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SOME ESTIMATES FOR THE BILINEAR FRACTIONAL INTEGRALS ON THE MORREY SPACE

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Abstract. In this paper, we are interested in the following bilinear fractional integral operator $B\mathscr{I}_\alpha$ defined by

$$B\mathscr{I}_\alpha(f,g)(x) = \int_{\mathbb{R}^n} \frac{f(x-y)g(x+y)}{|y|^{n-\alpha}}dy,$$

with $0<\alpha<n$. We prove the weighted boundedness of $B\mathscr{I}_\alpha$ on the Morrey type spaces. Moreover, an Olsen type inequality for $B\mathscr{I}_\alpha$ is also given.

1. Introduction

In 1992, Grafakos [14] studied the multilinear fractional integral operator $\mathscr{I}_{\alpha,\vec{\theta}}$ with its definition defined by

$$\mathscr{I}_{\alpha,\vec{\theta}}(\vec{f})(x) = \int_{\mathbb{R}^n} \frac{1}{|y|^{n-\alpha}} \prod_{i=1}^{m} f_i(x-\theta_i y)dy,$$

where

$$\vec{f} = (f_1,\cdots,f_m)$$

and

$$\vec{\theta} = (\theta_1,\theta_2,\ldots,\theta_m)$$

is a fixed vector with distinct nonzero components.

For a special case of $\mathscr{I}_{\alpha,\vec{\theta}}$, the following bilinear fractional integral was also studied by Kenig and Stein in [30].

$$B\mathscr{I}_\alpha(f,g)(x) = \int_{\mathbb{R}^n} \frac{f(x-y)g(x+y)}{|y|^{n-\alpha}}dy, \quad 0<\alpha<n.$$

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As the operator $B\mathscr{I}_\alpha$ can be regarded as a variant version of the bilinear Hilbert transform if we take $\alpha \to 0$, many authors pay much attention to such operator and they proved the boundedness of $B\mathscr{I}_\alpha$ on variant product function spaces. One may see [2, 3, 4, 5, 10, 43, 45] et al. for more details.

Meanwhile, it is well known that in the last 70s, Muckenhoupt and Wheeden ([36, 37]) introduced the A_p and $A_{(p,q)}$ weight classes which are very adopted for the weighted estimates of the singular integrals and fractional integrals. Now, let us introduce the definitions of $\omega \in A_p$ and $\omega \in A_{(p,q)}$ respectively.

DEFINITION 1. ([36]) We say a non-negative function $\omega(x)$ belongs to the Muckenhoupt class A_p with $1 < p < \infty$ if

$$[\omega]_{A_p} := \sup_Q \left(\frac{1}{|Q|}\int_Q \omega(x)dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{1-p'}dx\right)^{p-1} < \infty \tag{1}$$

for any cube Q and $1/p + 1/p' = 1$.

In case $p = 1$, $\omega \in A_1$ is understood as there exists a positive constant C such that

$$\frac{1}{|Q|}\int_Q \omega(y)dy \leqslant C\omega(x) \tag{2}$$

for a.e. $x \in Q$ and any cube Q. For the case $p = \infty$, we define $A_\infty = \bigcup\limits_{1<p<\infty} A_p$.

DEFINITION 2. ([37]) We say that a non-negative function $\omega(x)$ belongs to $A_{(p,q)}$ weight class with $1 < p < q < \infty$ if

$$[\omega]_{A_{p,q}} := \sup_Q \left(\frac{1}{|Q|}\int_Q \omega(x)^q dx\right)^{1/q}\left(\frac{1}{|Q|}\int_Q \omega(x)^{-p'}dx\right)^{1/p'} < \infty. \tag{3}$$

Since the last 90s, the multilinear theory for the singular integral operators was developed a lot. For example, in 2002, Grafakos and Torres [15] introduced the multilinear C-Z theory. Later, Lerner et al. [32] introduced a new kind of multiple weight which is very adopted for the weighted norm inequalities of the multilinear C-Z operator. Following their work, Chen and Xue [7], as well as Moen independently [33], introduced a new type of multiple fractional type $A_{(\vec{P},q)}$ weight class . Now, let us give the definition of $A_{(\vec{P},q)}$ weight class.

DEFINITION 3. ([7, 33]) Let $1 \leqslant p_1, \cdots, p_m, 1/p = 1/p_1 + \cdots + 1/p_m$ and $q > 0$. Suppose that $\vec{\omega} = (\omega_1, \cdots, \omega_m)$ and each ω_i is a nonnegative function on $\mathbb{R}^n$. We say that $\vec{\omega} \in A_{(\vec{p},q)}$ if it satisfies

$$[\vec{\omega}]_{A_{(\vec{P},q)}} := \sup_Q \left(\frac{1}{|Q|}\int_Q v_{\vec{\omega}}^q(x)dx\right)^{1/q}\prod_{i=1}^m\left(\frac{1}{|Q|}\int_Q \omega_i^{-p_i'}(x)dx\right)^{1/p_i'} < \infty, \tag{4}$$

where $v_{\vec{\omega}} = \prod_{i=1}^{m} \omega_i$. Moreover, for the case $p_i = 1$, $\left(\frac{1}{|Q|}\int_Q \omega_i^{-p_i'}\right)^{1/p_i'}$ is understood as $(\inf_Q \omega_i)^{-1}$.

Chen and Xue, as well as Moen independently, proved the following theorem.

THEOREM A. *([7, 33]) Suppose that* $0<\alpha<mn$, $1<p_1,\cdots,p_m<\infty$. *If* $1/p=\sum_{i=1}^{m} 1/p_i$ *and* $1/q=1/p-\alpha/n$. *Then,* $\vec{\omega}\in A_{(\vec{P},q)}$ *if and only if the following multiple weighted norm inequalities holds:*

$$\|\mathscr{I}_{\alpha,m}(\vec{f})\|_{L^q(v_{\vec{\omega}}^q)} \leqslant C\prod_{i=1}^{m}\|f_i\|_{L^{p_i}(\omega_i^{p_i})}.$$

Here, $\mathscr{I}_{\alpha,m}$ *denotes the multilinear fractional integral operator and its definition can be stated as*

$$\mathscr{I}_{\alpha,m}(\vec{f})(x)=\int_{(\mathbb{R}^n)^m}\frac{f_1(y_1)f_2(y_2),\cdots,f_m(y_m)}{(|x-y_1|+\cdots+|x-y_m|)^{mn-\alpha}}dy_1dy_2\cdots dy_m.$$

For the study of the weighted theory for $B\mathscr{I}_\alpha$ with the multiple fractional type weight class, Hoang and Moen [18, 34] did some excellent work to show that the operator $B\mathscr{I}_\alpha$ satisfy several weighted estimates on the product L^p spaces. Recently, Komori-Furuya [27, 28] also got some important weighted norm inequalities of $B\mathscr{I}_\alpha$ with power weights.

On the other hand, in order to study the local behavior of solutions to second order elliptical partial differential equations, Morrey [35] introduced the Morrey space. The Morrey space $\mathscr{M}_q^p(\mathbb{R}^n)$, $0<q\leqslant p<\infty$, is the collection of all measurable functions f with its definition defined by

$$\mathscr{M}_q^p(\mathbb{R}^n):=\left\{f\in\mathscr{M}_q^p(\mathbb{R}^n):\|f\|_{\mathscr{M}_q^p(\mathbb{R}^n)}=\sup_{\substack{Q\subset\mathbb{R}^n\\ Q:\text{cubes}}}|Q|^{1/p-1/q}\|f\chi_Q\|_{L^q(\mathbb{R}^n)}<\infty\right\}.$$

Many authors studied the weighted norm inequalities for integral operators on the Morrey type spaces, readers may see [20, 22, 25, 26, 29] et al. or the summary article [21] to find more details. Here we would like to mention that in [20, 22, 25], Iida et al. introduced the following new fractional type multiple weight condition as follows.

$$[\vec{\omega}]_{q_0,q,\vec{P}}:=\sup_{\substack{Q\subset Q'\\ Q,Q':\text{cubes}}}\left(\frac{|Q|}{|Q'|}\right)^{1/q_0}\left(\frac{1}{|Q|}\int_Q(\omega_1(x)\omega_2(x))^q dx\right)^{1/q}$$
$$\times\prod_{i=1}^{m}\left(\frac{1}{|Q'|}\int_{Q'}\omega_i(y_i)^{-p_i'}dy_i\right)^{1/p_j'}<\infty.$$

Iida et al. [20, 22, 25] found that the above multiple weight condition is very adopted for the weighted norm inequalities of the operator $\mathscr{I}_{\alpha,m}$ on the Morrey type space and they proved the following theorem.

THEOREM B. *([20, 25]) Let* $0<\alpha<mn,\ 1<p_1,\cdots,p_m<\infty,\ 1/p=\sum_{i=1}^{m}1/p_i$. *Then,we assume that* $0<p\leqslant p_0<\infty$ *and* $0<q\leqslant q_0<\infty$ *with* $1/q_0=1/p_0-\alpha/n$ *and* $q/q_0=p/p_0$. *Moreover, for* $\vec{f}=(f_1,\cdots,f_m)$ *and* $\vec{\omega}=(\omega_1,\omega_2,\cdots,\omega_m)$, *we denote*

$$\|\vec{f}\|_{\mathscr{M}^{p_0}_{\vec{P}}}:=\sup_{Q\subset\mathbb{R}^n}|Q|^{\frac{1}{p_0}}\prod_{i=1}^{m}\left(\frac{1}{|Q|}\int_Q|f_i(y_i)|^{p_i}dy_i\right)^{\frac{1}{p_i}},$$

and

$$\nu_{\vec{\omega}}(x)=\prod_{i=1}^{m}\omega_i(x).$$

If there exist $a>1$ *satisfying*

$$[\vec{\omega}]_{aq_0,q,\vec{P}}<\infty$$

where $\vec{P}=(p_1,\cdots,p_m)$ *and* $a>1$, *then there exist a positive constant* C *independent of* f_i, *such that*

$$\|\mathscr{I}_{\alpha,m}(\vec{f})\nu_{\vec{\omega}}\|_{\mathscr{M}^{q_0}_{q}}\leqslant C\|(f_1\omega_1,\cdots,f_m\omega_m)\|_{\mathscr{M}^{p_0}_{\vec{P}}}.$$

In [17], He and Yan studied the weighted bounedness of $B\mathscr{I}_\alpha$ on $\mathscr{M}^p_q(\mathbb{R}^n)$ with $0<q<1$. Thus, it is natural to ask whether we can prove the weighted norm inequalities for $B\mathscr{I}_\alpha$ on $\mathscr{M}^p_q(\mathbb{R}^n)$ with $q>1$? In this paper, we will give a positive answer to this question.

Motivated by the above backgrounds, in this paper, we will give the weighted boundedness of $B\mathscr{I}_\alpha$ on the Morrey type space with the fractional type multiple weights condition proposed by Iida et al. Our results can be stated as follows.

THEOREM 1. *Suppose* $0<\alpha<n,\ p_1>r>1,\ p_2>s>1,\ 1/r+1/s=1,\ 1/p=1/p_1+1/p_2,\ 1<p_1,p_2<\infty,\ 0<p\leqslant p_0<\infty,\ 0<q\leqslant q_0<\infty$. *Let*

$$1/q_0=1/p_0-\alpha/n,\ \ q/q_0=p/p_0\ \ \text{and}\ \ \nu_{\vec{\omega}}(x)=\prod_{i=1}^{2}\omega_i(x).$$

Moreover, assume that either p *or* q *satisfies one of the following condition:*

$$p>1\quad or\quad q>\frac{1}{2}.$$

If there exists $a>1$, *such that* $[\vec{\omega}]_{aq_0,q,\left(\frac{sp_1}{s+p_1},\frac{rp_2}{r+p_2}\right)}<\infty$, *that is*

$$\sup_{\substack{Q\subset Q'\\ Q,Q':cubes}}\left(\frac{|Q|}{|Q'|}\right)^{\frac{1}{aq_0}}\left(\frac{1}{|Q|}\int_Q\nu_{\vec{\omega}}(x)^q dx\right)^{1/q}\left(\frac{1}{|Q'|}\int_{Q'}\omega_1(x)^{-\frac{p_1r}{p_1-r}}\right)^{1/r-1/p_1}$$
$$\times\left(\frac{1}{|Q'|}\int_{Q'}\omega_2(x)^{-\frac{p_2s}{p_2-s}}\right)^{1/s-1/p_2}<\infty.$$

Then, there exists a positive constant C independent of f and g, such that

$$\|B\mathscr{I}_\alpha(f,g)\nu_{\vec{\omega}}\|_{\mathscr{M}_q^{q_0}} \leqslant C[\vec{\omega}]_{aq_0,q,\left(\frac{sp_1}{s+p_1},\frac{rp_2}{r+p_2}\right)}\|(f\omega_1,g\omega_2)\|_{\mathscr{M}_{\vec{P}}^{p_0}}. \tag{5}$$

REMARK 1. Note that for the operator $\mathscr{I}_{\alpha,2}$,

$$\mathscr{I}_{\alpha,2}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1)f_2(y_2)}{(|x-y_1|+|x-y_2|)^{2n-\alpha}}dy_1dy_2.$$

As mentioned in [34, p.629], if we denote δ is the point mass measure at the origin, then we know that the kernel of $\mathscr{I}_{\alpha,2}$,

$$K_\alpha(u,v) = (|u|+|v|)^{-2n+\alpha}$$

has a singularity at the origin in $\mathbb{R}^{2n}$ as opposed to the kernel of $B\mathscr{I}_\alpha$

$$k_\alpha(u,v) = \frac{\delta(u+v)}{|u|^{n-\alpha}},$$

which has a singularity along a line. Thus, we conclude that Theorem 1 parallel earlier results by the authors [25] for the less singular bilinear fractional integral operator $\mathscr{I}_{\alpha,2}$.

If we choose $p = p_0$ and $q = q_0$ in Theorem 1, we can easily obtain the following result proved by Hoang and Moen [18].

COROLLARY 1. ([18]) *Suppose that there exist real numbers $\alpha, p_1, r, p_2, s, p$ and q satisfying the same conditions as in Theorem 1. If $1/p_1 + 1/p_2 - 1/q = \alpha/n$ and $\vec{\omega} \in A_{\left(\left(\frac{p_1 s}{p_1+s},\frac{p_2 r}{p_2+r}\right),q\right)}$, then there exists a positive constant C independent of f and g, such that*

$$\|B\mathscr{I}_\alpha(f,g)\|_{L^q(\nu_{\vec{\omega}}^q)} \leqslant C\|f\|_{L^{p_1}(\omega_1^{p_1})}\|g\|_{L^{p_2}(\omega_2^{p_2})}. \tag{6}$$

Proof. By the definition of the Morrey space, it suffices to show

$$[\vec{\omega}]_{aq_0,q,\left(\frac{sp_1}{s+p_1},\frac{rp_2}{r+p_2}\right)} < \infty \quad (q = q_0). \tag{7}$$

In fact, as $\vec{\omega} \in A_{\left(\left(\frac{p_1 s}{p_1+s},\frac{p_2 r}{p_2+r}\right),q\right)}$, we have $\nu_{\vec{\omega}}^q = \prod_{i=1}^2 \omega_i^q \in A_{2q}$ or $\nu_{\vec{\omega}}^q \in A_{1+q(1-1/p)}$. Then, we know that $\nu_{\vec{\omega}}^q$ satisfies the reversed Hölder inequality (see Section 2). That is, if we choose $a = 1+\varepsilon$ where $\varepsilon \in \mathbb{R}^+$ and ε is small enough, there is

$$\left(\frac{1}{|Q|}\int_Q \nu_{\vec{\omega}}(x)^{aq}dx\right)^{1/aq} \leqslant C\left(\frac{1}{|Q|}\int_Q \nu_{\vec{\omega}}(x)^q dx\right)^{1/q}.$$

Recalling that $q = q_0$, we may have

$$\left(\frac{|Q|}{|Q'|}\right)^{\frac{1}{aq_0}}\left(\frac{1}{|Q|}\int_Q(\omega_1(x)\omega_2(x))^q dx\right)^{1/q}\left(\frac{1}{|Q'|}\int_{Q'}\omega_1(x)^{-\frac{p_1 r}{p_1-r}}dx\right)^{1/r-1/p_1}$$

$$\times\left(\frac{1}{|Q'|}\int_{Q'}\omega_2(x)^{-\frac{p_2 s}{p_2-s}}dx\right)^{1/s-1/p_2}$$

$$\leqslant\left(\frac{|Q|}{|Q'|}\right)^{\frac{1}{aq_0}}\left(\frac{1}{|Q|}\int_Q(\omega_1(x)\omega_2(x))^{aq}dx\right)^{1/aq}\left(\frac{1}{|Q'|}\int_{Q'}\omega_1(x)^{-\frac{p_1 r}{p_1-r}}dx\right)^{1/r-1/p_1}$$

$$\times\left(\frac{1}{|Q'|}\int_{Q'}\omega_2(x)^{-\frac{p_2 s}{p_2-s}}dx\right)^{1/s-1/p_2}$$

$$=\left(\frac{1}{|Q'|}\int_Q(\omega_1(x)\omega_2(x))^{aq}dx\right)^{1/aq}\left(\frac{1}{|Q'|}\int_{Q'}\omega_1(x)^{-\frac{p_1 r}{p_1-r}}dx\right)^{1/r-1/p_1}$$

$$\times\left(\frac{1}{|Q'|}\int_{Q'}\omega_2(x)^{-\frac{p_2 s}{p_2-s}}dx\right)^{1/s-1/p_2}$$

$$\leqslant\left(\frac{1}{|Q'|}\int_{Q'}(\omega_1(x)\omega_2(x))^{aq}dx\right)^{1/aq}\left(\frac{1}{|Q'|}\int_{Q'}\omega_1(x)^{-\frac{p_1 r}{p_1-r}}dx\right)^{1/r-1/p_1}$$

$$\times\left(\frac{1}{|Q'|}\int_{Q'}\omega_2(x)^{-\frac{p_2 s}{p_2-s}}dx\right)^{1/s-1/p_2}$$

$$\leqslant\left(\frac{1}{|Q'|}\int_{Q'}(\omega_1(x)\omega_2(x))^{q}dx\right)^{1/q}\left(\frac{1}{|Q'|}\int_{Q'}\omega_1(x)^{-\frac{p_1 r}{p_1-r}}dx\right)^{1/r-1/p_1}$$

$$\times\left(\frac{1}{|Q'|}\int_{Q'}\omega_2(x)^{-\frac{p_2 s}{p_2-s}}dx\right)^{1/s-1/p_2}<\infty,$$

where the second to last inequality follows from the reversed Hölder inequality for $\nu_{\vec{\omega}}^q=(\omega_1\omega_2)^q$ and we obtain (7).

REMARK 2. For the case $0<q<1$ in Theorem 1, our result is also different from [17, Theorem 4.6].

2. Preliminaries

In this section, we will give some lemmas and definitions that will be useful throughout this paper.

LEMMA 1. *(The reversed Hölder inequality, [16]) Let $1<p<\infty$ and $\omega\in A_p(\mathbb{R}^n)$. Then, there exist positive constants C and ε, depending only on p and the A_p condition of ω, such that for any cube Q, there is*

$$\left(\frac{1}{|Q|}\int_Q\omega(x)^{1+\varepsilon}dx\right)^{\frac{1}{1+\varepsilon}}\leqslant C\left(\frac{1}{|Q|}\int_Q\omega(x)dx\right). \tag{8}$$

LEMMA 2. *([6, 19]) Let $1\leqslant p_1,p_2,\cdots,p_m\leqslant\infty$, $1/p=\sum_{i=1}^{m}1/p_i$ and $0<q<\infty$. A vector $\vec{\omega}$ of weights satisfies $\vec{\omega}\in A_{(\vec{P},q)}$ if and only if* *(i)* $v_{\vec{\omega}}^{q} \in A_{1+q\left(m-\frac{1}{p}\right)}$;

(ii) $\omega_i^{-p_i'} \in A_{1+p_i' \cdot s_i} (i=1,\cdots,m)$ *where* $s_i = 1/q + m - 1/p - \frac{1}{p_i'}$.

Moreover, Moen [33] gave another characterization of $A_{(\vec{P},q)}$.

LEMMA 3. *([33]) Suppose* $1 < p_1,\cdots,p_m < \infty$ *and* $\vec{\omega} \in A_{(\vec{P},q)}$. *Then*

$$v_{\vec{\omega}}^{q} \in A_{mq} \quad and \quad \omega^{-p_i'} \in A_{mp_i'}.$$

From Lemmas 2 or 3, we know that if $[\vec{\omega}]_{aq_0,q,\left(\frac{sp_1}{s+p_1},\frac{rp_2}{r+p_2}\right)} < \infty$, then

$$v_{\vec{\omega}}^{q} \in A_{1+q(1-1/p)} (p>1), \omega_1^{-r\left(\frac{p_1}{r}\right)'} \in A_{1+r\left(\frac{p_1}{r}\right)'\left(\frac{1}{q}-\frac{1}{p_2}+\frac{1}{s}\right)}, \omega_2^{-s\left(\frac{p_2}{s}\right)'} \in A_{1+s\left(\frac{p_2}{s}\right)'\left(\frac{1}{q}-\frac{1}{p_1}+\frac{1}{r}\right)},$$

or

$$v_{\vec{\omega}}^{q} \in A_{2q} \ \left(q > \frac{1}{2}\right), \qquad \omega_1^{-r\left(\frac{p_1}{r}\right)'} \in A_{2r\left(\frac{p_1}{r}\right)'}, \quad \omega_2^{-s\left(\frac{p_2}{s}\right)'} \in A_{2s\left(\frac{p_2}{s}\right)'}.$$

Thus, we conclude that the functions $v_{\vec{\omega}}^{q}$, $\omega_1^{-r\left(\frac{p_1}{r}\right)'}$ and $\omega_2^{-s\left(\frac{p_2}{s}\right)'}$ all satisfy the reversed Hölder inequality throughout the proof of Theorem 1.

Next, we introduce some maximal functions (see [32] or [37]).

The maximal function M and the fractional maximal function M_α are defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|dy$$

and

$$M_\alpha f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)|dy, \quad (0 < \alpha < n)$$

with Q runs over all cubes containing x respectively.
Furthermore, for any $p > 1$, we denote

$$M^{(p)}f(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|} \int_Q |f(y)|^p dy \right)^{1/p}.$$

Before giving the next two lemmas which are the most important throughout this paper, we introduce some notations. First, we define the set of all dyadic grids. For more details about dyadic grids, one may see [31] et al. to find more details.

A dyadic grid $\mathscr{D}$ is a countable collection of cubes that satisfies the following properties:

(i) $Q \in \mathscr{D} \Rightarrow l(Q) = 2^{-k}$ for some $k \in \mathbb{Z}$.

(ii) For each $k \in \mathbb{Z}$, the set $\left\{Q \in \mathscr{D} : l(Q) = 2^{-k}\right\}$ forms a partition of $\mathbb{R}^n$.

(iii) $Q, P \in \mathscr{D} \Rightarrow Q \cap P \in \{P, Q, \emptyset\}$.

One very clear example (see [18, 31]) for this concept is the dyadic grid that is formed by translating and then dilating the unit cube $[0,1)^n$ all over $\mathbb{R}^n$. More precisely, it can be formulated as

$$\mathscr{D} = \left\{2^{-k}([0,1)^n + m) : k \in \mathbb{Z}, m \in \mathbb{Z}^n\right\}.$$

In practice, we also make extensive use of the family of dyadic grids as follows.

$$\mathscr{D}^t = \left\{2^{-k}\left([0,1)^n + m + (-1)^k t\right) : k \in \mathbb{Z}, m \in \mathbb{Z}^n\right\}, t \in \{0,1/3\}^n.$$

In [31], Lerner proved the following theorem.

LEMMA 4. *([31]) Given any cube in $\mathbb{R}^n$, there exists a $t \in \{0,1/3\}^n$ and a cube $Q_t \in \mathscr{D}^t$, such that $Q \subset Q_t$ and $l(Q_t) \leqslant 6l(Q)$.*

Next, let us give a decomposition result related to cubes. Suppose that Q_0 is a cube and let f be a locally integrable function. Then, we set

$$\mathscr{D}(Q_0) \equiv \{Q \in \mathscr{D} : Q \subset Q_0\}.$$

Moreover, suppose that $3Q_0$ is the unique cube concentric to Q_0 and have the volume $3^n|Q_0|$. Then, we denote

$$m_{3Q_0}(|f|^r, |g|^s) = \left(\frac{1}{|3Q_0|}\int_{3Q_0} |f(x)|^r dx\right)^{1/r} \left(\frac{1}{|3Q_0|}\int_{3Q_0} |g(x)|^s dx\right)^{1/s},$$

where $r,s > 1$ and $1/r + 1/s = 1$.

Next, we introduce the sparse family of Calderón-Zygmund cubes. That is, for each $k \in \mathbb{Z}^+$,

$$D_k \equiv \bigcup \left\{Q : Q \in \mathscr{D}(Q_0), m_{3Q_0}(|f|^r, |g|^s) > a^k\right\},$$

where a will be chosen later.

Considering the maximal cubes with respect to inclusion, we write

$$D_k = \bigcup_j Q_{k,j},$$

where the cubes $\{Q_{k,j}\} \subset \mathscr{D}(Q_0)$ are nonoverlapping. That is, $\{Q_{k,j}\}$ is a family of cubes satisfying

$$\sum_j \chi_{Q_{k,j}} \leqslant \chi_{Q_0} \tag{9}$$

for almost everywhere. By the maximality of $Q_{k,j}$, there is

$$a^k < m_{3Q_{k,j}}(|f|^r, |g|^s) < 2^{2n} a^k. \tag{10}$$

For the properties of $Q_{k,j}$, there is
(iv) For any fixed k, $Q_{k,j}$ are nonoverlapping for different j.
(v) If $k_1 < k_2$, then there exists i, such that $Q_{k_2,j} \subset Q_{k_1,i}$ for any $j \in \mathbb{Z}$.

Next, we will use the following decomposition of Q_0 from a clever idea proposed by Tanaka in [42].

Let $E_0 = Q_0 \setminus D_1, E_{k,j} = Q_{k,j} \setminus D_{k+1}$. Then, we have the following lemma.

LEMMA 5. *The set* $\{E_0\} \bigcup \{E_{k,j}\}$ *forms a disjoint family of sets, which decomposes* Q_0*, and satisfies*

$$|Q_0| \leqslant 2|E_0|, \quad |Q_{k,j}| \leqslant 2|E_{k,j}|. \tag{11}$$

Proof. We adopt some basic techniques from [18] to prove this lemma. By the definitions of $Q_{k,j}$ and D_{k+1}, there is

$$\begin{aligned}
&\left|Q_{k,j} \bigcap D_{k+1}\right| = \sum_{Q_{k+1,i} \subset Q_{k,j}} |Q_{k+1,i}| \\
\leqslant &\frac{1}{a^{k+1}} \sum_i \left[\left(|Q_{k+1,i}| \left(\frac{1}{|3Q_{k+1,i}|} \int_{3Q_{k+1,i}} |f(x)|^r dx \right) \right)^{1/r} \right. \\
&\left. \times \left(|Q_{k+1,i}| \left(\frac{1}{|3Q_{k+1,i}|} \int_{3Q_{k+1,i}} |g(x)|^s dx \right) \right)^{1/s} \right] \\
\leqslant &\frac{1}{a^{k+1}} \left(\sum_i |Q_{k+1,i}| \left(\frac{1}{|3Q_{k+1,i}|} \int_{3Q_{k+1,i}} |f(x)|^r dx \right) \right)^{1/r} \\
&\times \left(\sum_i |Q_{k+1,i}| \left(\frac{1}{|3Q_{k+1,i}|} \int_{3Q_{k+1,i}} |g(x)|^s dx \right) \right)^{1/s} \\
\leqslant &\frac{1}{a^{k+1}} \left(|Q_{k,j}| \left(\frac{1}{|3Q_{k,j}|} \int_{3Q_{k,j}} |f(x)|^r dx \right) \right)^{1/r} \left(|Q_{k,j}| \left(\frac{1}{|3Q_{k,j}|} \int_{3Q_{k,j}} |g(x)|^s dx \right) \right)^{1/s},
\end{aligned}$$

where the last inequality follows from the fact $Q_{k+1,i} \subset Q_{k,j}$ and $Q_{k,j}$ are nonoverlapping.
Then, using (10), we get

$$\begin{aligned}
&\left|Q_{k,j} \bigcap D_{k+1}\right| \\
\leqslant &\frac{1}{a^{k+1}} \left(|Q_{k,j}| \left(\frac{1}{|3Q_{k,j}|} \int_{3Q_{k,j}} |f(x)|^r dx \right) \right)^{1/r} \left(|Q_{k,j}| \left(\frac{1}{|3Q_{k,j}|} \int_{3Q_{k,j}} |g(x)|^s dx \right) \right)^{1/s} \\
\leqslant &\frac{2^{2n}}{a^{k+1}} |Q_{k,j}| a^k = \frac{2^{2n}}{a} |Q_{k,j}|.
\end{aligned}$$

Thus, if we choose $a = 2^{2n+1}$, we have

$$\left|Q_{k,j} \bigcap D_{k+1}\right| \leqslant \frac{1}{2}|Q_{k,j}|. \tag{12}$$

Similarly, we can also get

$$|D_1| \leqslant \frac{1}{2}|Q_0|. \tag{13}$$

Thus, we obtain (11) from (12) and (13).

LEMMA 6. *([1]) Let* $0 < \alpha < n$, $1 < q \leqslant p < \infty$ *and* $1 < t \leqslant s < \infty$. *Assume* $1/s = 1/p - \frac{\alpha}{n}$, $\frac{t}{s} = \frac{q}{p}$. *Then, there exists a positive constant* C *such that*

$$\|M_\alpha f\|_{\mathscr{M}_t^s} \leqslant \|I_\alpha f\|_{\mathscr{M}_t^s} \leqslant C\|f\|_{\mathscr{M}_q^p}.$$

LEMMA 7. *Suppose that there exists real numbers* t,q,p *satisfying* $1 < t < q \leqslant p < \infty$. *Then, we have* $\left\|f^\ell\right\|^{1/\ell}_{\mathscr{M}^{p/\ell}_{q/\ell}} = \|f\|_{\mathscr{M}_q^p}$ *with* $1 < \ell < q$.

Proof. Lemma 7 follows directly from the definition of the Morrey space and we omit the details here.

LEMMA 8. *([25]) Let* $0 \leqslant \alpha < mn$, $\vec{P} = (p_1,\cdots,p_m)$, $\vec{R} = (r_1,\cdots,r_m)$, $0 < r_i < p_i < \infty$, $0 < q \leqslant q_0 < \infty$, $0 < p \leqslant p_0 < \infty$, $1/q_0 = 1/p_0 - \alpha/n$, $\frac{q}{q_0} = \frac{p}{p_0}$ *and* $1/p = \sum_{i=1}^m 1/p_i$. *Then, we have*

$$\|\mathscr{M}_{\alpha,\vec{R}}(\vec{f})\|_{\mathscr{M}_q^{q_0}} \leqslant C\|\vec{f}\|_{\mathscr{M}_{\vec{P}}^{p_0}},$$

where

$$\mathscr{M}_{\alpha,\vec{R}}(\vec{f})(x) := \sup_{Q \ni x} l(Q)^\alpha \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q f_i(y_i)dy_i\right)^{1/r_i}.$$

3. Proof of Theorem 1

For the proof of (5), we decompose the proof into two cases: $q > 1$ and $q \leqslant 1$.

3.1. The case $q > 1$

Fix a cube $Q_0 = Q(x_0,\delta)$ with $\delta > 0$. Then, for any $x \in Q_0$, we may decompose $B\mathscr{I}_\alpha$ as

$$\begin{aligned} B\mathscr{I}_\alpha(f,g)(x) &= \int_{\mathbb{R}^n} \frac{f(x-t)g(x+t)}{|t|^{n-\alpha}}dt \\ &= \int_{|t|\leqslant 2\delta} \frac{f(x-t)g(x+t)}{|t|^{n-\alpha}}dt + \int_{|t|>2\delta} \frac{f(x-t)g(x+t)}{|t|^{n-\alpha}}dt \end{aligned}$$

$$=: I + II.$$

First, we decompose II as

$$\begin{aligned}
II &= \sum_{k=0}^{\infty} \int_{2\cdot 2^k\delta<|t|\leqslant 2\cdot 2^{k+1}\delta} \frac{f(x-t)g(x+t)}{|t|^{n-\alpha}} dt \\
&\leqslant \sum_{k=0}^{\infty} \frac{1}{(2\cdot 2^{k+1}\delta)^{n-\alpha}} \int_{|t|\leqslant 2\cdot 2^{k+1}\delta} |f(x-t)g(x+t)| dt \\
&\leqslant \sum_{k=0}^{\infty} \frac{1}{(2\cdot 2^{k+1}\delta)^{n-\alpha}} \left(\int_{|t|\leqslant 2\cdot 2^{k+1}\delta} |f(x-t)|^r dt\right)^{1/r} \left(\int_{|t|\leqslant 2\cdot 2^{k+1}\delta} |g(x+t)|^s dt\right)^{1/s}.
\end{aligned}$$

Then, by a change of variables and the fact $x \in Q_0 = Q(x_0, \delta)$, we obtain

$$\begin{aligned}
&|Q_0|^{\frac{1}{q_0}-\frac{1}{q}} \left(\int_{Q_0} |II|^q \left(\prod_{i=1}^{2} \omega_i(x)\right)^q dx\right)^{1/q} \\
\leqslant & C|Q_0|^{\frac{1}{q_0}-\frac{1}{q}} \sum_{k=0}^{\infty} \left(2\cdot 2^k\delta\right)^{\alpha-n} \left(\int_{Q_0} \prod_{i=1}^{2} \omega_i(x)^q dx\right)^{1/q} \\
&\times \left(\int_{2^{k+3}Q_0} |f(u)|^r du\right)^{1/r} \left(\int_{2^{k+3}Q_0} |g(v)|^s dv\right)^{1/s}.
\end{aligned}$$

For $\left(\int_{2^{k+3}Q_0} |f(u)|^r du\right)^{1/r}$, by the Hölder inequality, there is

$$\begin{aligned}
&\left(\int_{2^{k+3}Q_0} |f(u)|^r du\right)^{1/r} \\
\leqslant & \left(\int_{2^{k+3}Q_0} |f(u)\omega_1(u)|^{p_1} du\right)^{1/p_1} \left(\int_{2^{k+3}Q_0} |\omega_1(u)|^{-r\left(\frac{p_1}{r}\right)'} du\right)^{1/r-1/p_1}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\left(\int_{2^{k+3}Q_0} |g(v)|^s du\right)^{1/s} \\
\leqslant & \left(\int_{2^{k+3}Q_0} |g(v)\omega_2(v)|^{p_2} dv\right)^{1/p_2} \left(\int_{2^{k+3}Q_0} |\omega_2(v)|^{-s\left(\frac{p_2}{s}\right)'} dv\right)^{1/s-1/p_2}.
\end{aligned}$$

Thus, using the condition $[\vec{\omega}]_{aq_0,q,\left(\frac{sp_1}{s+p_1},\frac{rp_2}{r+p_2}\right)} < \infty$ and $a > 1$, we get

$$|Q_0|^{\frac{1}{q_0}-\frac{1}{q}} \left(\int_{Q_0} |II|^q \left(\prod_{i=1}^{2} \omega_i(x)\right)^q dx\right)^{1/q}$$

$$
\begin{aligned}
&\leqslant C\sum_{k=1}^{\infty}\left(2^{k+2}\delta\right)^{\alpha-n}|Q_0|^{1/q_0-1/q}\left(\int_{Q_0}\prod_{i=1}^{2}\omega_i(x)^q dx\right)^{1/q}\\
&\quad\times\left(\int_{2^{k+3}Q_0}|f(u)\omega_1(u)|^{p_1}du\right)^{1/p_1}\left(\int_{2^{k+3}Q_0}|\omega_1(u)|^{-r(\frac{p_1}{r})'}du\right)^{1/r-1/p_1}\\
&\quad\times\left(\int_{2^{k+3}Q_0}|g(v)\omega_2(v)|^{p_2}dv\right)^{1/p_2}\left(\int_{2^{k+3}Q_0}|\omega_2(v)|^{-s(\frac{p_2}{s})'}dv\right)^{1/s-1/p_2}\\
&\leqslant C\|(f\omega_1,g\omega_2)\|_{\mathscr{M}^{p_0}_{\vec{P}}}\sum_{k=1}^{\infty}\left(2^k\delta\right)^{\alpha-n}|Q_0|^{1/q_0-1/q+1/q}|2^{k+3}Q_0|^{1/p-1/p_0+1/r-1/p_1+1/s-1/p_2}\\
&\quad\times\left(\frac{|Q_0|}{|2^{k+3}Q_0|}\right)^{-\frac{1}{aq_0}}\left(\frac{|Q_0|}{|2^{k+3}Q_0|}\right)^{\frac{1}{aq_0}}\left(\frac{1}{|Q_0|}\int_{Q_0}\prod_{i=1}^{2}\omega_i(x)^q\right)^{1/q}\\
&\quad\times\left(\frac{1}{|2^{k+3}Q|}\int_{2^{k+3}Q_0}|f(u)\omega_1(u)|^{p_1}du\right)^{1/p_1}\left(\int_{2^{k+3}Q_0}|\omega_1(u)|^{-r(\frac{p_1}{r})'}du\right)^{1/r-1/p_1}\\
&\quad\times\left(\frac{1}{|2^{k+3}Q|}\int_{2^{k+3}Q_0}|g(v)\omega_2(v)|^{p_2}dv\right)^{1/p_2}\left(\int_{2^{k+3}Q_0}|\omega_2(v)|^{-s(\frac{p_2}{s})'}dv\right)^{1/s-1/p_2}\\
&\leqslant C[\vec{\omega}]_{aq_0,q,\left(\frac{sp_1}{s+p_1},\frac{rp_2}{r+p_2}\right)}\|(f\omega_1,g\omega_2)\|_{\mathscr{M}^{p_0}_{\vec{P}}},
\end{aligned}
$$

which implies

$$
\|II\cdot\nu_{\vec{\omega}}\|_{\mathscr{M}^{q_0}_q}\leqslant C[\vec{\omega}]_{aq_0,q,\left(\frac{sp_1}{s+p_1},\frac{rp_2}{r+p_2}\right)}\|(f\omega_1,g\omega_2)\|_{\mathscr{M}^{p_0}_{\vec{P}}}. \tag{14}
$$

Thus, it remains to give the estimates of $\|I\cdot\nu_{\vec{\omega}}\|_{\mathscr{M}^{q_0}_q}$. First, we prove the following lemma.

LEMMA 9. *Denote $I=\int_{|t|\leqslant 2\delta}\frac{f(x-t)g(x+t)}{|t|^{n-\alpha}}dt$ and $Q_0=Q(x_0,\delta)$ with $\delta>0$. There exists a positive constant independent of f and g, such that*

$$
I\leqslant C\sum_{Q\in\mathscr{D}(Q_0)}l(Q)^{\alpha}m_{3Q}(|f|^r,|g|^s)\chi_Q(x). \tag{15}
$$

Proof. By the definition I, we may get

$$
\begin{aligned}
I&=\int_{|t|\leqslant 2\delta}\frac{f(x-t)g(x+t)}{|t|^{n-\alpha}}dt\\
&=\sum_{k=0}^{+\infty}\int_{2\cdot 2^{-k-1}\delta<|t|\leqslant 2\cdot 2^{-k}\delta}\frac{f(x-t)g(x+t)}{|t|^{n-\alpha}}dt\\
&\leqslant\sum_{k=0}^{+\infty}\frac{1}{(2\cdot 2^k\delta)^{n-\alpha}}\int_{|t|\leqslant 2\cdot 2^{-k}\delta}f(x-t)g(x+t)dt
\end{aligned}
$$

$$\leqslant \sum_{k=0}^{+\infty} \frac{1}{(2\cdot 2^{-k}\delta)^{n-\alpha}} \left(\int_{|t|\leqslant 2\cdot 2^{-k}\delta} |f(x-t)|^r dt\right)^{1/r} \left(\int_{|t|\leqslant 2\cdot 2^{-k}\delta} |g(x+t)|^s dt\right)^{1/s}.$$

Then, by a change of variables and the fact $x \in Q_0$, it is easy to see

$$\begin{aligned} I \leqslant & C\sum_{k=0}^{+\infty} \frac{1}{(2\cdot 2^{-k}\delta)^{n-\alpha}} \left(\int_{|u-x|\leqslant 2\cdot 2^{-k}\delta} |f(u)|^r du\right)^{1/r} \left(\int_{|v-x|\leqslant 2\cdot 2^{-k}\delta} |g(v)|^s dv\right)^{1/s} \\ \leqslant & C\sum_{k=0}^{+\infty} \sum_{\substack{Q\in\mathscr{D}(Q_0)\\ l(Q)=2^{-k}\delta}} l(Q)^{\alpha-n} \left(\int_{|u-x|\leqslant 2l(Q)} |f(u)|^r du\right)^{1/r} \times \left(\int_{|v-x|\leqslant 2l(Q)} |g(v)|^s dv\right)^{1/s} \chi_Q(x) \\ \leqslant & C\sum_{k=0}^{+\infty} \sum_{\substack{Q\in\mathscr{D}(Q_0)\\ l(Q)=2^{-k}\delta}} l(Q)^{\alpha-n} \left(\int_{3Q} |f(u)|^r du\right)^{1/r} \left(\int_{3Q} |g(v)|^s dv\right)^{1/s} \chi_Q(x) \\ = & C \sum_{Q\in\mathscr{D}(Q_0)} l(Q)^{\alpha} m_{3Q}(|f|^r,|g|^s)\chi_Q(x), \end{aligned}$$

which implies

$$I \leqslant C \sum_{Q\in\mathscr{D}(Q_0)} l(Q)^{\alpha} m_{3Q}(|f|^r,|g|^s)\chi_Q(x).$$

Thus, the proof of Lemma 9 has been finished.

Next, we recall some notations from Section 2. For $r,s>1$ with $1/r+1/s=1$, we set

$$\mathscr{D}_0(Q_0) \equiv \{Q\in\mathscr{D}(Q_0) : m_{3Q}(|f|^r,|g|^s) \leqslant a\}$$

and

$$\mathscr{D}_{k,j}(Q_0) \equiv \left\{Q\in\mathscr{D}(Q_0) : Q\subset Q_{k,j}, a^k < m_{3Q}(|f|^r,|g|^s) \leqslant a^{k+1}\right\},$$

where a is the same as in Section 2. Thus, we have

$$\mathscr{D}(Q_0) = \mathscr{D}_0(Q_0)\bigcup\left(\bigcup_{k,j}\mathscr{D}_{k,j}(Q_0)\right).$$

As $q>1$, by duality, there is

$$\left(\int_{Q_0} |I|^q (\omega_1(x)\omega_2(x))^q dx\right)^{1/q} = \sup_{\|h\|_{L^{q'}(Q_0)\leqslant 1}} \|I\omega_1\omega_2 h\|_{L^1(Q_0)}.$$

Then, we denote

$$I_0 := \sum_{Q\in\mathscr{D}_0(Q_0)} l(Q)^{\alpha} m_{3Q}(|f|^r,|g|^s) \int_Q \omega_1(x)\omega_2(x)h(x)dx,$$

and

$$I_{k,j} := \sum_{Q\in\mathscr{D}_{k,j}(Q_0)} l(Q)^{\alpha} m_{3Q}(|f|^r,|g|^s) \int_Q \omega_1(x)\omega_2(x)h(x)dx.$$

From $\mathscr{D}(Q_0)=\mathscr{D}_0(Q_0)\bigcup\left(\bigcup_{k,j}\mathscr{D}_{k,j}(Q_0)\right)$ and (15), we get

$$\left(\int_{Q_0}|I|^q|\omega_1(x)\omega_2(x)|^q dx\right)^{1/q}\leqslant I_0+I_{k,j}. \tag{16}$$

For $I_{k,j}$, recall that $q>1$, $a>1$ and $\alpha>0$. Then, using (10), the Hölder inequality, Lemmas 5 and the property of $\mathscr{D}$, we obtain

$$\begin{aligned}
I_{k,j}\leqslant & a^{k+1}\sum_{Q\in\mathscr{D}_{k,j}(Q_0)} l(Q)^\alpha\int_Q\omega_1(x)\omega_2(x)h(x)dx\\
\leqslant & Ca^{k+1}l(Q_{k,j})^\alpha\int_{Q_{k,j}}\omega_1(x)\omega_2(x)h(x)dx\\
\leqslant & Cam_{3Q_{k,j}}(|f|^r,|g|^s)l(Q_{k,j})^\alpha\int_{Q_{k,j}}\omega_1(x)\omega_2(x)h(x)dx\\
\leqslant & Cam_{3Q_{k,j}}(|f|^r,|g|^s)l(Q_{k,j})^\alpha|Q_{k,j}|\\
&\times\left(\frac{1}{|Q_{k,j}|}\int_{Q_{k,j}}(\omega_1(x)\omega_2(x))^{aq}dx\right)^{1/aq}\left(\frac{1}{|Q_{k,j}|}\int_{Q_{k,j}}|h(x)|^{(aq)'}dx\right)^{1/(aq)'}\\
\leqslant & Ca|E_{k,j}|m_{3Q_{k,j}}(|f|^r,|g|^s)l(Q_{k,j})^\alpha\\
&\times\left(\frac{1}{|Q_{k,j}|}\int_{Q_{k,j}}(\omega_1(x)\omega_2(x))^{aq}dx\right)^{1/aq}\left(\frac{1}{|Q_{k,j}|}\int_{Q_{k,j}}|h(x)|^{(aq)'}dx\right)^{1/(aq)'}\\
\leqslant & Ca\int_{E_{k,j}}m_{3Q_{k,j}}(|f|^r,|g|^s)l(Q_{k,j})^\alpha\left(\frac{1}{|Q_{k,j}|}\int_{Q_{k,j}}(\omega_1(y)\omega_2(y))^{aq}dy\right)^{1/aq}\\
&\times\left(\frac{1}{|Q_{k,j}|}\int_{Q_{k,j}}|h(y)|^{(aq)'}dy\right)^{1/(aq)'}dx\\
\leqslant & Ca\int_{E_{k,j}}\left[M(h^{(aq)'})(x)\right]^{\frac{1}{(aq)'}}\tilde{\mathscr{M}}^{aq}_{\alpha,r,s}(f,g,\vec{\omega})(x)dx,
\end{aligned}$$

where

$$\tilde{\mathscr{M}}^{aq}_{\alpha,r,s}(f,g,\vec{\omega})(x)=\sup_{Q\ni x}l(Q)^\alpha m_{3Q}(|f|^r,|g|^s)\left(\frac{1}{|Q|}\int_Q(\omega_1(y)\omega_2(y))^{aq}dy\right)^{1/aq}.$$

Similarly, there is

$$I_0\leqslant Ca\int_{E_0}\left[M\left(h^{(aq)'}\right)(x)\right]^{\frac{1}{(aq)'}}\tilde{\mathscr{M}}^{aq}_{\alpha,r,s}(f,g,\vec{\omega})(x)dx.$$

Thus, using the boundedness of the Hardy-Littlewood maximal function, the Hölder inequality and the fact $q'>(aq)'$, we obtain

$$I_0+\sum_{k,j}I_{k,j}\leqslant C\left(\int_{Q_0}|\tilde{\mathscr{M}}^{aq}_{\alpha,r,s}(f,g,\vec{\omega})(x)|^q dx\right)^{1/q}\left(\int_{Q_0}\left[M\left(h^{(aq)'}\right)(x)\right]^{\frac{q'}{(aq)'}}\right)^{1/q'}$$

$$\leqslant C\left(\int_{Q_0} |\tilde{\mathscr{M}}^{aq}_{\alpha,r,s}(f,g,\vec{\omega})(x)|^q dx\right)^{1/q}\left(\int_{Q_0}|h(x)|^{q'}dx\right)^{1/q'}$$
$$\leqslant C\left\|\tilde{\mathscr{M}}^{aq}_{\alpha,r,s}(f,g,\vec{\omega})\right\|_{L^q(Q_0)},$$

which implies

$$I_0+\sum_{k,j} I_{k,j} \leqslant C\left\|\tilde{\mathscr{M}}^{aq}_{\alpha,r,s}(f,g,\vec{\omega})\right\|_{L^q(Q_0)}. \tag{17}$$

From the Hölder inequality and the reversed Hölder inequality for $\omega_1^{-r\left(\frac{p_1}{r}\right)'}$ and $\omega_2^{-s\left(\frac{p_2}{s}\right)'}$, there is

$$m_{3Q}(|f|^r,|g|^s)=\left(\frac{1}{|3Q|}\int_{3Q}|f|^r dx\right)^{1/r}\left(\frac{1}{|3Q|}\int_{3Q}|g|^s dx\right)^{1/s}$$
$$\leqslant |3Q|^{-1}\left(\int_{3Q}(|f(x)|^r\omega_1(x)^r)^{\frac{p_1 r}{ar}}dx\right)^{\frac{1}{r}\frac{ar}{p_1}}\left(\int_{3Q}\omega_1(x)^{-r\frac{p_1}{p_1-ar}}dx\right)^{\frac{1}{r}-\frac{a}{p_1}}$$
$$\times\left(\int_{3Q}(|g(x)|^s\omega_2(x)^s)^{\frac{p_2 s}{as}}dx\right)^{\frac{1}{s}\frac{as}{p_2}}\left(\int_{3Q}\omega_2(x)^{-s\frac{p_2}{p_2-as}}dx\right)^{\frac{1}{s}-\frac{a}{p_2}}$$
$$\leqslant |3Q|^{-1}|3Q|^{a/p_1+a/p_2+1/r-a/p_1+1/s-a/p_2}\left(\frac{1}{|3Q|}\int_{3Q}(|f(x)|^r\omega_1(x)^r)^{\frac{p_1}{ar}r}dx\right)^{\frac{1}{r}\frac{ar}{p_1}}$$
$$\times\left(\frac{1}{|3Q|}\int_{3Q}(|g(x)|^s\omega_2(x)^s)^{\frac{p_2}{as}s}dx\right)^{\frac{1}{s}\frac{as}{p_2}}\left(\frac{1}{|3Q|}\int_{3Q}\omega_2(x)^{-s\frac{p_2}{p_2-as}}dx\right)^{\frac{1}{s}-\frac{a}{p_2}}$$
$$\times\left(\frac{1}{|3Q|}\int_{3Q}\omega_1(x)^{-r\frac{p_1}{p_1-ar}}dx\right)^{\frac{1}{r}-\frac{a}{p_1}}$$
$$\leqslant\left(\frac{1}{|3Q|}\int_{3Q}|f(x)\omega_1(x)|^{\frac{p_1}{a}}dx\right)^{\frac{a}{p_1}}\left(\frac{1}{|3Q|}\int_{3Q}|g(x)\omega_2(x)|^{\frac{p_2}{a}}dx\right)^{\frac{a}{p_2}}$$
$$\times\left(\frac{1}{|3Q|}\int_{3Q}\omega_1(x)^{-\frac{rp_1}{p_1-r}}dx\right)^{\frac{1}{r}-\frac{1}{p_1}}\left(\frac{1}{|3Q|}\int_{3Q}\omega_2(x)^{-\frac{sp_2}{p_2-s}}dx\right)^{\frac{1}{s}-\frac{1}{p_2}}.$$

Thus, recalling the definition of $\mathscr{M}_{\alpha,\vec{R}}(\vec{f})(x)$ in Section 2, we obtain

$$\tilde{\mathscr{M}}^{aq}_{\alpha,r,s}(f,g,\vec{\omega})(x)\leqslant C[\vec{\omega}]_{aq_0,q,\left(\frac{sp_1}{s+p_1},\frac{rp_2}{r+p_2}\right)}\mathscr{M}_{\alpha,\frac{\vec{P}}{a}}(f\omega_1,g\omega_2)(x). \tag{18}$$

Using Lemma 8 and (16)-(18), we have

$$\|I\cdot v_{\vec{\omega}}\|_{\mathscr{M}^{q_0}_q}\leqslant C[\vec{\omega}]_{aq_0,q,\left(\frac{sp_1}{s+p_1},\frac{rp_2}{r+p_2}\right)}\left\|\mathscr{M}_{\alpha,\frac{\vec{P}}{a}}(f\omega_1,g\omega_2)\right\|_{\mathscr{M}^{q_0}_q}$$
$$\leqslant C[\vec{\omega}]_{aq_0,q,\left(\frac{sp_1}{s+p_1},\frac{rp_2}{r+p_2}\right)}\|(f\omega_1,g\omega_2)\|_{\mathscr{M}^{p_0}_{\vec{P}}},$$

which implies

$$\|I\cdot \nu_{\vec{\omega}}\|_{\mathscr{M}_q^{q_0}} \leqslant C[\vec{\omega}]_{aq_0,q,\left(\frac{sp_1}{s+p_1},\frac{rp_2}{r+p_2}\right)}\|(f\omega_1,g\omega_2)\|_{\mathscr{M}_{\vec{P}}^{p_0}}. \tag{19}$$

Combining (14) and (19), we finish the proof of Theorem 1 for the case $q>1$.

3.2. The case $q\leqslant 1$

First, we denote

$$L:=\left(\sum_{Q\in\mathscr{D}(Q_0)} l(Q)^{\alpha} m_{3Q}(|f|^r,|g|^s)\chi_Q(x)\right)^q.$$

Since $q\leqslant 1$, we have

$$\begin{aligned}
L&\leqslant \sum_{Q\in\mathscr{D}(Q_0)} l(Q)^{q\alpha} m_{3Q}(|f|^r,|g|^s)^q\chi_Q(x)\\
&\leqslant\left(\sum_{Q\in\mathscr{D}_0(Q_0)}+\sum_{k,j}\sum_{Q\in\mathscr{D}_{k,j}(Q_0)}\right) l(Q)^{q\alpha} m_{3Q}(|f|^r,|g|^s)^q\chi_Q(x).
\end{aligned}$$

Recall that $\nu_{\vec{\omega}}(x)=\omega_1(x)\omega_2(x)$. Then, we obtain

$$\begin{aligned}
&\int_{Q_0}|B\mathscr{I}_\alpha(f,g)(x)|^q(\omega_1(x)\omega_2(x))^q dx\\
\leqslant& C\left(\sum_{Q\in\mathscr{D}_0(Q_0)}+\sum_{k,j}\sum_{Q\in\mathscr{D}_{k,j}(Q_0)}\right) l(Q)^{\alpha q} m_{3Q}(|f|^r,|g|^s)^q\int_Q(\omega_1(x)\omega_2(x))^q dx\\
:=&C(I_0'+\sum_{k,j}I_{k,j}').
\end{aligned}$$

For $I_{k,j}'$, there is

$$\begin{aligned}
I_{k,j}'&=\sum_{Q\in\mathscr{D}_{k,j}(Q_0)} l(Q)^{\alpha q} m_{3Q}(|f|^r,|g|^s)^q\int_Q(\omega_1(x)\omega_2(x))^q dx\\
&\leqslant l(Q_{k,j})^{\alpha q}\left(a^{k+1}\right)^q\int_{Q_{k,j}}(\omega_1(x)\omega_2(x))^q dx\\
&\leqslant Ca|Q_{k,j}|l(Q_{k,j})^{\alpha q}\left(a^{k+1}\right)^q m_{3Q_{k,j}}(|f|^r,|g|^s)^q\left(\frac{1}{|Q_{k,j}|}\int_{Q_{k,j}}(\omega_1(x)\omega_2(x))^q dx\right)\\
&\leqslant Ca|E_{k,j}|l(Q_{k,j})^{\alpha q} m_{3Q_{k,j}}(|f|^r,|g|^s)^q\left(\frac{1}{|Q_{k,j}|}\int_{Q_{k,j}}(\omega_1(x)\omega_2(x))^q dx\right)\\
&\leqslant Ca\int_{E_{k,j}}\left[l(Q_{k,j})^{\alpha} m_{3Q_{k,j}}(|f|^r,|g|^s)\left(\frac{1}{|Q_{k,j}|}\int_{Q_{k,j}}(\omega_1(y)\omega_2(y))^q dy\right)^{1/q}\right]^q dx
\end{aligned}$$

$$\leqslant Ca\int_{E_{k,j}}\tilde{\mathscr{M}}^q_{\alpha,r,s}(f,g,\omega_1,\omega_2)(x)^q dx,$$

where

$$\tilde{\mathscr{M}}^q_{\alpha,r,s}(f,g,\omega_1,\omega_2)(x)=\sup_{Q\ni x} l(Q)^{\alpha} m_{3Q}(|f|^r,|g|^s)\left(\frac{1}{|Q|}\int_Q(\omega_1(y)\omega_2(y))^q dy\right)^{1/q}.$$

Similarly, there is

$$I_0'\leqslant Ca\int_{E_0}\tilde{\mathscr{M}}^q_{\alpha,r,s}(f,g,\vec{\omega})(x)^q dx.$$

Thus, we obtain

$$I_0'+\sum_{k,j}I_{k,j}'\leqslant C\int_{Q_0}\tilde{\mathscr{M}}^q_{\alpha,r,s}(f,g,\vec{\omega})(x)^q dx.$$

Then, by a similar argument as in the proof of (18), there is

$$\tilde{\mathscr{M}}^q_{\alpha,r,s}(f,g,\vec{\omega})(x)\leqslant[\vec{\omega}]_{aq_0,q,\left(\frac{sp_1}{s+p_1},\frac{rp_2}{r+p_2}\right)}\mathscr{M}_{\alpha,\frac{\vec{P}}{a}}(f\omega_1,g\omega_2)(x). \tag{20}$$

Now, using Lemma 8 and the definition of the Morrey space, we finish the proof of Theorem 1 with $q\leqslant 1$.

4. Two-weight norm inequalities for $B\mathscr{I}_\alpha$

In this section, we are going to give the two-weight norm inequalities for $B\mathscr{I}_\alpha$ on the Morrey type spaces. Suppose that v and $\vec{\omega}=(\omega_1,\omega_2)$ satisfy the following condition:

$$[v,\vec{\omega}]_{q_0,q,\vec{P}}:=\sup_{\substack{Q\subset Q'\\ Q,Q':\text{cubes}}}\left(\frac{|Q|}{|Q'|}\right)^{1/q_0}\left(\frac{1}{|Q|}\int_Q v(x)^q dx\right)^{1/q}\prod_{i=1}^{2}\left(\frac{1}{|Q|}\int_{Q'}\omega_i(y_i)^{-p_i'}dy_i\right)^{1/p_i'}.$$

Obviously, if $[v,\vec{\omega}]_{q_0,q,\left(\frac{sp_1}{s+p_1},\frac{rp_2}{r+p_2}\right)}<\infty$, we cannot get the reversed Hölder inequality for v, $\omega_1^{-r\left(\frac{p_1}{r}\right)'}$ and $\omega_2^{-s\left(\frac{p_2}{s}\right)'}$.

By checking the proof of Theorem 1, we obtain

THEOREM 2. *Suppose* $0<\alpha<n$, $p_1>r>1$, $p_2>s>1$, $1/r+1/s=1$, $1/p=1/p_1+1/p_2$, $1<p_1,p_2<\infty$, $0<p\leqslant p_0<\infty$, $0<q\leqslant q_0<\infty$. *Assume that*

$$1/q_0=1/p_0-\alpha/n,\ \ q/q_0=p/p_0.$$

Case 1. *If* $q>1$, *suppose that there exists* a *satisfying* $1<a<\min\left\{\frac{p_1}{s'},\frac{p_2}{r'}\right\}$, *such that*

$$[v,\vec{\omega}]_{aq_0,aq,\left(\frac{sp_1}{as+p_1},\frac{rp_2}{ar+p_2}\right)}<\infty.$$

Then, there exists a positive constant C independent of f and g, such that

$$\|B\mathscr{I}_\alpha(f,g)v\|_{\mathscr{M}_q^{q_0}} \leqslant C[v,\vec{\omega}]_{aq_0,aq,\left(\frac{sp_1}{as+p_1},\frac{rp_2}{ar+p_2}\right)}\|(f\omega_1,g\omega_2)\|_{\mathscr{M}_{\vec{P}}^{p_0}}. \tag{21}$$

Case 2. *If $0<q\leqslant 1$, suppose that there exists a satisfying $1<a<\min\left\{\frac{p_1}{s'},\frac{p_2}{r'}\right\}$, such that*

$$[v,\vec{\omega}]_{aq_0,q,\left(\frac{sp_1}{as+p_1},\frac{rp_2}{ar+p_2}\right)}<\infty.$$

Then, there exists a positive constant C independent of f and g, such that

$$\|B\mathscr{I}_\alpha(f,g)v\|_{\mathscr{M}_q^{q_0}} \leqslant C[v,\vec{\omega}]_{aq_0,q,\left(\frac{sp_1}{as+p_1},\frac{rp_2}{ar+p_2}\right)}\|(f\omega_1,g\omega_2)\|_{\mathscr{M}_{\vec{P}}^{p_0}}. \tag{22}$$

In order to prove Theorem 2, recalling the definition of $\tilde{\mathscr{M}}_{\alpha,r,s}^{aq}(f,g,\vec{\omega})(x)$ and $\tilde{\mathscr{M}}_{\alpha,r,s}^{q}(f,g,\vec{\omega})(x)$ in Section 3, we need the following lemma.

LEMMA 10. *Under the same conditions as in Theorem 2, we have the following estimates for $\tilde{\mathscr{M}}_{\alpha,r,s}^{aq}$ and $\tilde{\mathscr{M}}_{\alpha,r,s}^{q}$.*

Case 1. *For the case $q>1$, suppose that there exists a satisfying $1<a<\min\left\{\frac{p_1}{s'},\frac{p_2}{r'}\right\}$, such that*

$$[v,\vec{\omega}]_{aq_0,aq,\left(\frac{sp_1}{as+p_1},\frac{rp_2}{ar+p_2}\right)}<\infty.$$

Then

$$\tilde{\mathscr{M}}_{\alpha,r,s}^{aq}(f,g,\vec{\omega})(x) \leqslant C[v,\vec{\omega}]_{aq_0,aq,\left(\frac{sp_1}{as+p_1},\frac{rp_2}{ar+p_2}\right)}\mathscr{M}_{\alpha,\frac{\vec{P}}{a}}(f\omega_1,g\omega_2)(x).$$

Case 2. *For the case $q\leqslant 1$, suppose that there exists a satisfying $1<a<\min\left\{\frac{p_1}{s'},\frac{p_2}{r'}\right\}$, such that*

$$[v,\vec{\omega}]_{aq_0,q,\left(\frac{sp_1}{as+p_1},\frac{rp_2}{ar+p_2}\right)}<\infty.$$

Then

$$\tilde{\mathscr{M}}_{\alpha,r,s}^{q}(f,g,\vec{\omega})(x) \leqslant C[v,\vec{\omega}]_{aq_0,q,\left(\frac{sp_1}{as+p_1},\frac{rp_2}{ar+p_2}\right)}\mathscr{M}_{\alpha,\frac{\vec{P}}{a}}(f\omega_1,g\omega_2)(x).$$

If we check the proof of (18) and (20) carefully, we can easily get Lemma 10 and we omit the details here.

Moreover, we can generalize Theorem 2 to a more general case.

Suppose that another quantity of two-weight type multiple weights $[v,\vec{\omega}]_{q_0,r_0,q,\vec{P}}$ is defined as follows.

$$[v,\vec{\omega}]_{q_0,r_0,q,\vec{P}} := \sup_{\substack{Q\subset Q'\\ Q,Q':\text{cubes}}}\left(\frac{|Q|}{|Q'|}\right)^{1/q_0}|Q'|^{1/r_0}\left(\frac{1}{|Q|}\int_Q v(x)^q dx\right)^{1/q}\prod_{i=1}^{2}\left(\frac{1}{|Q|}\int_{Q'}\omega_i(y_i)^{-p_i'}dy_i\right)^{1/p_i'}<\infty.$$

By checking the proof of Theorem 1 again, we have

THEOREM 3. *Suppose* $0<\alpha<n$, $p_1>r>1$, $p_2>s>1$, $1/r+1/s=1$, $1/p=1/p_1+1/p_2$, $1<p_1,p_2<\infty$, $0<p\leqslant p_0<\infty$, $0<q\leqslant q_0<\infty$. *Assume that*

$$q/q_0=p/p_0,\ \ 1/q_0=1/p_0+1/r_0-\alpha/n,\ \ r_0\geqslant \frac{n}{\alpha}.$$

Case 1. *If* $q>1$, *suppose that there exists* a *satisfying* $1<a<\min\left\{\frac{r_0}{q_0},\frac{p_1}{s'},\frac{p_2}{r'}\right\}$, *such that*

$$[v,\vec{\omega}]_{aq_0,r_0,aq,\left(\frac{sp_1}{as+p_1},\frac{rp_2}{ar+p_2}\right)}<\infty.$$

Then, there exists a positive constant C *independent of* f *and* g, *such that*

$$\|B\mathscr{I}_\alpha(f,g)v\|_{\mathscr{M}_q^{q_0}}\leqslant C[v,\vec{\omega}]_{aq_0,r_0,aq,\left(\frac{sp_1}{as+p_1},\frac{rp_2}{ar+p_2}\right)}\|(f\omega_1,g\omega_2)\|_{\mathscr{M}_{\vec{P}}^{p_0}}. \tag{23}$$

Case 2. *If* $0<q\leqslant 1$, *suppose that there exists* a *satisfying* $1<a<\min\left\{\frac{r_0}{q_0},\frac{p_1}{s'},\frac{p_2}{r'}\right\}$, *such that*

$$[v,\vec{\omega}]_{aq_0,r_0,q,\left(\frac{sp_1}{as+p_1},\frac{rp_2}{ar+p_2}\right)}<\infty.$$

Then, there exists a positive constant C *independent of* f *and* g, *such that*

$$\|B\mathscr{I}_\alpha(f,g)v\|_{\mathscr{M}_q^{q_0}}\leqslant C[v,\vec{\omega}]_{aq_0,r_0,q,\left(\frac{sp_1}{as+p_1},\frac{rp_2}{ar+p_2}\right)}\|(f\omega_1,g\omega_2)\|_{\mathscr{M}_{\vec{P}}^{p_0}}. \tag{24}$$

Similarly, to prove Theorem 2, we need the following lemma.

LEMMA 11. *Under the same conditions as in Theorem 3, we have the following estimates for* $\tilde{\mathscr{M}}^{aq}_{\alpha,r,s}$ *and* $\tilde{\mathscr{M}}^{q}_{\alpha,r,s}$.

Case 1. *For the case* $q>1$, *suppose that there exists* a *satisfying* $1<a<\min\left\{\frac{r_0}{q_0},\frac{p_1}{s'},\frac{p_2}{r'}\right\}$, *such that*

$$[v,\vec{\omega}]_{aq_0,r_0,aq,\left(\frac{sp_1}{as+p_1},\frac{rp_2}{ar+p_2}\right)}<\infty.$$

Then

$$\tilde{\mathscr{M}}^{aq}_{\alpha,r,s}(f,g,\vec{\omega})(x)\leqslant C[v,\vec{\omega}]_{aq_0,r_0,aq,\left(\frac{sp_1}{as+p_1},\frac{rp_2}{ar+p_2}\right)}\mathscr{M}_{\alpha-\frac{n}{r_0},\frac{\vec{P}}{a}}(f\omega_1,g\omega_2)(x).$$

Case 2. *For the case* $0<q\leqslant 1$, *suppose that there exists* a *satisfying* $1<a<\min\left\{\frac{r_0}{q_0},\frac{p_1}{s'},\frac{p_2}{r'}\right\}$, *such that*

$$[v,\vec{\omega}]_{aq_0,r_0,q,\left(\frac{sp_1}{as+p_1},\frac{rp_2}{ar+p_2}\right)}<\infty.$$

Then

$$\tilde{\mathscr{M}}^{q}_{\alpha,r,s}(f,g,\vec{\omega})(x)\leqslant C[v,\vec{\omega}]_{aq_0,r_0,q,\left(\frac{sp_1}{as+p_1},\frac{rp_2}{ar+p_2}\right)}\mathscr{M}_{\alpha-\frac{n}{r_0},\frac{\vec{P}}{a}}(f\omega_1,g\omega_2)(x).$$

REMARK 3. For the case $0<q\leqslant 1$, the results of (22) and (24) are still different from [17, Theorem 4.2].

5. An Olsen type inequality for $B\mathscr{I}_\alpha$

In this section, we will give an Olsen type inequality for $B\mathscr{I}_\alpha$. Recall the fractional integral

$$I_\alpha(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0 < \alpha < n.$$

For the study of I_α on the Morrey space, one may see [1, 23, 24] et al. to find more details.

Particularly, Sawano, Sugano and Tanaka obtained the following result.

THEOREM C. *([40]) Suppose that the indices $\alpha, p_0, q_0, r_0, p, q, r_1$ satisfy*

$$1 < p \leqslant p_0 < \infty,\ 1 < q \leqslant q_0 < \infty, 1 < r_1 \leqslant r_0 < \infty$$

and

$$r_1 > q,\ 1/p_0 > \alpha/n \geqslant 1/r_0.$$

Also assume

$$q/q_0 = p/p_0,\ 1/p_0 + 1/r_0 - \alpha/n = 1/q_0.$$

Then, for all $f \in \mathscr{M}_p^{p_0}(\mathbb{R}^n)$ and $h \in \mathscr{M}_{r_1}^{r_0}(\mathbb{R}^n)$, there is

$$\|h \cdot I_\alpha(f)\|_{\mathscr{M}_q^{q_0}(\mathbb{R}^n)} \leqslant C \|f\|_{\mathscr{M}_p^{p_0}(\mathbb{R}^n)} \|h\|_{\mathscr{M}_{r_1}^{r_0}(\mathbb{R}^n)}, \tag{25}$$

where C is a positive constant independet of f and g.

The above inequality was first proposed by Olsen in [38] and Olsen found that (25) plays an important role in the study of Schrödinger equation. Conlon and Redondo proved (25) for the case $n = 3$ in [9] essentially. In fact, some analogous inequalities on a generalized case were obtained in [40, 41, 44] et al. Moreover, we would like to mention that readers may see [12, 13] et al. to find more applications about Olsen type inequalities in the study of PDEs.

For the Olsen type inequality of $B\mathscr{I}_\alpha$, we would like to mention that if we take $v = h$ and $\vec{\omega} = (1, 1, \cdots, 1)$ in Theorem 3, we may obtain

THEOREM 4. *Under the same conditions as in Theorem 3, there is*
Case 1. *For the $q > 1$, we have*

$$\|h \cdot B\mathscr{I}_\alpha(f,g)\|_{\mathscr{M}_q^{q_0}} \leqslant C\|h\|_{\mathscr{M}_{r_1}^{r_0}} \|(f,g)\|_{\mathscr{M}_{\vec{P}}^{p_0}} \leqslant C\|h\|_{\mathscr{M}_{r_1}^{r_0}} \|f\|_{\mathscr{M}_{p_1}^{q_1}} \|g\|_{\mathscr{M}_{p_2}^{q_2}}, \tag{26}$$

for all $h \in \mathscr{M}_{r_1}^{r_0}$, $1/q_1 + 1/q_2 = 1/p_0$ and $r_1 = aq$.
Case 2. *For the case $0 < q \leqslant 1$, we have*

$$\|h \cdot B\mathscr{I}_\alpha(f,g)\|_{\mathscr{M}_q^{q_0}} \leqslant C\|h\|_{\mathscr{M}_q^{r_0}} \|\vec{f}\|_{\mathscr{M}_{\vec{P}}^{p_0}} \leqslant C\|h\|_{\mathscr{M}_q^{r_0}} \|f\|_{\mathscr{M}_{p_1}^{q_1}} \|g\|_{\mathscr{M}_{p_2}^{q_2}}, \tag{27}$$

for all $h \in \mathscr{M}_q^{r_0}$ and $1/q_1 + 1/q_2 = 1/p_0$.

According to the conditions of Theorem 3, we find that the exponent $r_1 = aq$ in (26) should satisfy the condition $r_1 \in \left(q, q\cdot\min\left\{\frac{r_0}{q_0}, \frac{p_1}{s'}, \frac{p_2}{r'}\right\}\right) \subsetneq (q, r_0)$. Then, comparing (25) with (26), it is natural to ask whether we can get the following Olsen type inequality for $B\mathscr{I}_\alpha$,

$$\|h\cdot B\mathscr{I}_\alpha(f,g)\|_{\mathscr{M}_q^{q_0}} \leqslant C\|h\|_{\mathscr{M}_{r_1}^{r_0}}\|f\|_{\mathscr{M}_{p_1}^{q_1}}\|g\|_{\mathscr{M}_{p_2}^{q_2}} \tag{28}$$

with any $r_1 \in (q, r_0]$ and $q > 1$. In this section, we will give a positive answer to this question. The main result of this section is

THEOREM 5. *Suppose that there exist real numbers α, q_i, p_i $(i=1,2), r_0,\ r_1, s, q_0$ and q satisfying $0<\alpha<n, 1<q_i \leqslant p_i<\infty, 1<q\leqslant q_0<\infty, 1<r_1\leqslant r_0, p_1>r>1, p_2>s>1$ and*

$$r_1 > q, 1/r_0 < \alpha/n < 1/q_1 + 1/q_2 < 1, 1/s + 1/r = 1.$$

Furthermore, we assume that

$$1/q_0 = 1/r_0 + 1/q_1 + 1/q_2 - \alpha/n$$

and

$$\frac{q}{q_0} = \frac{p_1}{q_1} = \frac{p_2}{q_2}. \tag{29}$$

Then, there exists a positive constant C independent of f and g, such that

$$\|h\cdot B\mathscr{I}_\alpha(f,g)\|_{\mathscr{M}_q^{q_0}} \leqslant C\|h\|_{\mathscr{M}_{r_1}^{r_0}}\|f\|_{\mathscr{M}_{p_1}^{q_1}}\|g\|_{\mathscr{M}_{p_2}^{q_2}}$$

for any $h \in \mathscr{M}_{r_1}^{r_0}(\mathbb{R}^n)$.

The method for the proof of Theorem 5 is also adapted to the case $q = \infty$ and $h \equiv 1$. Thus, we may obtain the following Spanne type estimates for $B\mathscr{I}_\alpha$ and it is also a new result with its independent interest as far as we know.

COROLLARY 2. *(The Spanne type estimate for $B\mathscr{I}_\alpha$) Suppose that there exist real numbers α, q_i, p_i $(i=1,2), r, s, q_0$ and q satisfying $0<\alpha<n, 1<p_i\leqslant q_i<\infty, 1<q\leqslant q_0<\infty, p_1>r>1, p_2>s>1$ and*

$$\alpha/n < 1/q_1 + 1/q_2 < 1, \ \ 1/s + 1/r = 1.$$

Furthermore, we assume that

$$1/q_0 = 1/q_1 + 1/q_2 - \alpha/n$$

and

$$\frac{q_0}{q} = \frac{q_1}{p_1} = \frac{q_2}{p_2}.$$

Then, there exists a positive constant C independent of f and g, such that

$$\|B\mathscr{I}_\alpha(f,g)\|_{\mathscr{M}_q^{q_0}} \leqslant C\|f\|_{\mathscr{M}_{p_1}^{q_1}}\|g\|_{\mathscr{M}_{p_2}^{q_2}}.$$

REMARK 4. Here we would like to mention that we cannot get Theorems 5 directly from Corollary 2 and the Hölder inequality for functions on the Morrey spaces ([26, p.1377]). Readers may see [39, 40, 44] for details. In fact, from Corollary 2 and the Hölder inequality for functions on the Morrey spaces, there is

$$\|h\cdot B\mathscr{I}_\alpha(f,g)\|_{\mathscr{M}_q^{q_0}} \leqslant C\|h\|_{\mathscr{M}_{r_1}^{r_0}}\|f\|_{\mathscr{M}_{p_1}^{q_1}}\|g\|_{\mathscr{M}_{p_2}^{q_2}}, \tag{30}$$

where

$$\frac{r_0}{r_1}=\frac{q_0}{q}=\frac{q_1}{p_1}=\frac{q_2}{p_2} \tag{31}$$

and the other conditions are the same as in Theorem 5.

REMARK 5. Comparing (29) with (31), we find that the restriction of (31) is much more stronger than (29).

REMARK 6. In [11], Fan and Gao [11, Corollary 2.5] got an Olsen type inequality for $B\mathscr{I}_\alpha$ which is similar to (30). If we check [11, Corollary 2.5] carefully, we find that the exponents $q,q_0,r_1,r_0,q_1,p_1,q_2,p_2$ in [11, Corollary 2.5] also satisfy (31). However, our result shows that the condition (31) is unnecessary as the method used in this paper is quite different and more difficult from [11].

5.1. Proof of Theorem 5

Without loss of generality, we may assume that both f and g are non-negative functions. From Lemma 4 and the fact $q\leqslant q_0$, then for any cube $Q\subset\mathbb{R}^n$, there is

$$\begin{aligned}&|Q|^{1/q_0-1/q}\left(\int_Q |h(x)B\mathscr{I}_\alpha(f,g)(x)|^q dx\right)^{1/q} \\ \leqslant& 6^n\sum_{t=1}^{3^n}|Q_t|^{1/q_0-1/q}\left(\int_{Q_t}|h(x)B\mathscr{I}_\alpha(f,g)(x)|^q dx\right)^{1/q},\end{aligned} \tag{32}$$

where $Q_t\in\mathscr{D}^t$, $Q\subset Q_t$ and $l(Q_t)\leqslant 6l(Q)$.

Thus, we only need to estimate $|Q_0|^{1/q_0-1/q}\left(\int_{Q_0}|h(x)B\mathscr{I}_\alpha(f,g)(x)|^q dx\right)^{1/q}$ with $Q_0\in\mathscr{D}^t$.

From (ii) in Section 2, we know that the set $\{Q\in\mathscr{D}^t : l(Q)=2^{-\nu}\}$ forms a partition of $\mathbb{R}^n$ with a fixed t and each $\nu\in\mathbb{Z}$. Moreover, we denote $Q\in\mathscr{D}_\nu^t$ with $l(Q)=2^{-\nu}$ and let $3Q$ be made up of 3^n dyadic grids of equal size and have the same center of Q. Then, using the notations as in Section 2, we can decompose $B\mathscr{I}_\alpha$ as follows.

$$\begin{aligned}B\mathscr{I}_\alpha(f,g)(x)&=\int_{\mathbb{R}^n}\frac{f(x-y)g(x+y)}{|y|^{n-\alpha}}dy=\sum_{\nu\in\mathbb{Z}}\int_{2^{-\nu-1}<|y|\leqslant 2^{-\nu}}\frac{f(x-y)g(x+y)}{|y|^{n-\alpha}}dy\\ &\leqslant\sum_{\nu\in\mathbb{Z}}\sum_{Q\in\mathscr{D}_\nu^t}2^{\nu(n-\alpha)}\chi_Q(x)\int_{2^{-\nu-1}<|y|\leqslant 2^{-\nu}}f(x-y)g(x+y)dy.\end{aligned}$$

Then, by a geometric observation, we have $B(x,2^{-\nu}) \subset 3Q$ if $x \in Q \in \mathscr{D}_\nu^t$. Thus, using the Hölder inequality with $1/r + 1/s = 1$ $(r,s>1)$ and a change of variables, there is

$$\begin{aligned}
&\int_{2^{-\nu-1}<|y|\leqslant 2^{-\nu}} f(x-y)g(x+y)dy \\
&\leqslant \left(\int_{2^{-\nu-1}<|y|\leqslant 2^{-\nu}} |f(x-y)|^r dy\right)^{1/r} \left(\int_{2^{-\nu-1}<|y|\leqslant 2^{-\nu}} |g(x+y)|^s dy\right)^{1/s} \\
&\leqslant \left(\int_{2^{-\nu-1}<|x-u|\leqslant 2^{-\nu}} |f(u)|^r du\right)^{1/r} \left(\int_{2^{-\nu-1}<|x-z|\leqslant 2^{-\nu}} |g(z)|^s dz\right)^{1/s} \\
&\leqslant \left(\int_{B(x,2^{-\nu})} |f(u)|^r du\right)^{1/r} \left(\int_{B(x,2^{-\nu})} |g(z)|^s dz\right)^{1/s} \\
&\leqslant \left(\int_{3Q} |f(u)|^r du\right)^{1/r} \left(\int_{3Q} |g(z)|^s dz\right)^{1/s}.
\end{aligned}$$

Then, for any cube fixed cube $Q_0 \in \mathscr{D}^t$, as $Q \in \mathscr{D}_\nu^t$, we denote

$$I = h(x)\sum_{\nu\in\mathbb{Z}}\sum_{Q\in\mathscr{D}_\nu^t, Q\supset Q_0} \chi_Q(x)2^{\nu(n-\alpha)}\left(\int_{3Q}|f(u)|^r du\right)^{1/r}\left(\int_{3Q}|g(z)|^s dz\right)^{1/s}$$

and

$$II = h(x)\sum_{\nu\in\mathbb{Z}}\sum_{Q\in\mathscr{D}_\nu^t, Q\subset Q_0} \chi_Q(x)2^{\nu(n-\alpha)}\left(\int_{3Q}|f(u)|^r du\right)^{1/r}\left(\int_{3Q}|g(z)|^s dz\right)^{1/s}.$$

Thus, it is easy to see

$$h(x)\cdot B\mathscr{I}_\alpha(f,g)(x) \leqslant I + II.$$

For I, let Q_k be the unique cube containing Q_0 and satisfy $|Q_k| = 2^{kn}|Q_0|$. Set $\nu = -\log_2|Q_k|^{\frac{1}{n}}$. Then, we denote

$$\begin{aligned}
E_k = |Q_0|^{1/q_0-1/q}\Bigg\{\int_{Q_0}\Bigg|&2^{\nu(n-\alpha)}\chi_{Q_k}(x)h(x)\left(\int_{3Q_k}|f(u)|^r du\right)^{1/r} \\
&\times\left(\int_{3Q_k}|g(z)|^s dz\right)^{1/s}\Bigg|^q dx\Bigg\}^{1/q}.
\end{aligned}$$

Next, we will give the estimates of E_k. By the definition of the Morrey space and the condition $1/r + 1/s = 1$ with $r,s>1$, we see that

$$\begin{aligned}
&\left(\int_{3Q_k}|f(u)|^r du\right)^{1/r}\left(\int_{3Q_k}|g(z)|^s dz\right)^{1/s} \\
&\leqslant \left(\int_{3Q_k}|f(u)|^{p_1} du\right)^{1/p_1}|3Q_k|^{1/r-1/p_1}\left(\int_{3Q_k}|g(z)|^{p_2} dz\right)^{1/p_2}|3Q_k|^{1/s-1/p_2}
\end{aligned}$$

$$\leqslant \|f\|_{\mathscr{M}^{q_1}_{p_1}} |3Q_k|^{1/r-1/p_1-1/q_1+1/p_1} \|g\|_{\mathscr{M}^{q_2}_{p_2}} |3Q_k|^{1/s-1/p_2+1/p_2-1/q_2}$$
$$\leqslant \|f\|_{\mathscr{M}^{q_1}_{p_1}} \|g\|_{\mathscr{M}^{q_2}_{p_2}} |3Q_k|^{1-1/q_1-1/q_2}.$$

Thus, we obtain

$$\begin{aligned} E_k &\leqslant \|f\|_{\mathscr{M}^{q_1}_{p_1}} \|g\|_{\mathscr{M}^{q_2}_{p_2}} |3Q_k|^{1-1/q_1-1/q_2} |Q_0|^{1/q_0-1/q} 2^{\nu(n-\alpha)} \left(\int_{Q_0} |h(x)|^q dx \right)^{1/q} \\ &\leqslant \|f\|_{\mathscr{M}^{q_1}_{p_1}} \|g\|_{\mathscr{M}^{q_2}_{p_2}} |3Q_k|^{1-1/q_1-1/q_2} |Q_0|^{1/q_0-1/q+1/q-1/r_1} 2^{\nu(n-\alpha)} \left(\int_{Q_0} |h(x)|^{r_1} dx \right)^{1/r_1} \\ &\leqslant \|h\|_{\mathscr{M}^{r_0}_{r_1}} \|f\|_{\mathscr{M}^{q_1}_{p_1}} \|g\|_{\mathscr{M}^{q_2}_{p_2}} |Q_0|^{1/q_0-1/r_0} |Q_k|^{1-1/q_1-1/q_2} 2^{\nu(n-\alpha)}. \end{aligned}$$

By the facts $2^{\nu(n-\alpha)} = \left(2^{-\log_2 |Q_k|^{\frac{1}{n}}}\right)^{n-\alpha} = |Q_k|^{-\frac{1}{n}(n-\alpha)} = |Q_k|^{\frac{\alpha}{n}-1}$ and $|Q_k| = 2^{kn}|Q_0|$, we get

$$\begin{aligned} E_k &\leqslant \|h\|_{\mathscr{M}^{r_0}_{r_1}} \|f\|_{\mathscr{M}^{q_1}_{p_1}} \|g\|_{\mathscr{M}^{q_2}_{p_2}} |Q_0|^{1/q_0-1/r_0+\alpha/n-1/q_1-1/q_2} 2^{kn(\alpha/n-1/q_1-1/q_2)} \\ &\leqslant \|h\|_{\mathscr{M}^{r_0}_{r_1}} \|f\|_{\mathscr{M}^{q_1}_{p_1}} \|g\|_{\mathscr{M}^{q_2}_{p_2}} 2^{kn(\alpha/n-1/q_1-1/q_2)}. \end{aligned}$$

Recall that Q_k is the unique cube containing Q_0. By the condition that $1/q_1 + 1/q_2 - \alpha/n > 0$, and the definitions of I and E_k, we obtain

$$|Q_0|^{1/q_0-1/q} \left(\int_{Q_0} |I|^q dx \right)^{1/q} \leqslant C \|h\|_{\mathscr{M}^{r_0}_{r_1}} \|f\|_{\mathscr{M}^{q_1}_{p_1}} \|g\|_{\mathscr{M}^{q_2}_{p_2}}. \tag{33}$$

Next, we recall some notations from Section 3. For $r,s > 1$ with $1/r + 1/s = 1$, we set

$$\mathscr{D}^t_0(Q_0) \equiv \{Q \in \mathscr{D}^t(Q_0) : m_{3Q}(|f|^r, |g|^s) \leqslant a\}$$

and

$$\mathscr{D}^t_{k,j}(Q_0) \equiv \left\{ Q \in \mathscr{D}^t(Q_0) : Q \subset Q_{k,j}, a^k < m_{3Q}(|f|^r, |g|^s) \leqslant a^{k+1} \right\},$$

where a is the same as in Section 2 and $\mathscr{D}^t(Q_0) \equiv \{Q \in \mathscr{D}^t : Q \subset Q_0\}$.
Thus, we have

$$\mathscr{D}^t(Q_0) = \mathscr{D}^t_0(Q_0) \cup \bigcup_{k,j} \mathscr{D}^t_{k,j}(Q_0).$$

By the duality theory, we may choose a function $\omega \in L^{q'}$, such that

$$\left(\int_{Q_0} |II|^q dx \right)^{1/q} \leqslant 2 \int_{Q_0} |II| \omega(x) dx. \tag{34}$$

Thus, we get

$$\left(\int_{Q_0} |II|^q dx \right)^{1/q}$$

$$
\begin{aligned}
&= \sum_{Q\in\mathscr{D}_0^t(Q_0)} 2^{\nu(n-\alpha)}\int_Q h(x)\omega(x)dx\left(\int_{3Q}|f(u)|^r du\right)^{1/r}\left(\int_{3Q}|g(z)|^s dz\right)^{1/s}\\
&\quad+\sum_{k,j}\sum_{Q\in\mathscr{D}_{k,j}^t(Q_0)} 2^{\nu(n-\alpha)}\int_Q h(x)\omega(x)dx\left(\int_{3Q}|f(u)|^r du\right)^{1/r}\left(\int_{3Q}|g(z)|^s dz\right)^{1/s}\\
&:= II_1+II_2.
\end{aligned}
$$

To estimate II_2, using (10), Lemma 5, the definition of $\mathscr{D}_{k,j}(Q_0)$, the geometric property of $\mathscr{D}$ and the fact $0<\frac{\alpha}{n}<1$, there is

$$
\begin{aligned}
II_2 &\leqslant \sum_{k,j}\sum_{Q\in\mathscr{D}_{k,j}^t(Q_0)} 2^{\nu(n-\alpha)}\int_Q h(x)\omega(x)dx|3Q|m_{3Q}(|f|^r,|g|^s)\\
&\leqslant C\sum_{k,j}\sum_{Q\in\mathscr{D}_{k,j}^t(Q_0)}|Q|^{\frac{\alpha}{n}}m_{3Q}(|f|^r,|g|^s)\int_Q h(x)\omega(x)dx\\
&= C\sum_{k,j}\sum_{Q\in\mathscr{D}_{k,j}^t(Q_0)}|Q|^{\frac{\alpha}{n}}m_{3Q}(|f|^r,|g|^s)\frac{|Q|}{|Q|}\int_Q h(x)\omega(x)dx\\
&\leqslant C\sum_{k,j}\sum_{Q\in\mathscr{D}_{k,j}^t(Q_0)}|Q|^{\frac{\alpha}{n}}m_{3Q}(|f|^r,|g|^s)\int_Q M(h\omega)(x)dx\\
&\leqslant C\sum_{k,j}|Q_{k,j}|^{\frac{\alpha}{n}}m_{3Q_{k,j}}(|f|^r,|g|^s)\int_{Q_{k,j}} M(h\omega)(x)dx\\
&\leqslant C\sum_{k,j}|Q_{k,j}|^{\frac{\alpha}{n}}m_{3Q_{k,j}}(|f|^r,|g|^s)m_{Q_{k,j}}[M(h\omega)]|Q_{k,j}|\\
&\leqslant C\sum_{k,j}|Q_{k,j}|^{\frac{\alpha}{n}}m_{3Q_{k,j}}(|f|^r,|g|^s)m_{Q_{k,j}}[M(h\omega)]|E_{k,j}|.
\end{aligned}
$$

Thus, for any θ satisfying $1<q<\theta<r_1$, we have

$$
\begin{aligned}
II_2 \leqslant& C\sum_{k,j}|Q_{k,j}|^{\frac{\alpha}{n}}m_{3Q_{k,j}}(|f|^r,|g|^s)\\
&\times|E_{k,j}|\left(m_{Q_{k,j}}\left(\left(M^{(\theta')}\omega\right)^{r_1'}\right)\right)^{1/r_1'}\left(m_{Q_{k,j}}\left(\left(M^{(\theta)}h\right)^{r_1}\right)\right)^{1/r_1}\\
=&C\sum_{k,j}|Q_{k,j}|^{\frac{\alpha}{n}-1/r_0}m_{3Q_{k,j}}(|f|^r,|g|^s)|E_{k,j}|\left(m_{Q_{k,j}}\left(\left(M^{(\theta')}\omega\right)^{r_1'}\right)\right)^{1/r_1'}\\
&\times|Q_{k,j}|^{1/r_0}\left(m_{Q_{k,j}}\left(\left(M^{(\theta)}h\right)^{r_1}\right)\right)^{1/r_1}\\
=&\sum_{k,j}|Q_{k,j}|^{\frac{\alpha}{n}-1/r_0}m_{3Q_{k,j}}(|f|^r,|g|^s)|E_{k,j}|\left(m_{Q_{k,j}}\left(\left(M^{(\theta')}\omega\right)^{r_1'}\right)\right)^{1/r_1'}\\
&\times\left(|Q_{k,j}|^{\frac{\theta}{r_0}}\left(\frac{1}{|Q_{k,j}|}\int_{Q_{k,j}}M\left(|h|^\theta\right)(x)^{r_1/\theta}dx\right)^{\theta/r_1}\right)^{1/\theta}
\end{aligned}
$$

$$\begin{aligned}
&\leqslant C\sum_{k,j}|Q_{k,j}|^{\frac{\alpha}{n}-1/r_0}m_{3Q_{k,j}}(|f|^r,|g|^s)|E_{k,j}|\left(m_{Q_{k,j}}\left(\left(M^{(\theta)}\omega\right)^{r_1'}\right)\right)^{1/r_1'}\\
&\quad\times|Q_{k,j}|^{1/r_0-1/r_1}\left(\int_{Q_{k,j}}|h(x)|^{r_1}dx\right)^{1/r_1}\\
&\leqslant C\|h\|_{\mathscr{M}^{r_0}_{r_1}}\sum_{k,j}|Q_{k,j}|^{\frac{\alpha}{n}-1/r_0}m_{3Q_{k,j}}(|f|^r,|g|^s)|E_{k,j}|\left(m_{Q_{k,j}}\left(\left(M^{(\theta')}\omega\right)^{r_1'}\right)\right)^{1/r_1'},
\end{aligned}$$

where the definition of $M^{(\theta')}\omega$ can be found in Section 2.

Similarly, for the estimates of II_1, there is

$$II_1\leqslant C\|h\|_{\mathscr{M}^{r_0}_{r_1}}|Q_0|^{\frac{\alpha}{n}-1/r_0}m_{3Q_0}(|f|^r,|g|^s)|E_0|\left(m_{Q_0}\left(\left(M^{(\theta')}\omega\right)^{r_1'}\right)\right)^{1/r_1'}.$$

Combing the estimates of II_1 and II_2 and recalling the fact that $\{E_0\}\bigcup\{E_{k,j}\}$ forms a disjoint family of decomposition for Q_0, the definition of $Q_{k,j}$ and the fact $\alpha/n>1/r_0$, we get

$$\begin{aligned}
&|Q_0|^{1/q_0-1/q}\int_{Q_0}|II|\omega(x)dx\\
&\leqslant C|Q_0|^{1/q_0-1/q}\|h\|_{\mathscr{M}^{r_0}_{r_1}}\int_{Q_0}M^{(r_1')}\left(M^{(\theta')}\omega\right)(x)M_{\beta_1}(|f|^r)(x)^{1/r}M_{\beta_2}(|g|^s)(x)^{1/s}dx\\
&\leqslant C|Q_0|^{1/q_0-1/q}\|h\|_{\mathscr{M}^{r_0}_{r_1}}\left(\int_{Q_0}M^{(r_1')}\left(M^{(\theta')}\omega\right)(x)^{q'}dx\right)^{1/q'}\\
&\quad\times\left(\int_{Q_0}\left(M_{\beta_1}(|f|^r)(x)^{1/r}M_{\beta_2}(|g|^s)(x)^{1/s}\right)^q dx\right)^{1/q},
\end{aligned}$$

where M_{β_i} denotes the fractional maximal function and $\beta_1=\alpha_1 r-\frac{nr}{2r_0}>0$, $\beta_2=\alpha_2 s-\frac{ns}{2r_0}>0$ with $\alpha_1=\alpha_2=\frac{\alpha}{2}$.

As $\frac{q'}{\theta'}>1$ and $\frac{q'}{r_1'}>1$, we can easily get $\left(\int_{Q_0}M^{(r_1')}\left(M^{(\theta')}\omega\right)(x)^{q'}dx\right)^{1/q'}\leqslant C$ and it remains to give the estimate of

$$|Q_0|^{1/q_0-1/q}\left(\int_{Q_0}\left(M_{\beta_1}(|f|^r)(x)^{1/r}M_{\beta_2}(|g|^s)(x)^{1/s}\right)^q dx\right)^{1/q}.$$

By the Hölder inequality on Morrey spaces and Lemmas 6-7, there is

$$\begin{aligned}
&|Q_0|^{1/q_0-1/q}\left(\int_{Q_0}\left(M_{\beta_1}(|f|^r)(x)^{1/r}M_{\beta_2}(|g|^s)(x)^{1/s}\right)^q dx\right)^{1/q}\\
&\leqslant\|M_{\beta_1}(|f|^r)^{1/r}M_{\beta_2}(|g|^s)^{1/s}\|_{\mathscr{M}^{q_0}_q}\\
&\leqslant\|M_{\beta_1}(|f|^r)^{1/r}\|_{\mathscr{M}^{\mu_1}_{\nu_1}}\|M_{\beta_2}(|g|^s)^{1/s}\|_{\mathscr{M}^{\mu_2}_{\nu_2}}
\end{aligned}$$

$$=\|M_{\beta_1}(|f|^r)\|^{1/r}_{\mathscr{M}^{\frac{\mu_1}{r}}_{\frac{\nu_1}{r}}}\|M_{\beta_2}(|g|^s)\|^{1/s}_{\mathscr{M}^{\frac{\mu_2}{s}}_{\frac{\nu_2}{s}}}$$

$$\leqslant C\||f|^r\|^{1/r}_{\mathscr{M}^{\frac{q_1}{r}}_{\frac{p_1}{r}}}\||g|^s\|^{1/s}_{\mathscr{M}^{\frac{q_2}{s}}_{\frac{p_2}{s}}}=C\|f\|_{\mathscr{M}^{q_1}_{p_1}}\|g\|_{\mathscr{M}^{q_2}_{p_2}},$$

where $\frac{\mu_1}{\nu_1}=\frac{\mu_2}{\nu_2}=\frac{q_0}{q}=\frac{q_1}{p_1}=\frac{q_2}{p_2}$, $\frac{r}{q_1}-\frac{r}{\mu_1}=\frac{r\alpha_1}{n}-\frac{r}{2r_0}=\frac{\beta_1}{n}$ and $\frac{s}{q_2}-\frac{s}{\mu_2}=\frac{s\alpha_2}{n}-\frac{s}{2r_0}=\frac{\beta_2}{n}$.
Thus, we have

$$|Q_0|^{1/q_0-1/q}\int_{Q_0}|II|\omega(x)dx\leqslant C\|h\|_{\mathscr{M}^{r_0}_{r_1}}\|f\|_{\mathscr{M}^{q_1}_{p_1}}\|g\|_{\mathscr{M}^{q_2}_{p_2}}. \tag{35}$$

Then, combing (32)-(35), we conclude that

$$\|h\cdot B\mathscr{I}_\alpha(f,g)\|_{\mathscr{M}^{q_0}_{q}}\leqslant C\|h\|_{\mathscr{M}^{r_0}_{r_1}}\|f\|_{\mathscr{M}^{q_1}_{p_1}}\|g\|_{\mathscr{M}^{q_2}_{p_2}}. \tag{36}$$

Consequently, the proof of Theorem 5 has been finished.

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REFERENCES

[1] R. ADAMS, *A note on Riesz potential*, Duke Math. J., **42**, 4 (1975), 765–778.
[2] F. BERNICOT, D. MALDONADO, K. MOEN AND V. NAIBO, *Bilinear Sobolev-Poincaré inequalities and Leibniz-type rules*, J. Geom. Anal., **24**, 2 (2014), 1144–1180.
[3] J. CHEN AND D. FAN, *Some bilinear estimates*, J. Korean Math. Soc., **46**, 3 (2009), 609–620.
[4] J. CHEN AND D. FAN, *Rough bilinear fractional integrals with variable kernels*, Front. Math. China, **5**, 3 (2010), 369–378.
[5] J. CHEN AND D. FAN, *A bilinear fractional integral on compact Lie groups*, Canad. Math. Bull., **54**, 2 (2011), 207–216.
[6] S. CHEN, H. WU AND Q. XUE, *A note on multilinear Muckenhoupt classes for multiple weights*, Studia Math., **223**, 1 (2014), 1–18.
[7] X. CHEN AND Q. XUE, *Weighted estimates for a class of fractional type operators*, J. Math. Anal. Appl., **362**, 2 (2010), 355–373.
[8] F. CHIARENZA AND M. FRASCA, *Morrey spaces and Hardy-Littlewood maximal function*, Rend. Math. Appl., **7**, 3-4 (1987), 273–279.
[9] J. CONLON AND J. REDONDO, *Estimates on the solution of an elliptic equation related to Brownian motion with drift*, Rev. Mat. Iberoam., **11**, 1 (1995), 1–65.
[10] Y. DING AND C. LIN, *Rough bilinear fractional integrals*, Math. Nachr., **246/247**, (2002), 47–52.
[11] Y. FAN AND G. GAO, *Some Estimates of Rough Bilinear Fractional Integral*, J. Func. Spaces Appl., **406540**, (2012), 17 pages.
[12] S. GALA, Y. SAWANO AND H. TANAKA, *A new Beale-Kato-Majda criteria for the 3D magneto-micropolar fluid equations in the Orlicz-Morrey space*, Math. Methods Appl. Sci., **35**, 11 (2012), 1321–1334.
[13] S. GALA, Y. SAWANO AND H. TANAKA, *On the uniqueness of weak solutions of the 3D MHD equations in the Orlicz-Morrey space*, Appl. Anal., **92**, 4 (2013), 776–783.
[14] L. GRAFAKOS, *On multilinear fractional integrals*, Studia Math., **102**, 1 (1992), 49–56.
[15] L. GRAFAKOS AND R. TORRES, *Multilinear Calderón-Zygmund theory*, Adv. Math., **165**, 1 (2002), 124–164.
[16] J. GARCIA-CUERVA AND J.L. RUBIO DE FRANCIA, *Weighted Norm Inequalities and related topics*, North Holland Mathematics Studies **116**, North Holland, Amsterdam, 1985.

[17] Q. HE AND D. YAN, *Bilinear fractional integral operators on Morrey spaces*, arXiv:1805.01846v2 (2018).

[18] C. HOANG AND K. MOEN, *Weighted estimates for bilinear fractional integral and their commutators*, Indiana Univ. Math. J., **67**, 1 (2018), 397–428.

[19] T. IIDA, *A Characterization of a Multiple Weights Class*, Tokyo J. Math., **35**, 2 (2012), 375–383.

[20] T. IIDA, *Weighted norm inequalities on Morrey spaces for linear and multilinear fractional integrals with homogeneous kernels*, Taiwan. J. Math., **18**, 1 (2014), 147–185.

[21] T. IIDA, *Multilinear fractional integral operators on weighted Morrey spaces*, Harmonic analysis and nonlinear partial differential equations, RIMS Kôkyûroku Bessatsu, **B49**, (2014), 13–31.

[22] T. IIDA, *Weighted estimates of higher order commutators generated by BMO-functions and the fractional integral operator on Morrey spaces*, J. Inequal. Appl., **2016**, (2016), 23 pp.

[23] T. IIDA, Y. KOMORI-FURUYA AND E. SATO, *New multiple weights and the Adams inequality on weighted Morrey spaces*, Sci. Math. Jpn., **74**, 2-3 (2011), 145–157.

[24] T. IIDA, Y. KOMORI-FURUYA AND E. SATO, *The Adams inequality on weighted Morrey spaces*, Tokyo J. Math., **34**, 2 (2011), 535–545.

[25] T. IIDA, E. SATO, Y. SAWANO AND H. TANAKA, *Weighted norm inequalities for multilinear fractional operators on Morrey spaces*, Studia Math., **205**, 2 (2011), 139–170.

[26] T. IIDA, E. SATO, Y. SAWANO AND H. TANAKA, *Multilinear Fractional Integrals on Morrey Spaces*, Acta Math. Sin. (Engl. Ser.), **28**, 7 (2012), 1375–1384.

[27] Y. KOMORI-FURUYA, *Weighted estimates for bilinear fractional integral operators: a necessary and sufficient condition for power weights*, Collect. Math., **71**, 1 (2019), 25–37.

[28] Y. KOMORI-FURUYA, *Weighted estimates for bilinear fractional integral operators*, Math. Nachr., **292**, 8 (2019), 1751–1762.

[29] Y. KOMORI-FURUYA AND S. SHIRAI, *Weighted Morrey spaces and a singular integral operator*, Math. Nachr., **282**, 2 (2009), 219–231.

[30] C.E. KENIG AND E.M. STEIN, *Multilinear estimates and fractional integration*, Math. Res. Lett., **6**, 1 (1999), 1–15.

[31] A. LERNER, *On an estimate of Calderón-Zygmund operators by dyadic positive operators*, J. Anal. Math., **121**, (2013), 141–161.

[32] A. LERNER, S. OMBROSI, C. PÉREZ ET AL., *New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory*, Adv. Math., **220**, 4 (2009), 1222–1264.

[33] K. MOEN, *Weighted inequalities for multilinear fractional integral operators*, Collect. Math., **60**, 2 (2009), 213–238.

[34] K. MOEN, *New weighted estimates for bilinear fractional integral operators*, Trans. Amer. Math. Soc., **366**, 2 (2014), 627–646.

[35] C. MORREY, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc., **43**, 1 (1938), 126–166.

[36] B. MUCKENHOUPT, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc., **165**, (1972), 207–226.

[37] B. MUCKENHOUPT AND R. WHEEDEN, *Weighted norm inequalities for fractional integrals*, Trans. Amer. Math. Soc., **192**, (1974), 261–274.

[38] P. OLSEN, *Fractional integration, Morrey spaces and Schrödinger equation*, Comm. Partial. Differ. Equa., **20**, 11-12 (1995), 2005–2055.

[39] Y. SAWANO, S. SUGANO AND H. TANAKA, *A note on generalized fractional integral on generalized Morrey spaces*, Bound. Value Probl., **835865**, (2009), 18 pages.

[40] Y. SAWANO, S. SUGANO AND H. TANAKA, *Generalized Fractional Integral Operators and Fractional Maximal Operators in the Framework of Morrey Spaces*, Trans. Amer. Math. Soc., **363**, 12 (2011), 6481–6503.

[41] Y. SAWANO, S. SUGANO AND H. TANAKA, *A bilinear eatimate for commutators of fractional integral operators*, RIMS Kôkyûroku Bessatsu, **B43**, (2013), 155–170.

[42] H. TANAKA, *Morrey spaces and fractional integrals*, J. Aust. Math. Soc., **88**, 2 (2010), 247–259.

[43] X. WU AND J. CHEN, *Boundedness of fractional integral operators on α-modulation spaces*, Appl. Math. J. Chinese Univ., Ser. B, **29**, 3 (2014), 339–351.

[44] X. YU AND S. LU, *Olsen type inequalities for the generalized commutator of multilinear fractional integrals*, Turk. J. Math., **42**, 5 (2018), 2348–2370.
[45] Y. ZHONG AND J. CHEN, *Modulation space estimates for the fractional integral operators*, Sci. China Math., **54**, 7 (2011), 1479–1489.

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