we then write $f \sim g$. For any $s \in \mathbb{R}$, let $\lfloor s \rfloor$ (resp. $\lceil s \rceil$) be the largest integer not greater than s (resp. the smallest integer not smaller than s).

2. Proofs of Theorem 1 and Corollaries 1 through 3

This section is devoted to the proofs of Theorem 1 and Corollaries 1 through 3. We first prove Theorem 1. To this end, we need some basic properties of the weighted vector-valued Lebesgue space. Recall that for any $a \in \mathbb{R}$, $m \in \mathbb{N}$, Ω being a bounded open domain in $\mathbb{R}^n$ and δ a nonnegative measurable function on Ω, the *weighted vector-valued Lebesgue space* $L^p(\delta^a, \Omega; \mathbb{R}^m)$ is defined to be

$$L^p(\delta^a, \Omega; \mathbb{R}^m) := \left\{ \vec{u} : \Omega \to \mathbb{R}^m \text{ is measurable } : \ \|\vec{u}\|_{L^p(\delta^a, \Omega; \mathbb{R}^m)} < \infty \right\},$$

where

$$\|\vec{u}\|_{L^p(\delta^a, \Omega; \mathbb{R}^m)} := \left\{ \int_\Omega |\vec{u}(x)|^p \, \delta(x)^a \, dx \right\}^{1/p}.$$

The following lemma gives some classical results on the density and duality of the weighted vector-valued Lebesgue space. Here we provide a sketch of the proof, since we can't find an exact reference.

LEMMA 1. *Let $a \in \mathbb{R}$, $m \in \mathbb{N}$ and Ω be a bounded open domain in $\mathbb{R}^n$. Assume that Assumption* **(A1)** *holds. Then for any $p \in (1, \infty)$,*

(i) *$C_c^\infty(\Omega; \mathbb{R}^m)$ is dense in $L^p(\delta^a, \Omega; \mathbb{R}^m)$;*

(ii) *$[L^p(\delta^{ap}, \Omega; \mathbb{R}^m)]^* = L^{p'}(\delta^{-ap'}, \Omega; \mathbb{R}^m)$ with $p' = p/(p-1)$. In particular, for any $\vec{u} \in L^{p'}(\delta^{-ap'}, \Omega; \mathbb{R}^m)$, its action $\langle \vec{u}, \vec{v} \rangle$ on every $\vec{v} \in L^p(\delta^{ap}, \Omega; \mathbb{R}^m)$ is as follows*

$$\langle \vec{u}, \vec{v} \rangle = \int_\Omega \vec{u}(x) \cdot \vec{v}(x) \, dx.$$

Proof. To prove (i), by Assumption **(A1)** and a straightforward calculation, we see $C_c^\infty(\Omega; \mathbb{R}^m)$ is in $L^p(\delta^a, \Omega; \mathbb{R}^m)$ for any $a \in \mathbb{R}$. To show the density of $C_c^\infty(\Omega; \mathbb{R}^m)$ in $L^p(\delta^a, \Omega; \mathbb{R}^m)$, for any $\vec{u} \in L^p(\delta^a, \Omega; \mathbb{R}^m)$ and any $k \in \mathbb{N}$, let $\Omega_k := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geqslant 1/k\}$ be a sequence of bounded closed subsets that exhaust Ω as $k \to \infty$. It is easy to see $\chi_{\Omega_k} \vec{u}$ has a compact support and $\lim_{k\to\infty} \chi_{\Omega_k} \vec{u} = \vec{u}$ in $L^p(\delta^a, \Omega; \mathbb{R}^m)$.

Now for each $k \in \mathbb{N}$ and any $\varepsilon \in (0, \infty)$, take $\vec{\varphi} \in C_c^\infty(\Omega; \mathbb{R}^m)$ such that $\|\vec{\varphi} - \chi_{\Omega_k} \vec{u}\|_{L^p(\Omega; \mathbb{R}^m)} < \frac{\varepsilon}{M}$, where $M := \max\{M_\delta^{a/p}, m_\delta^{a/p}\}$ with M_δ and m_δ being respectively the maximal and minimal values of δ in Ω_k. As Ω_k is compact and Assumption **(A1)**, we know that $M \in (0, \infty)$. This combined with an elementary calculation shows that $\|\vec{\varphi}-\chi_{\Omega_k}\vec{u}\|_{L^p(\delta^a,\Omega;\mathbb{R}^m)} \leqslant M\|\vec{\varphi}-\chi_{\Omega_k}\vec{u}\|_{L^p(\Omega;\mathbb{R}^m)} < \varepsilon$, which shows that $\chi_{\Omega_k}\vec{u}$ can be approximated arbitrarily by functions in $C_c^\infty(\Omega;\mathbb{R}^m)$ and hence verifies (i).

We now prove (ii). The inclusion $L^{p'}(\delta^{-ap'},\Omega;\mathbb{R}^m) \subset [L^p(\delta^{ap},\Omega;\mathbb{R}^m)]^*$ follows from an easy application of Hölder's inequality. To prove the converse inclusion, let $I_b(\vec{u})(x) := [\delta(x)]^b\vec{u}(x)$ be a multiplication operator on the vector-valued function $\vec{u}$ on Ω with $b\in\mathbb{R}$. From its definition, it follows that I_b is an isometry from $L^p(\delta^{bp+c},\Omega;\mathbb{R}^m)$ to $L^p(\delta^c,\Omega;\mathbb{R}^m)$ for any $c\in\mathbb{R}$. Now for any $L\in[L^p(\delta^{ap},\Omega;\mathbb{R}^m)]^*$, we induce a new linear functional L_a on $L^p(\Omega;\mathbb{R}^m)$ by setting for any $\vec{u}\in L^p(\Omega;\mathbb{R}^m)$, $L_a(\vec{u}) := L(I_{-a}(\vec{u}))$. Using Riesz's representation theorem for $L^p(\Omega;\mathbb{R}^m)$ and the isometricity of I_b, we know that there exists $\vec{v}\in L^{p'}(\delta^{-ap'},\Omega;\mathbb{R}^m)$ corresponding to L. This proves the converse inclusion and hence finishes the proof of (ii).

With the help of Lemma 1, we now turn to the proof of Theorem 1.

Proof of Theorem 1. We first show (i), namely, the equivalence between (α,p,c)-**HR** and (α,p,c)-**HV**. Indeed, if (α,p,c)-**HR** holds true, then for any $\vec{u}\in G(\Omega;\mathbb{R}^n)$, by the definition of $G(\Omega;\mathbb{R}^n)$ in (1.17), we know that there exists $f\in C_c^\infty(\Omega)$ such that $\vec{u}=\nabla f$. This, combined with (α,p,c)-**HR** (1.15), shows that

$$\int_\Omega \frac{|\operatorname{div}\vec{u}(x)|^p}{\delta(x)^{\alpha p}}\,dx = \int_\Omega \frac{|\Delta f(x)|^p}{\delta(x)^{\alpha p}}\,dx \geqslant c^{-p}\int_\Omega \frac{|\nabla f(x)|^p}{\delta(x)^{(\alpha+1)p}}\,dx = c^{-p}\int_\Omega \frac{|\vec{u}(x)|^p}{\delta(x)^{(\alpha+1)p}}\,dx,$$

which implies that (α,p,c)-**HV** holds true.

On the other hand, if (α,p,c)-**HV** holds true, then for any $f\in C_c^\infty(\Omega)$, since $\nabla f\in G(\Omega;\mathbb{R}^n)$ and $\Delta f=\operatorname{div}(\nabla f)$, it follows immediately (α,p,c)-**HR** holds true and hence (i) holds true.

We now prove (ii), that is, under Assumption **(A1)**, (α,p,c)-**HV** implies (α,p',c)-**HD**. Indeed, assume that (α,p,c)-**HV** holds true. Then for any $g\in C_c^\infty(\Omega)$, let φ be the unique function in $C_c^\infty(\Omega)$ satisfying $\Delta\varphi\equiv g$ and set $\vec{v}:=\nabla\varphi$. By (1.17), it is easy to see $\vec{v}\in \mathrm{G}(\Omega;\mathbb{R}^n)$ and $\operatorname{div}\vec{v}=g$, which combined with Hölder's inequality and (α,p,c)-**HV** (1.16) implies that for any $f\in C_c^\infty(\Omega)$,

$$\begin{aligned}\left|\int_\Omega f(x)\,g(x)\,dx\right| &= \left|\int_\Omega f(x)\operatorname{div}\vec{v}(x)\,dx\right| = \left|\int_\Omega \nabla f(x)\cdot\vec{v}(x)\,dx\right|\\ &\leqslant \left\|\delta^{-(\alpha+1)}\vec{v}\right\|_{L^p(\Omega;\mathbb{R}^n)}\left\|\delta^{\alpha+1}\nabla f\right\|_{L^{p'}(\Omega;\mathbb{R}^n)}\\ &\leqslant c\left\|\delta^{-\alpha}\operatorname{div}\vec{v}\right\|_{L^p(\Omega;\mathbb{R}^n)}\left\|\delta^{\alpha+1}\nabla f\right\|_{L^{p'}(\Omega;\mathbb{R}^n)}\\ &= c\,\|g\|_{L^p(\delta^{-\alpha p},\Omega)}\left\|\delta^{\alpha+1}\nabla f\right\|_{L^{p'}(\Omega;\mathbb{R}^n)},\end{aligned}$$

which together with the arbitrariness of g in $C_c^\infty(\Omega)$ and Lemma 1(i) implies that for any $f\in C_c^\infty(\Omega)$,

$$\|f\|_{L^{p'}(\delta^{\alpha p},\Omega)} = \sup_{\substack{g\in C_c^\infty(\Omega),\\ \|g\|_{L^p(\delta^{-\alpha p},\Omega)}\leqslant 1}}\left|\int_\Omega f(x)\,g(x)\,dx\right| \leqslant c\left\|\delta^{\alpha+1}\nabla f\right\|_{L^{p'}(\Omega;\mathbb{R}^n)}.$$

This shows the validity of (α,p',c)-**HD** and hence (ii) holds true.

Finally, we prove (iii), namely, (α, p', c)-**HD** implies (α, p, cc_p)-**HV** under Assumptions **(A2)**. Assume that (α, p', c)-**HD** holds true. Then for any $f \in C_c^\infty(\Omega)$, we define a linear functional L_f on $\mathrm{G}(\Omega;\mathbb{R}^n)$, induced by f, such that for any $\vec{v} \in \mathrm{G}(\Omega;\mathbb{R}^n)$,

$$L_f(\vec{v}) := \int_\Omega f(x)\, g_{\vec{v}}(x)\, dx,$$

where $g_{\vec{v}} \in C_c^\infty(\Omega)$ satisfying $\nabla g_{\vec{v}} = \vec{v}$. Observe that in (1.17), if there exists another $\widetilde{g}_{\vec{v}} \in C_c^\infty(\Omega)$ satisfying $\nabla \widetilde{g}_{\vec{v}} = \vec{v}$, then $\nabla(g_{\vec{v}} - \widetilde{g}_{\vec{v}}) \equiv 0$ in $\mathbb{R}^n$. This combined with the condition $g_{\vec{v}} - \widetilde{g}_{\vec{v}} \in C_c^\infty(\Omega)$ shows that $\widetilde{g}_{\vec{v}} \equiv g_{\vec{v}}$ and hence $L_f(\vec{v})$ is well-defined. Moreover, by Hölder's inequality and (α, p', c)-**HD** (1.18), we find

$$\begin{aligned} \left|L_f(\vec{v})\right| &\leqslant \|\delta^{-\alpha} f\|_{L^p(\Omega)} \|\delta^{\alpha} g_{\vec{v}}\|_{L^{p'}(\Omega)} \leqslant c\|\delta^{-\alpha} f\|_{L^p(\Omega)} \|\delta^{\alpha+1} \nabla g_{\vec{v}}\|_{L^{p'}(\Omega;\mathbb{R}^n)} \\ &= c\|\delta^{-\alpha} f\|_{L^p(\Omega)} \|\vec{v}\|_{L^{p'}(\delta^{(\alpha+1)p'},\Omega;\mathbb{R}^n)}, \end{aligned} \tag{2.1}$$

which implies that L_f is a linear functional on $G(\Omega;\mathbb{R}^n) \subset L^{p'}(\delta^{(\alpha+1)p'},\Omega;\mathbb{R}^n)$ and can be extended to a bounded linear functional on $L^{p'}(\delta^{(\alpha+1)p'},\Omega;\mathbb{R}^n)$. Thus, by Lemma 1(ii), we see that there exists a unique $\vec{u} \in L^p(\delta^{-(\alpha+1)p},\Omega;\mathbb{R}^n)$ such that for any $\vec{v} \in G(\Omega;\mathbb{R}^n) \subset C_c^\infty(\Omega;\mathbb{R}^n) \subset L^{p'}(\delta^{(\alpha+1)p'},\Omega;\mathbb{R}^n)$,

$$\int_\Omega f(x) g_{\vec{v}}(x)\, dx = L_f(\vec{v}) = \int_\Omega \vec{u}(x)\cdot\vec{v}(x)\, dx = \int_\Omega \vec{u}(x)\cdot\nabla g_{\vec{v}}(x)\, dx \tag{2.2}$$

with

$$\|\vec{u}\|_{L^p(\delta^{-(\alpha+1)p},\Omega;\mathbb{R}^n)} = \|L_f\| \leqslant c\|\delta^{-\alpha} f\|_{L^p(\Omega)}. \tag{2.3}$$

Using the fact $G(\Omega;\mathbb{R}^n) = \nabla(C_c^\infty(\Omega))$ and the arbitrariness of $\vec{v} \in G(\Omega;\mathbb{R}^n)$ in (2.1) and (2.2), we know that (2.2) holds true for all $g_{\vec{v}} \in C_c^\infty(\Omega)$. This together with the fact $f \in C_c^\infty(\Omega)$ shows that $-\mathrm{div}\,\vec{u} = f$. By this and (2.3), we conclude that for any $f \in C_c^\infty(\Omega)$, there exists a unique $\vec{u} \in C^\infty(\Omega;\mathbb{R}^n) \cap L^p(\delta^{-(\alpha+1)p},\Omega;\mathbb{R}^n)$ satisfying $-\mathrm{div}\,\vec{u} = f$ such that

$$\|\delta^{-(\alpha+1)}\vec{u}\|_{L^p(\Omega;\mathbb{R}^n)} \leqslant c\|\delta^{-\alpha}\mathrm{div}\,\vec{u}\|_{L^p(\Omega)}. \tag{2.4}$$

We now want to extend (2.4) from $\vec{u} \in C^\infty(\Omega;\mathbb{R}^n) \cap L^p(\delta^{-(\alpha+1)p},\Omega;\mathbb{R}^n)$ to all $\vec{w} \in G(\Omega;\mathbb{R}^n)$ via Helmholtz decomposition. Indeed, for any $\vec{w} \in G(\Omega;\mathbb{R}^n)$, let $\varphi_{\vec{w}} \in C_c^\infty(\Omega)$ satisfy $\vec{w} = \nabla\varphi_{\vec{w}}$ and set $f := -\Delta\varphi_{\vec{w}}$. It is easy to see that $f \in C_c^\infty(\Omega)$ and $-\mathrm{div}\,\vec{w} = f$. Then by applying the same argument in the proof of (2.4), we know that there exists a unique $\vec{u} \in C^\infty(\Omega;\mathbb{R}^n) \cap L^p(\delta^{-(\alpha+1)p},\Omega;\mathbb{R}^n)$ satisfying $-\mathrm{div}\,\vec{u} = f$ such that (2.4) holds true. On the other hand, for any $\vec{v} \in G(\Omega;\mathbb{R}^n)$, let $\varphi_{\vec{v}} \in C_c^\infty(\Omega)$ satisfy $\vec{v} = \nabla\varphi_{\vec{v}}$. We write

$$\int_\Omega -\mathrm{div}\,\vec{u}(x)\varphi_{\vec{v}}(x)\, dx = L_f(\vec{v}) = \int_\Omega f(x)\varphi_{\vec{v}}(x)\, dx = \int_\Omega -\mathrm{div}\,\vec{w}(x)\varphi_{\vec{v}}(x)\, dx,$$

which combined with the arbitrariness of $\varphi_{\vec{v}} \in C_c^\infty(\Omega)$ implies that

$$\operatorname{div}(\vec{u}-\vec{w}) \equiv 0. \tag{2.5}$$

Thus,

$$\vec{u} = (\vec{u}-\vec{w}) + \vec{w}$$

is a Helmholtz decomposition of $\vec{u} \in C^\infty(\Omega;\mathbb{R}^n) \cap L^p(\delta^{-(\alpha+1)p},\Omega;\mathbb{R}^n)$ as in (1.19). By Assumption **(A2)** and (1.20), we know that

$$\|\vec{w}\|_{L^p(\delta^{-(\alpha+1)p},\Omega;\mathbb{R}^n)} \leqslant c_p \|\vec{u}\|_{L^p(\delta^{-(\alpha+1)p},\Omega;\mathbb{R}^n)}.$$

By this, (2.4) and (2.5), we know that (α,p,cc_p)-**HV** holds true for all $\vec{w} \in G(\Omega;\mathbb{R}^n)$. This shows that (α,p',c)-**HD** implies (α,p,cc_p)-**HV** and hence finish the proof of Theorem 1.

Next we turn to the proof of Corollary 1.

Proof of Corollary 1. Without loss of generality, we only prove (ii), as (i) is a special case of (ii) with $p=2$. Note also that in the latter case, the Helmholtz decomposition holds for any weighted vector-valued Lebesgue space $L^2(|x|^\beta,\Omega;\mathbb{R}^n)$ with $\beta \in \mathbb{R}$.

To prove (ii), let $p \in (1,\infty)$, $\alpha \in (-n/p'-1, n/p-1)$ and $w(x) := |x|^{-(\alpha+1)p}$. We first claim that in this case Assumptions **(A1)** and **(A2)** hold true. Indeed, since $0 \notin \Omega$ it is easy to see that $|x|$ is continuous and positive in Ω. Thus, to show the claim, it remains to show that the space $L^p(|x|^{-(\alpha+1)p},\Omega;\mathbb{R}^n)$ admits the Helmholtz decomposition. Indeed, by [16, Lemma 2.3], we know that for any $p \in (1,\infty)$ and $\beta \in \mathbb{R}$, $|x|^\beta \in A_p(\mathbb{R}^n)$ if and only $\beta \in (-n, n(p-1))$. This combined with Remark 1(ii) shows that $L^p(|x|^\beta,\Omega;\mathbb{R}^n)$ admits the Helmholtz decomposition (1.19). Thus, by letting $\beta = -(\alpha+1)p$, we know that the claim holds true.

On the other hand, using the weighted n-dimensional Hardy inequality (1.5), we know that for any $\alpha \in \mathbb{R}$, $\alpha \neq -n/p'$, $p \in (1,\infty)$ and $f \in C_c^\infty(\Omega)$,

$$\int_\Omega |f(x)|^{p'} |x|^{\alpha p'}\,dx \leqslant \left(\frac{p'}{|n+\alpha p'|}\right)^{p'} \int_\Omega |\nabla f(x)|^{p'} |x|^{(\alpha+1)p'}\,dx,$$

namely, the weighted dual Hardy inequality (α,p',c)-**HD** holds true with $c = \frac{p'}{|n+\alpha p'|}$. This combined with Theorem 1(iii) shows that (α,p,cc_p)-**HV** holds true. By Theorem 1(i), we conclude that the weighted Hardy-Rellich inequality (α,p,cc_p)-**HR** holds true. This implies (1.22) and hence finishes the proof of Corollary 1.

Using the same idea for the proof of Corollary 1, we now prove Corollary 2.

Proof of Corollary 2. Following the proof of Corollary 1, we only need to establish two facts: 1) $d(x)^{-(\alpha+1)p} := [\operatorname{dist}(x,K)]^{-(\alpha+1)p} \in A_p(\mathbb{R}^n)$ for any $\alpha \in (-\frac{k}{p'}-1, \frac{k}{p}-1)$; 2) under the assumptions of a) and b) in Corollary 2, the weighted dual Hardy inequality holds.

The first fact follows from that $[\operatorname{dist}(x,K)]^{\beta} \in A_p(\mathbb{R}^n)$ if and only if $\beta \in (-k, k(p-1))$ (see [16, Lemma 2.2]).

For the later fact, in the case (i), by [6, Theorem 1], we have that under the assumptions of a) and b), it holds that for any $f \in C_c^{\infty}(\Omega \setminus K)$,

$$\int_{\Omega} |f(x)|^{p'} d(x)^{\alpha p'}\, dx \leqslant \left(\frac{p'}{|k+\alpha p'|}\right)^{p'} \int_{\Omega} |\nabla f(x)|^{p'} d(x)^{(\alpha+1)p'}\, dx.$$

In the case (ii) when $K = \partial\Omega$, by (1.8) with $\varepsilon = (1+\alpha)p'$, we obtain that for any $p' \in (1,\infty)$, $\alpha \in (-\infty, -\frac{1}{p'})$ and $f \in C_c^{\infty}(\Omega)$,

$$\int_{\Omega} |f(x)|^{p'} d(x)^{\alpha p'}\, dx \leqslant \left(\frac{p'}{|1+\alpha p'|}\right)^{p'} \int_{\Omega} |\nabla f(x)|^{p'} d(x)^{(\alpha+1)p'}\, dx. \tag{2.6}$$

Combining the above two inequalities, we conclude that (1.23) and (1.24) hold true, which completes the proof of Corollary 2.

Proof of Corollary 3. By Corollary 2, we know that the weighted Hardy-Rellich inequality (1.24) holds true. This combined with the weighted Hardy inequality (2.6) (with p' and α therein respectively replaced by p and $-\alpha-2$) finishes the proof of Corollary 3.

3. Proofs of Theorems 2 and 3

In this section, we prove Theorems 2 and 3, which establish the L^p Rellich inequalities with homogeneous weights for any $p \in (1,\infty)$. To this end, we need some technical lemmas. The following Lemma 2 was established by Dolbeault et al. [12], which gives the sharp estimates for the first negative eigenvalue of a Schrödinger operator with negative integrable potential on the sphere $\mathbb{S}^{n-1}$.

LEMMA 2. ([12]) *Let $n \geqslant 2$ and $q \in (\max\{1, \frac{n-1}{2}\}, \infty)$. There exists an increasing function $\alpha : \mathbb{R}^+ := [0,\infty) \to \mathbb{R}^+$ satisfying the following two conditions:*

(i) *for any $\mu \in [0, \frac{1}{2}(n-1)(q-1)]$, $\alpha(\mu) = \mu$;*

(ii) *α is convex in $(\frac{1}{2}(n-1)(q-1), \infty)$*

such that for any $0 \leqslant V \in L^q(\mathbb{S}^{n-1})$,

$$|\lambda_1(-\Delta_\theta - V)| \leqslant \alpha\left(\frac{\|V\|_{L^q(\mathbb{S}^{n-1})}}{|\mathbb{S}^{n-1}|^{\frac{1}{q}}}\right), \tag{3.1}$$

where $\lambda_1(-\Delta_\theta - V)$ denotes the first eigenvalue of the Schrödinger operator $-\Delta_\theta - V$ on the unit sphere $\mathbb{S}^{n-1}$.

Moreover, if $n \geqslant 4$ and $q = \frac{n-1}{2}$, then (3.1) is also satisfied with $\alpha(\mu) = \mu$ for $\mu \in [0, (n-1)(n-3)/2]$.

The following lemma gives the weighted Hardy inequalities with homogeneous weights, which generalizes the corresponding Hardy inequalities with homogeneous weights in [22, Theorem 1.1].

LEMMA 3. *Let $n \geqslant 3$ and $q \in (\frac{n-1}{2}, \infty)$. Assume that $\beta \in [0, \infty)$ satisfies $\beta + 2 \neq n$ and $0 \leqslant \Phi \in L^q(\mathbb{S}^{n-1})$.*

(i) *If $n \geqslant 2 + \frac{\beta}{2} + \frac{1}{2\beta}$, then for any $f \in C_c^\infty(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} \frac{|\nabla f(x)|^2}{|x|^\beta} dx \geqslant \tau \int_{\mathbb{R}^n} \frac{\Phi(x/|x|)|f(x)|^2}{|x|^{\beta+2}} dx, \tag{3.2}$$

where

$$\tau := \left(\frac{n-\beta-2}{2}\right)^2 \frac{|\mathbb{S}^{n-1}|^{\frac{1}{q}}}{\|\Phi\|_{L^q(\mathbb{S}^{n-1})}}. \tag{3.3}$$

(ii) *If $n < 2 + \frac{\beta}{2} + \frac{1}{2\beta}$ and $p \in [\frac{(n-\beta-2)^2}{2(n-1)} + 1, \infty)$, then for any $f \in C_c^\infty(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} \frac{|\nabla f(x)|^2}{|x|^\beta} dx \geqslant \tau \int_{\mathbb{R}^n} \frac{\Phi(x/|x|)|f(x)|^2}{|x|^{\beta+2}} dx, \tag{3.4}$$

where τ is as in (3.3).

(iii) *If $n < 2 + \frac{\beta}{2} + \frac{1}{2\beta}$ and $q \in (\frac{n-1}{2}, \frac{(n-\beta-2)^2}{2(n-1)} + 1)$, then for any $f \in C_c^\infty(\mathbb{R}^n)$,*

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|\nabla f(x)|^2}{|x|^\beta} dx \; &\geqslant (1-\nu_0)\left(\frac{n-\beta-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|^{\beta+2}} dx \\ &\quad + \widetilde{\tau} \int_{\mathbb{R}^n} \frac{\Phi(x/|x|)|f(x)|^2}{|x|^{\beta+2}} dx,\end{aligned} \tag{3.5}$$

where $\nu_0 := \frac{2(n-1)(q-1)}{(n-\beta-2)^2}$ and

$$\widetilde{\tau} := \nu_0 \left(\frac{n-\beta-2}{2}\right)^2 \frac{|\mathbb{S}^{n-1}|^{\frac{1}{q}}}{\|\Phi\|_{L^q(\mathbb{S}^{n-1})}} = \frac{(n-1)(q-1)}{2} \frac{|\mathbb{S}^{n-1}|^{\frac{1}{q}}}{\|\Phi\|_{L^q(\mathbb{S}^{n-1})}}. \tag{3.6}$$

Proof. We first prove (i). Observe that using the polar coordinates in $\mathbb{R}^n$, we have $|\nabla f|^2 = |\partial_r f|^2 + \frac{1}{r^2}|\nabla_\theta f|^2$, where $r := |x|$ and ∇_θ denotes the first-order Beltrami operator on the sphere $\mathbb{S}^{n-1}$. From this, we deduce

$$\int_{\mathbb{R}^n} \frac{|\nabla f|^2}{|x|^\beta} dx \; = \int_0^\infty \left[\int_{\mathbb{S}^{n-1}} \left(\frac{|\partial_r f|^2}{r^\beta} + \frac{|\nabla_\theta f|^2}{r^{\beta+2}}\right) r^{n-1} d\theta\right] dr =: \mathrm{A} + \mathrm{B}. \tag{3.7}$$

For A, using the one-dimensional weighted Hardy inequality (1.2) with $p=2$, we know that for any $\gamma\in\mathbb{R}$ with $\gamma\neq 1$ and $g\in C_c^\infty(0,\infty)$,

$$\int_0^\infty |g'(t)|^2 t^\gamma\,dt \geqslant \left[\frac{\gamma-1}{2}\right]^2 \int_0^\infty |g(t)|^2 t^{\gamma-2}\,dt. \tag{3.8}$$

By letting $g\equiv f$ and $\gamma=n-\beta-1$ in (3.8), we immediately conclude that

$$\begin{aligned} \mathrm{A} &\geqslant \left(\frac{n-\beta-2}{2}\right)^2 \int_{\mathbb{S}^{n-1}}\left[\int_0^\infty |f|^2 r^{n-\beta-3}\,dr\right]d\theta \\ &= \left(\frac{n-\beta-2}{2}\right)^2 \int_{\mathbb{R}^n}\frac{|f(x)|^2}{|x|^{\beta+2}}\,dx. \end{aligned} \tag{3.9}$$

To estimate B, from the fact that $-\Delta_\theta=\nabla_\theta\cdot\nabla_\theta$, it follows that

$$\begin{aligned} \mathrm{B} &= \int_0^\infty \frac{1}{r^{\beta-n+3}}\left[\int_{\mathbb{S}^{n-1}} -\Delta_\theta(f)\,f\,d\theta\right]dr \\ &= \int_0^\infty \frac{1}{r^{\beta-n+3}}\left[\int_{\mathbb{S}^{n-1}} (-\Delta_\theta-\tau\Phi)(f)\,f\,d\theta\right]dr \\ &\quad + \int_0^\infty \frac{1}{r^{\beta-n+3}}\left[\int_{\mathbb{S}^{n-1}} \tau\Phi f^2\,d\theta\right]dr =: \mathrm{B}_1+\mathrm{B}_2, \end{aligned} \tag{3.10}$$

where τ is as in (3.3). By an elementary calculation, we see that

$$\mathrm{B}_2=\tau\int_{\mathbb{R}^n}\frac{\Phi(x/|x|)|f(x)|^2}{|x|^{\beta+2}}\,dx. \tag{3.11}$$

To estimate B_1, using Lemma 2 and the fact that $\lambda_1(-\Delta_\theta-\tau\Phi)\leqslant 0$ (this follows from the assumption that $\Phi\geqslant 0$), we know that

$$\begin{aligned} \mathrm{B}_1 &\geqslant \int_0^\infty \frac{1}{r^{\beta-n+3}}\lambda_1(-\Delta_\theta-\tau\Phi)\left[\int_{\mathbb{S}^{n-1}}|f|^2\,d\theta\right]dr \\ &= \lambda_1(-\Delta_\theta-\tau\Phi)\int_{\mathbb{R}^n}\frac{|f(x)|^2}{|x|^{\beta+2}}\,dx \\ &\geqslant -\alpha\left(\frac{\|\tau\Phi\|_{L^q(\mathbb{S}^{n-1})}}{|\mathbb{S}^{n-1}|^{\frac{1}{q}}}\right)\int_{\mathbb{R}^n}\frac{|f(x)|^2}{|x|^{\beta+2}}\,dx. \end{aligned} \tag{3.12}$$

Moreover, by the assumption that $n\geqslant 2+\frac{\beta}{2}+\frac{1}{2\beta}$ and an elementary calculation, we see that

$$\frac{(n-\beta-2)^2}{2(n-1)}+1\leqslant\frac{n-1}{2}=\max\left\{1,\frac{n-1}{2}\right\}$$

and hence when $q\in(\max\{1,\frac{n-1}{2}\},\infty)$, we have

$$\left(\frac{n-\beta-2}{2}\right)^2\leqslant\frac{1}{2}(n-1)(q-1), \tag{3.13}$$

which together with the assumption that $\tau = (\frac{n-\beta-2}{2})^2|\mathbb{S}^{n-1}|^{\frac{1}{q}}/\|\Phi\|_{L^q(\mathbb{S}^{n-1})}$, (3.12) and Lemma 2(i) implies that

$$\mathrm{B}_1 \geqslant -\alpha\left(\left[\frac{n-\beta-2}{2}\right]^2\right)\int_{\mathbb{R}^n}\frac{|f(x)|^2}{|x|^{\beta+2}}\,dx = -\left(\frac{n-\beta-2}{2}\right)^2\int_{\mathbb{R}^n}\frac{|f(x)|^2}{|x|^{\beta+2}}\,dx. \quad (3.14)$$

Thus, combined (3.7), (3.9) through (3.12) and (3.14), we conclude that (3.2) holds true, which proves (i).

To prove (ii), we follow the same argument used to prove (i). It is easy to see that the estimates (3.7), (3.9) and (3.10) through (3.12) are still valid in this case, as the proofs of these estimates don't need the assumption $n \geqslant 2+\beta/2+1/(2\beta)$ and holds for any $n \geqslant 3$. Thus, to finish the proof of (3.4), it suffices to reprove (3.14) in this case. However, from the assumption $q \in [\frac{(n-\beta-2)^2}{2(n-1)}+1, \infty)$, it follows immediately that (3.13) still holds true in this case, which shows that (3.14) is valid. Thus, (ii) holds true.

We now prove (iii). In this case, by the assumption that $q \in (\frac{n-1}{2}, \frac{(n-\beta-2)^2}{2(n-1)}+1)$, we know

$$\left(\frac{n-\beta-2}{2}\right)^2 > \frac{1}{2}(n-1)(q-1).$$

Moreover let $\nu_0 := \frac{2(n-1)(p-1)}{(n-\beta-2)^2} \in (0,1)$. We then have

$$\nu_0\left(\frac{n-\beta-2}{2}\right)^2 = \frac{1}{2}(n-1)(q-1),$$

which together with Lemma 2(i) implies

$$\alpha\left(\nu_0\left[\frac{n-\beta-2}{2}\right]^2\right) = \nu_0\left[\frac{n-\beta-2}{2}\right]^2. \quad (3.15)$$

Now, by following the same arguments used in the proof of (i) with τ therein replaced by $\widetilde{\tau}$ in (3.6) (namely, using (3.7) and (3.9) through (3.12)), and (3.15), we conclude that

$$\begin{aligned}\int_{\mathbb{R}^n}\frac{|\nabla f(x)|^2}{|x|^\beta}\,dx &\geqslant \left(\frac{n-\beta-2}{2}\right)^2\int_{\mathbb{R}^n}\frac{|f(x)|^2}{|x|^{\beta+2}}\,dx + \widetilde{\tau}\int_{\mathbb{R}^n}\frac{\Phi(x/|x|)|f(x)|^2}{|x|^{\beta+2}}\,dx \\ &\quad -\alpha\left(\frac{\|\widetilde{\tau}\Phi\|_{L^q(\mathbb{S}^{n-1})}}{|\mathbb{S}^{n-1}|^{\frac{1}{q}}}\right)\int_{\mathbb{R}^n}\frac{|f(x)|^2}{|x|^{\beta+2}}\,dx \\ &\geqslant (1-\nu_0)\left(\frac{n-\beta-2}{2}\right)^2\int_{\mathbb{R}^n}\frac{|f(x)|^2}{|x|^{\beta+2}}\,dx + \widetilde{\tau}\int_{\mathbb{R}^n}\frac{\Phi(x/|x|)|f(x)|^2}{|x|^{\beta+2}}\,dx,\end{aligned}$$

which shows that (iii) holds true and hence completes the proof of Lemma 3.

The following Lemma 4 is useful in the proof of Theorem 3. It also gives an extension of the weighted L^2 Hardy inequalities in Lemma 3 to any $p \in (1,\infty)$.

LEMMA 4. *Let $p\in(1,\infty)$, $\beta\in\mathbb{R}$ with $\beta\neq n/p$ and $\Phi=\Phi(\cdot/|\cdot|)\geqslant 0$ is a measurable function defined on $\mathbb{S}^{n-1}$ such that $\Phi\in L^{n/(\beta p)}(\mathbb{S}^{n-1})$. Then for any $u\in C_0^\infty(\mathbb{R}^n)$, it holds that*

$$\int_{\mathbb{R}^n}\frac{|\nabla f(x)|^p}{|x|^{(\beta-1)p}}dx\geqslant\frac{|\mathbb{S}^{n-1}|^{\frac{\beta p}{n}}}{\|\Phi\|_{L^{\frac{n}{\beta p}}(\mathbb{S}^{n-1})}}\left(\frac{|n-\beta p|}{p}\right)^p\int_{\mathbb{R}^n}\frac{\Phi(x/|x|)|f(x)|^p}{|x|^{\beta p}}dx. \tag{3.16}$$

Proof. For any $f\in C_0^\infty(\mathbb{R}^n)$, recall in [27] the definition of the *symmetric decreasing rearrangement* $f^*:=\mathbb{R}^n\to[0,\infty)$ of f that for any $x\in\mathbb{R}^n$,

$$f^*(x):=\int_0^\infty\chi_{\{y\in\mathbb{R}^n:\ |f(y)|>t\}^*}(x)\,dt, \tag{3.17}$$

where for any measurable set $E\subset\mathbb{R}^n$, $E^*:=\{x\in\mathbb{R}^n:\ |x|<r\}$ is its symmetric rearrangement with

$$\frac{|\mathbb{S}^{n-1}|}{n}r^n=|E^*| \tag{3.18}$$

and $|\mathbb{S}^{n-1}|$ the surface area of the unite sphere $\mathbb{S}^{n-1}$.

By the Hardy-Littlewood rearrangement inequality and the fact that $(|u|^p)^*=(u^*)^p$ (see [27, pages 81-82]), we know that

$$\int_{\mathbb{R}^n}\frac{\Phi(x/|x|)|u(x)|^p}{|x|^{\beta p}}dx\leqslant\int_{\mathbb{R}^n}\left[\frac{\Phi(x/|x|)}{|x|^{\beta p}}\right]^*|u^*(x)|^p\,dx. \tag{3.19}$$

We first estimate $[\frac{\Phi(x/|x|)}{|x|^{\beta p}}]^*$. By Chebyshev's inequality, we have that for any $t\in(0,\infty)$

$$\begin{aligned}\left|\left\{y\in\mathbb{R}^n:\ \frac{\Phi(y/|y|)}{|y|^{\beta p}}>t\right\}\right|&=\left|\left\{y\in\mathbb{R}^n:\ |y|<\left[\frac{\Phi(y/|y|)}{t}\right]^{\frac{1}{\beta p}}\right\}\right|\\&=\int_{\mathbb{S}^{n-1}}\left[\int_0^{[\Phi(\theta)/t]^{\frac{1}{\beta p}}}r^{n-1}\,dr\right]d\theta\\&=\frac{1}{nt^{\frac{n}{\beta p}}}\int_{\mathbb{S}^{n-1}}[\Phi(\theta)]^{\frac{n}{\beta p}}\,d\theta,\end{aligned}$$

which combined with (3.18) implies that the symmetric rearrangement of the above level set can be written as follows

$$\begin{aligned}\left\{y\in\mathbb{R}^n:\ \frac{\Phi(y/|y|)}{|y|^{\beta p}}>t\right\}^*&=\left\{y\in\mathbb{R}^n:\ \frac{|\mathbb{S}^{n-1}|}{n}|y|^n<\frac{1}{nt^{\frac{n}{\beta p}}}\int_{\mathbb{S}^{n-1}}[\Phi(\theta)]^{\frac{n}{\beta p}}\,d\theta\right\}\\&=\left\{y\in\mathbb{R}^n:\ |y|<\left\{\frac{1}{|\mathbb{S}^{n-1}|t^{\frac{n}{\beta p}}}\int_{\mathbb{S}^{n-1}}[\Phi(\theta)]^{\frac{n}{\beta p}}\,d\theta\right\}^{\frac{1}{n}}\right\}.\end{aligned}$$

Thus, we conclude that

$$\begin{aligned}\left[\frac{\Phi(x/|x|)}{|x|^{\beta p}}\right]^* &= \int_0^\infty \chi_{\left\{y\in\mathbb{R}^n:\ \frac{\Phi(y/|y|)}{|y|^{\beta p}}>t\right\}^*}(x)\,dt \\ &= \int_0^\infty \chi_{\left\{y\in\mathbb{R}^n:\ |y|<\left\{\frac{1}{|\mathbb{S}^{n-1}|t^{\frac{n}{\beta p}}}\int_{\mathbb{S}^{n-1}}[\Phi(\theta)]^{\frac{n}{\beta p}}\,d\theta\right\}^{\frac{1}{n}}\right\}}(x)\,dt \\ &= \frac{\|\Phi\|_{L^{\frac{n}{\beta p}}(\mathbb{S}^{n-1})}}{|x|^{\beta p}|\mathbb{S}^{n-1}|^{\frac{\beta p}{n}}},\end{aligned}$$

which together with (3.19) shows that

$$\int_{\mathbb{R}^n} \frac{\Phi(x/|x|)|f(x)|^p}{|x|^{\beta p}}\,dx \leqslant \frac{\|\Phi\|_{L^{\frac{n}{\beta p}}(\mathbb{S}^{n-1})}}{|\mathbb{S}^{n-1}|^{\frac{\beta p}{n}}}\int_{\mathbb{R}^n}\frac{|f^*(x)|^p}{|x|^{\beta p}}\,dx.$$

This combined with the weighted n-dimensional Hardy inequality (1.5) with $\varepsilon = p-\beta p$ and the Pólya-Szegö inequality (namely $\|\nabla(g^*)\|_{L^p(\mathbb{R}^n)} \leqslant \|\nabla g\|_{L^p(\mathbb{R}^n)}$ for any $g \in C_c^\infty(\mathbb{R}^n)$), we further see

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{\Phi(x/|x|)|f(x)|^p}{|x|^{\beta p}}\,dx &\leqslant \frac{\|\Phi\|_{L^{\frac{n}{\beta p}}(\mathbb{S}^{n-1})}}{|\mathbb{S}^{n-1}|^{\frac{\beta p}{n}}}\left(\frac{p}{|n-\beta p|}\right)^p\int_{\mathbb{R}^n}\frac{|\nabla(f^*)(x)|^p}{|x|^{(\beta-1)p}}\,dx \\ &\leqslant \frac{\|\Phi\|_{L^{\frac{n}{\beta p}}(\mathbb{S}^{n-1})}}{|\mathbb{S}^{n-1}|^{\frac{\beta p}{n}}}\left(\frac{p}{|n-\beta p|}\right)^p\int_{\mathbb{R}^n}\frac{|\nabla f(x)|^p}{|x|^{(\beta-1)p}}\,dx.\end{aligned}$$

This finishes the proof of Lemma 4.

With the help of Lemmas 2 and 3, we now give a proof of Theorem 2.

Proof of Theorem 2. Take $\beta = 2$. From the assumption that $n \geqslant 5$, we know $n \geqslant 2+\frac{\beta}{2}+\frac{1}{2\beta}$. Thus, by Lemma 3(i), we conclude that for any $f \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n}\frac{|\nabla f(x)|^2}{|x|^2}\,dx \geqslant \tau\int_{\mathbb{R}^n}\frac{\Phi(x/|x|)|f(x)|^2}{|x|^4}\,dx, \tag{3.20}$$

with $\tau = \left(\frac{n-4}{2}\right)^2|\mathbb{S}^{n-1}|^{\frac{1}{q}}/\|\Phi\|_{L^q(\mathbb{S}^{n-1})}$. By this and the L^2 Hardy-Rellich inequality (1.12), we further conclude that (1.27) holds true and hence finish the proof of Theorem 2.

Finally we turn to the proof of Theorem 3 by using Lemma 4.

Proof of Theorem 3. By Corollary 1(ii) with $\alpha = 0$, we know that for any $f \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n}|\Delta f(x)|^p\,dx \geqslant \left(\frac{n(p-1)}{c_p p}\right)^p\int_{\mathbb{R}^n}\frac{|\nabla f(x)|^p}{|x|^p}\,dx. \tag{3.21}$$

On the other hand, from Lemma 4 with $\beta = 2$, we deduce that

$$\int_{\mathbb{R}^n} \frac{|\nabla f(x)|^p}{|x|^p}\,dx \geqslant \frac{|\mathbb{S}^{n-1}|^{\frac{2p}{n}}}{\|\Phi\|_{L^{\frac{n}{2p}}(\mathbb{S}^{n-1})}} \left(\frac{|n-2p|}{p}\right)^p \int_{\mathbb{R}^n} \frac{\Phi(x/|x|)|f(x)|^p}{|x|^{2p}}\,dx,$$

which together with (3.21) completes the proof of Theorem 3.

REFERENCES

[1] F. G. AVKHADIEV, *Hardy type inequalities in higher dimensions with explicit estimate of constants*, Lobachevskii J. Math., **21**, (2006), 3–31.

[2] F. G. AVKHADIEV, *Hardy-Rellich inequalities in domains of the Euclidean space*, J. Math. Anal. Appl., **442**, (2016), 469–484.

[3] F. G. AVKHADIEV AND I. K. SHAFIGULLIN, *Sharp estimates of Hardy constants for domains with special boundary properties*, Translation of Izv. Vyssh. Uchebn. Zaved. Mat. 2014, 69–73, Russian Math. (Iz. VUZ), **58**, (2014), 58–61.

[4] A. BALINSKY, W. D. EVANS AND R. T. LEWIS, *The Analysis and Geometry of Hardy's Inequality*, Universitext, Springer, Cham, 2015.

[5] G. BARBATIS, *Best constants for higher-order Rellich inequalities in $L^p(\Omega)$*, Math. Z., **255**, (2007), 877–896.

[6] G. BARBATIS AND A. TERTIKAS, *On a class of Rellich inequalities*, J. Comput. Appl. Math., **194**, (2006), 156–172.

[7] J. BOURGAIN AND H. BREZIS, *New estimates for the Laplacian, the div-curl, and related Hodge systems*, C. R. Math. Acad. Sci. Paris, **338**, (2004), 539–543.

[8] P. CALDIROLI AND R. MUSINA, *Rellich inequalities with weights*, Calc. Var., **45**, (2012), 147–164.

[9] D. G. COSTA, *On Hardy-Rellich type inequalities in $\mathbb{R}^n$*, Appl. Math. Lett., **22**, (2009), 902–905.

[10] E. B. DAVIES AND A. M. HINZ, *Explicit constants for Rellich inequalities in $L_p(\Omega)$*, Math. Z., **227**, (1998), 511–523.

[11] Y. DI, L. JIANG, S. SHEN AND Y. JIN, *A note on a class of Hardy-Rellich type inequalities*, J. Inequal. Appl., **84**, (2013), 6 pp.

[12] J. DOLBEAULT, M. ESTEBAN AND A. LAPTEV, *Spectral estimates on the sphere*, Anal. PDE, **7**, (2014), 435–460.

[13] D. E. EDMUNDS AND W. D. EVANS, *The Rellich inequality*, Rev. Mat. Complut., **29**, (2016), 511–530.

[14] D. E. EDMUNDS AND R. HURRI-SYRJÄNEN, *Weighted Hardy inequalities*, J. Math. Anal. Appl., **310**, (2005), 424–435.

[15] E. FABES, O. MENDEZ AND M. MITREA, *Boundary layers on Sobolev-Besov spaces and Poisson's equation for the Laplacian in Lipschitz domains*, J. Funct. Anal., **159**, (1998), 323–368.

[16] R. FARWIG AND H. SOHR, *Weighted L^q-theory for the Stokes resolvent in exterior domains*, J. Math. Soc. Japan, **49**, (1997), 251–288.

[17] A. FRÖHLICH, *The Helmholtz decomposition of weighted L^q-spaces for Muckenhoupt weights, Navier-Stokes equations and related nonlinear problems*, Ann. Univ. Ferrara Sez. VII (N.S.), **46**, (2000), 11–19.

[18] G. P. GALDI, *An Introduction to the Mathematical Theory of the Navier-Stokes equations. Vol. I: Linearezed Steady Problem*, Springer Tracts Nat. Philos. **38**, Springer-Verlag, New York, 1994.

[19] P. HAJŁASZ AND P. KOSKELA, *Isoperimetric inequalities and imbedding theorems in irregular domains*, J. London Math. Soc. (2), **58**, (1998), 425–450.

[20] G. H. HARDY, *Note on a theorem of Hilbert*, Math. Z., **6**, (1920), 314–317.

[21] G. H. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1988.

[22] T. HOFFMANN-OSTENHOF AND A. LAPTEV, *Hardy inequalities with homogeneous weights*, J. Funct. Anal., **268**, (2015), 3278–3289.

[23] A. KUFNER AND L.-E. PERSSON, *Weighted Inequalities of Hardy Type*, World Scientific Publishing Co., Inc., River Edge, NJ, 2003.

[24] L. LANZANI AND E. M. STEIN, *A note on div curl inequalities*, Math. Res. Lett., **12**, (2005), 57–61.

[25] J. LEHRBÄCK, *Weighted Hardy inequalities and the size of the boundary*, Manuscripta Math., **127**, (2008), 249–273.

[26] J. L. LEWIS, *Uniformly fat sets*, Trans. Am. Math. Soc., **308**, (1988), 177–196.

[27] E. H. LIEB AND M. LOSS, *Analysis*, Graduate Studies in Mathematics **14**, American Mathematical Society, Providence, RI, 2001.

[28] M. MARCUS, V. J. MIZEL AND Y. PINCHOVER, *On the best constant for Hardy's inequality in* R^n, Trans. Amer. Math. Soc., **350** (1998), 3237–3255.

[29] T. MATSKEWICH AND P. E. SOBOLEVSKII, *The best possible constant in generalized Hardy's inequality for convex domain in* $\mathbb{R}^n$, Nonlinear Anal., **28**, (1997), 1601–1610.

[30] G. METAFUNE, M. SOBAJIMA AND C. SPINA, *Weighted Calderón-Zygmund and Rellich inequalities in* L^p, Math. Ann., **361**, (2015), 313–366.

[31] B. MUCKENHOUPT, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc., **165**, (1972), 207–226.

[32] J. NEČAS, *Sur une méthode pour résoudre les équations aux dérivées partielles du type elliptique, voisine de la variationnelle*, Ann. Scuola Norm. Sup. Pisa (3), **16**, (1962), 305–326.

[33] N. OKAZAWA, L^∞*-estimates for Sobolev functions in terms of* ∇ *and* Δ, Math. Z., **262**, (2009), 475–515.

[34] B. OPIC AND A. KUFNER, *Hardy-Type Inequalities*, Pitman Research Notes in Mathematics Series **219**, Longman Scientific Technical, Harlow, 1990.

[35] M. P. OWEN, *The Hardy-Rellich inequality for polyharmonic operators*, Proc. Roy. Soc. Edinburgh Sect. A, **129** (1999), 825–839.

[36] F. RELLICH, *Halbbeschränkte Differentialoperatoren Höherer Ordnung*, Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 243-250, Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956.

[37] A. TERTIKAS AND N. B. ZOGRAPHOPOULOS, *Best constants in the Hardy-Rellich inequalities and related improvements*, Adv. Math., **209**, (2007), 407–459.

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