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WEIGHTED ITERATED DISCRETE HARDY–TYPE INEQUALITIES

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Abstract. Necessary and sufficient conditions on functions u and ω are established ensuring boundedness of a discrete Hardy-type operator from a weighted sequence space $l_{p,u}$ to a weighted sequence space for a wide range of the numerical parameters p,u and θ.

1. Introduction

The original form of Hardy's integral inequality (see [7]) from 1925 reads: If $p>1$ and f is a non-negative p-integrable function over $(0,\infty)$, then

$$\left(\int_0^\infty\left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx\right)^{\frac{1}{p}} \leqslant \frac{p}{p-1}\left(\int_0^\infty (f(x))^p dx\right)^{\frac{1}{p}}, \tag{1.1}$$

where the constant $\frac{p}{p-1}=p'$ is sharp. This means that the Hardy operator H , defined by $H(f)(x):=\frac{1}{x}\int_0^x f(t)dt$ maps $L^p(0,\infty)$ to $L^p(0,\infty)$ with the norm $=p'$. The dramatic prehistory until G.H. Hardy discovered (1.1) is described in detail in [9]. After that it has been an almost unbelievable amount of research to develop (1.1) to what today is called Hardy-type inequalities. The history of this development up to 2007 was described in detail in the book [8]. But this development has continued also after that and the most important steps in this development can be found in the new book [10].

Most of the developments of (1.1) so far has been concentrated on the problem to characterize weighted versions of (1.1) and where p on one side is replaced by q which can be different from q and with $\frac{p}{p-1}$ replaced by a finite positive constant C. This means that Hardy-type inequalities so far have been mostly concentrated on the problem to characterize the weights so that the Hardy operator maps different weighted L^p -spaces to other weighted L^q spaces. However, there are also some results of this type also for other function spaces than weighted L^p spaces. A number of new such results are described in Chapter 7.6 of the book [10], for example when the weighted L^p spaces are replaced by Orlicz spaces, Lorentz spaces, r.i. invariant spaces, general Banach

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function spaces, Morrey-type spaces, Hölder-type spaces, variable $L^{p(\cdot)}$ - spaces, etc. For this paper the most interesting part is about weighted Morrey-type spaces (see [10], 7.6.2).

It is easy to see that (1.1) implies the following discrete form of Hardy's inequality:

$$\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}a_i\right)^p\right)^{\frac{1}{p}} \leqslant \frac{p}{p-1}\left(\sum_{n=1}^{\infty}a_n^p\right)^{\frac{1}{p}}, \quad p>1, \tag{1.2}$$

still with $p' = \frac{p}{p-1}$ as the sharp constant, and where $\{a_i\}_{i=1}^{\infty}$ is a sequence of non-negative numbers.

It has been a parallel development of (1.2) as what has been described above concerning the development of (1.1) to the theory of Hardy-type inequalities. Also in this case the development has been concentrated around mapping properties of the discrete Hardy operator between weighted l_p- spaces. This development up to 2007 is described in detail in Chapter 6 of the book [8]. See also the references therein. Here we just mention some important papers in this connection: [2],[3],[4],[6], [11] and [13].

In recent years, after the publication of [5], it begun some new interesting research concerning discrete Hardy-type inequalities with weighted discrete Hardy operators involved (see e.g. [14]). In this paper we consider the following case:

Let $\varphi = \{\varphi_k\}_{k=1}^{\infty}$ be a non-negative sequence of real numbers and consider the following Hardy-type operator H_φ defined as follows for $\forall f \in l_1$:

$$(H_\varphi f)_k := \varphi_k \sum_{i=1}^{k} f_i, \tag{1.3}$$

where $k \in \mathbb{N}$.

Let $1<p<\infty$, $\frac{1}{p}+\frac{1}{p'}=1$ and $u=\{u_i\}_{i=1}^{\infty}$ and $\omega=\{\omega_i\}_{i=1}^{\infty}$ be positive sequences of real numbers, which will be referred to as weight sequences. We denote by $l_{p,u}$ the space of sequences $f=\{f_j\}_{j=1}^{\infty}$ of real numbers such that

$$\|f\|_{p,u} = \left(\sum_{j=1}^{\infty}|u_j f_j|^p\right)^{\frac{1}{p}} < \infty.$$

Let $1<p<\infty$ and $0<q,\theta<\infty$. The aim of this paper is to characterize the following discrete Hardy-type inequalities:

$$\left(\sum_{n=1}^{\infty}\omega_n^{\theta}\left(\sum_{k=1}^{n}\left|\varphi_k\sum_{i=1}^{k}f_i\right|^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}} \leqslant C\left(\sum_{j=1}^{\infty}|u_j f_j|^p\right)^{\frac{1}{p}}, \quad \forall f \in l_{p,u}, \tag{1.4}$$

where C is a positive constant independent of f for the following cases:

a) $1<p\leqslant \min\{q,\theta\}<\infty$ (see Theorem 1 in Section 2);

b) $0<q<p\leqslant\theta<\infty$, $p>1$ (see Theorem 2 in Section 3);

c) $0<q<\theta<p<\infty$, $p>1$ (see Theorem 3 in Section 4).

Convention: The symbol $M \ll K$ means that $M \leqslant cK$, where $c > 0$ is a constant depending only on unessential parameters. If $M \ll K \ll M$, then we write $M \approx K$.

In the proofs of our main results we will need the following well-known results on the discrete weighted Hardy inequality (see [2], [3]) and boundedness of matrix operators (see [11]-[12] or [15]). More exactly, see [2], Theorem 1 (viii) and also [8], Theorem 7 (iii).

THEOREM A. *Let* $0 < q < p < \infty$, $1 < p < \infty$. *Then the inequality*

$$\left(\sum_{i=1}^{\infty}\left|\sum_{j=1}^{i} f_j\right|^q v_i^q\right)^{\frac{1}{q}} \leqslant C\left(\sum_{i=1}^{\infty}|u_i f_i|^p\right)^{\frac{1}{p}} \tag{1.5}$$

holds for some $C < \infty$ *if and only if*

$$H = \left(\sum_{k=1}^{\infty}\left(\sum_{i=k}^{\infty} v_i^q\right)^{\frac{p}{p-q}}\left(\sum_{j=1}^{k} u_j^{-p'}\right)^{\frac{p(q-1)}{p-q}} u_k^{-p'}\right)^{\frac{p-q}{pq}} < \infty.$$

Moreover, $H \approx C$, *where* C *is the best constant in* (1.5).

DEFINITION. The matrix $\{a_{i,j}\}_{j=1}^{\infty}$, $i \geqslant j$ satisfies the (discrete) Oinarov condition, if there exist $d \geqslant 1$, a non-negative matrix $(a_{i,j})$, whose entries $a_{i,j}$ are almost non-decreasing in i and almost non-increasing in j such that the inequalities

$$\frac{1}{d}(a_{i,k} + a_{k,j}) \leqslant a_{i,j} \leqslant d(a_{i,k} + a_{k,j}),$$

or $a_{i,j} \approx a_{i.k} + a_{k,j}$ hold for all $i \geqslant k \geqslant j \geqslant 1$.

THEOREM B. *(see [12] or [15]). Let* $1 \leqslant p \leqslant q < \infty$ *and the entries of the matrix* $(a_{i,j})$ *satisfies the discrete Oinarov condition. Then the inequality*

$$\left(\sum_{j=1}^{\infty}\left|\sum_{i=j}^{\infty} a_{i,j} f_i\right|^q u_i^q\right)^{\frac{1}{q}} \leqslant C\left(\sum_{i=1}^{\infty}|v_i f_i|^p\right)^{\frac{1}{p}} \tag{1.6}$$

holds for some $C < \infty$ *if and only if* $M = max\{M_1, M_2\} < \infty$, *where*

$$M_1 = \sup_{k\geqslant 1}\left(\sum_{j=1}^{k} u_j^q\right)^{\frac{1}{q}}\left(\sum_{i=k}^{\infty} a_{i,k}^{p'} v_i^{-p'}\right)^{\frac{1}{p'}},$$

$$M_2 = \sup_{k\geqslant 1}\left(\sum_{j=1}^{k} a_{k,j}^q u_j^q\right)^{\frac{1}{q}}\left(\sum_{i=k}^{\infty} v_i^{-p'}\right)^{\frac{1}{p'}}.$$

Moreover, $M \approx C$, *where* C *is the best constant in* (1.6).

THEOREM C. *(see [11]). Let $1 \leqslant q < p < \infty$ and the entries of the matrix $(a_{i,j})$ satisfies the discrete Oinarov condition. Then the inequality (1.6) holds for some $C < \infty$ if and only if $M^* = max\{M_1^*, M_2^*\} < \infty$, where*

$$M_1^* = \left(\sum_{k=1}^{\infty} \left(\sum_{i=k}^{\infty} a_{i,k}^{p'} v_i^{-p'}\right)^{\frac{q(p-1)}{p-q}} \left(\sum_{j=1}^{k} u_j^q\right)^{\frac{q}{p-q}} u_k^q\right)^{\frac{p-q}{pq}},$$

$$M_2^* = \left(\sum_{k=1}^{\infty} \left(\sum_{j=1}^{k} a_{k,j}^q u_j^q\right)^{\frac{p}{p-q}} \left(\sum_{i=k}^{\infty} v_i^{-p'}\right)^{\frac{p(q-1)}{p-q}} v_k^{-p'}\right)^{\frac{p-q}{pq}}.$$

Moreover, $M^* \approx C$, where C is the best constant in (1.6).

For the proofs we need the following Lemma:

LEMMA 1. *Let $\gamma > 0$ and $\{B_k\}$ be a nonnegative sequence. Then*

$$\left(\sum_{k=1}^{j} B_k\right)^{\gamma} \approx \sum_{k=1}^{j} B_k \left(\sum_{i=1}^{k} B_i\right)^{\gamma-1}, \ j \geqslant 1. \tag{1.7}$$

If $\sum_k B_k < \infty$, $1 \leqslant j,k < N \leqslant \infty$, then

$$\left(\sum_{k=j}^{N} B_k\right)^{\gamma} \approx \sum_{k=j}^{N} B_k \left(\sum_{i=k}^{N} B_i\right)^{\gamma-1}. \tag{1.8}$$

REMARK 1. The estimates (1.7) and (1.8), due to K.F. Andersen and H.P. Heinig [[1], p. 844], have been used by many authors including K. G. Grosse-Erdmann [[6], p. 12] and G. Bennett [[3], Lemmas 2 and 3].

We also need the following well-known version of the discrete Minkowski inequality:

LEMMA 2. *Let $\{a_{i,j}\}$, $i = 1,2,...,n \leqslant +\infty$, $j = 1,2,...,m$, be a positive matrix. Then the inequalities*

$$\left(\sum_{i=1}^{n} \left|\sum_{j=1}^{m} a_{i,j}\right|^{\sigma}\right)^{\frac{1}{\sigma}} \leqslant \sum_{j=1}^{m} \left(\sum_{i=1}^{n} |a_{i,j}|^{\sigma}\right)^{\frac{1}{\sigma}}, \tag{1.9}$$

and

$$\left(\sum_{i=1}^{n} \left|\sum_{j=1}^{i} a_{i,j}\right|^{\sigma}\right)^{\frac{1}{\sigma}} \leqslant \sum_{j=1}^{n} \left(\sum_{i=j}^{n} |a_{i,j}|^{\sigma}\right)^{\frac{1}{\sigma}}, \tag{1.10}$$

holds, where $\sigma \geqslant 1$.

We also need the following elementary inequalities: If $a_i > 0$, $i = 1,2,\ldots,k$, then

$$\Big(\sum_{m=1}^{k} a_i\Big)^{\alpha} \leqslant \sum_{m=1}^{k} a_i^{\alpha}, \;\; 0 < \alpha \leqslant 1, \tag{1.11}$$

and

$$\Big(\sum_{m=1}^{k} a_i\Big)^{\alpha} \geqslant \sum_{m=1}^{k} a_i^{\alpha}, \;\; \alpha \geqslant 1. \tag{1.12}$$

2. The case $1 < p \leqslant q, \theta < \infty$

The main result in this section reads:

THEOREM 1. *Let* $1 < p \leqslant \min\{q,\theta\} < \infty$. *Then the inequality* (1.4) *holds for some* $C < \infty$ *if and only if* $B_1 < \infty$, *where*

$$B_1 := \sup_{r\geqslant 1}\left(\sum_{n=r}^{\infty}\omega_n^{\theta}\Big(\sum_{k=r}^{n}\varphi_k^{q}\Big)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}}\Big(\sum_{i=1}^{r}u_i^{-p'}\Big)^{\frac{1}{p'}}. \tag{2.1}$$

Moreover, $C \approx B_1$, *where* C *is the best constant in* (1.4).

Proof. *Necessity:* Suppose that the inequality (1.4) holds with the best constant $C > 0$. Let us show that $B_1 < \infty$. We choose $r \geqslant 1$ arbitrary and take a test sequence $\widetilde{f_r} = \{\widetilde{f_{r,s}}\}_{s=1}^{\infty}$ defined by $\widetilde{f_{r,s}} = u_s^{-p'}$ for $1 \leqslant s \leqslant r$ and $\widetilde{f_{r,s}} = 0$ for $s > r$.
Then

$$\|\widetilde{f_r}\|_{l_{p,u}} = \Big(\sum_{s=1}^{\infty}|\widetilde{f_r}\cdot u_s|^p\Big)^{\frac{1}{p}} = \Big(\sum_{s=1}^{r}|u_s^{-p'}\cdot u_s|^p\Big)^{\frac{1}{p}} = \Big(\sum_{s=1}^{r}u_s^{-p'}\Big)^{\frac{1}{p}} < \infty. \tag{2.2}$$

Substituting $\widetilde{f_r}$ in the left hand side of inequality (1.4), we can deduce that

$$\begin{aligned} I(\widetilde{f}) &\equiv \left(\sum_{n=1}^{\infty}\omega_n^{\theta}\Big(\sum_{k=1}^{n}\Big|\varphi_k\sum_{i=1}^{k}\widetilde{f_i}\Big|^q\Big)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}} \\ &\geqslant \left(\sum_{n=r}^{\infty}\omega_n^{\theta}\Big(\sum_{k=r}^{n}\Big|\varphi_k\sum_{i=1}^{r}u_i^{-p'}\Big|^q\Big)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}} \geqslant \left(\sum_{n=r}^{\infty}\omega_n^{\theta}\Big(\sum_{k=r}^{n}\varphi_k^{q}\Big)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}}\Big(\sum_{i=1}^{r}u_i^{-p'}\Big). \end{aligned}$$

i.e. that

$$I(\widetilde{f}) \geqslant \left(\sum_{n=r}^{\infty}\omega_n^{\theta}\Big(\sum_{k=r}^{n}\varphi_k^{q}\Big)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}}\Big(\sum_{i=1}^{r}u_i^{-p'}\Big). \tag{2.3}$$

From (2.2), (2.3) and (1.4) it follows that

$$C \geqslant \left(\sum_{n=r}^{\infty}\omega_n^{\theta}\Big(\sum_{k=r}^{n}\varphi_k^{q}\Big)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}}\Big(\sum_{i=1}^{r}u_i^{-p'}\Big)^{\frac{1}{p'}}, \;\; \forall r \geqslant 1.$$

Since $r \geqslant 1$ is arbitrary we have that

$$B_1 = \sup_{r\geqslant 1}\left(\sum_{n=r}^{\infty}\omega_n^{\theta}\left(\sum_{k=r}^{n}\varphi_k^{q}\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}}\left(\sum_{i=1}^{r}u_i^{-p'}\right)^{\frac{1}{p'}} \leqslant C < \infty. \tag{2.4}$$

Sufficiency: Let $B_1 < \infty$. Now, we prove that (1.4) holds for a finite constant C. Let $0 \leqslant f \in l_{p,u}$.

Let $f_1 \neq 0$. If $f_i = 0$, $1 \leqslant i < k$, $f_k \neq 0$, then $\sum\limits_{i=k}^{\infty} f_i = \sum\limits_{i=1}^{\infty} f_{k-1+i} = \sum\limits_{i=1}^{\infty} \widetilde{f}_i$ and therefore $\widetilde{f}_1 \neq 0$.

Let

$$\sup\{k \in Z : 2^k \leqslant f_1\} = k_1,$$

then

$$2^{k_1} \leqslant f_1 < 2^{k_1+1}.$$

Hence,

$$k_\infty := \sup\left\{k \geqslant 1 : 2^{k_1+k-1} \leqslant \sum_{i=1}^{\infty} f_i\right\}.$$

If $\sum\limits_{i=1}^{\infty} f_i < \infty$, then $k_\infty < \infty$. If $\sum\limits_{i=1}^{\infty} f_i = \infty$, then $k_\infty = \infty$.

We consider the sequence $\{j_k\}$, where j_k are defined by

$$j_k := \min\left\{j \geqslant 1 : \sum_{i=1}^{j} f_i \geqslant 2^{k_1+k-1}\right\},\quad 1 \leqslant k \leqslant k_\infty.$$

We note that

$$j_1 = \min\left\{j \geqslant 1 : \sum_{i=1}^{j} f_i \geqslant 2^{k_1}\right\} = 1,$$

and then obviously $j_{k_\infty} = \infty$ and if $k_\infty < \infty$, then

$$2^{k_1+k_\infty-1} \leqslant \sum_{i=1}^{\infty} f_i < 2^{k_1+k_\infty}.$$

For all $k \geqslant 1$ it yields that

$$\sum_{i=1}^{j_k-1} f_i < 2^{k_1+k-1} \leqslant \sum_{i=1}^{j_k} f_i. \tag{2.5}$$

Therefore the set of natural numbers $\mathbb{N}$ can be written

$$\mathbb{N} = \bigcup_{k=2}^{k_\infty}[j_{k-1}, j_k - 1],$$

Moreover,

$$2^{k_1+m-1} \leqslant \sum_{i=1}^{j_m} f_i = \sum_{i=1}^{j_{m-1}-1} f_i + \sum_{i=j_{m-1}}^{j_m} f_i < 2^{k_1+m-2} + \sum_{i=j_{m-1}}^{j_m} f_i, \ m \geqslant 3.$$

$$2^{k_1+m-2} \leqslant \sum_{i=j_{m-1}}^{j_m} f_i, \ m \geqslant 3.$$

By substitution m by $m-1$ we find that

$$2^{k_1+m-3} \leqslant \sum_{i=j_{m-2}}^{j_{m-1}} f_i, \ m \geqslant 4.$$

Then we obtain that

$$2^{k_1+m-1} \leqslant 4 \sum_{i=j_{m-2}}^{j_{m-1}} f_i, \ m \geqslant 4. \tag{2.6}$$

Let us consider special cases: if $m=2$, then we have that

$$2^{k_1+2-1} = 2^{k_1+1} = 2^{k_1} \cdot 2 \leqslant 2f_1 = 2 \sum_{i=j_0}^{j_1} f_i, \ f_{j_o} = 0,$$

$$2^{k_1+2-1} \leqslant 4 \sum_{i=j_0}^{j_1} f_i; \tag{2.7}$$

if $m=3$, we apply (2.5) and get that

$$2^{k_1+3-1} = 2 \cdot 2^{k_1+2-1} \leqslant 2 \sum_{i=1}^{j_2} f_i \leqslant 4 \sum_{i=j_{3-2}}^{j_{3-1}} f_i. \tag{2.8}$$

By combining (2.6)-(2.8) we obtain that

$$2^{k_1+m-1} \leqslant 4 \sum_{i=j_{m-2}}^{j_{m-1}} f_i, \ m \geqslant 2. \tag{2.9}$$

Therefore, by force of (2.5)

$$I^\theta(f) := \sum_{n=1}^{\infty} \omega_n^\theta \left(\sum_{s=1}^{n} \left| \varphi_s \sum_{i=1}^{s} f_i \right|^q \right)^{\frac{\theta}{q}} = \sum_{k=2}^{k_\infty} \sum_{n=j_{k-1}}^{j_k-1} \omega_n^\theta \left(\sum_{s=1}^{n} \left| \varphi_s \sum_{i=1}^{s} f_i \right|^q \right)^{\frac{\theta}{q}}$$

$$\leqslant \sum_{k=2}^{k_\infty} \sum_{n=j_{k-1}}^{j_k-1} \omega_n^\theta \left(\sum_{m=2}^{k} \sum_{s=j_{m-1}}^{\min(n,j_m-1)} \varphi_s^q \Big(\sum_{i=1}^{s} f_i \Big)^q \right)^{\frac{\theta}{q}}$$

$$\leqslant \sum_{k=2}^{k_\infty} \sum_{n=j_{k-1}}^{j_k-1} \omega_n^\theta \left(\sum_{m=2}^{k} \sum_{s=j_{m-1}}^{\min(n,j_m-1)} \varphi_s^q \left(\sum_{i=1}^{j_m-1} f_i \right)^q \right)^{\frac{\theta}{q}}$$

$$\leqslant \sum_{k=2}^{k_\infty} \sum_{n=j_{k-1}}^{j_k-1} \omega_n^\theta \left(\sum_{m=2}^{k} \sum_{s=j_{m-1}}^{\min(n,j_m-1)} \varphi_s^q \left(2^{k_1+m-1}\right)^q \right)^{\frac{\theta}{q}}.$$

Hence, by applying (2.9) we have that

$$I^\theta(f) \leqslant 4^\theta \sum_{k=2}^{k_\infty} \sum_{n=j_{k-1}}^{j_k-1} \omega_n^\theta \left(\sum_{m=2}^{k} \sum_{s=j_{m-1}}^{\min(n,j_m-1)} \varphi_s^q \left(\sum_{i=j_{m-2}}^{j_{m-1}} f_i \right)^q \right)^{\frac{\theta}{q}}. \tag{2.10}$$

We must now consider the cases $\theta \leqslant q$ and $\theta > q$ separately.

2.1. The case $\theta \leqslant q$

We consider the inequality $I(f)$ and note that $\frac{\theta}{q} \leqslant 1$. By applying the elementary inequality (1.11) and using (2.10), we find that

$$\begin{aligned} I^\theta(f) &\leqslant 4^\theta \sum_{k=2}^{k_\infty} \sum_{n=j_{k-1}}^{j_k-1} \omega_n^\theta \sum_{m=2}^{k} \left(\sum_{s=j_{m-1}}^{\min(n,j_m-1)} \varphi_s^q \right)^{\frac{\theta}{q}} \left(\sum_{i=j_{m-2}}^{j_{m-1}} f_i \right)^\theta \\ &= 4^\theta \sum_{k=2}^{k_\infty} \sum_{m=2}^{k} \left(\sum_{i=j_{m-2}}^{j_{m-1}} f_i \right)^\theta \sum_{n=j_{k-1}}^{j_k-1} \omega_n^\theta \left(\sum_{s=j_{m-1}}^{\min(n,j_m-1)} \varphi_s^q \right)^{\frac{\theta}{q}}. \end{aligned}$$

Thus, by changing the orders of sums, we get that

$$\begin{aligned} I^\theta(f) &= 4^\theta \sum_{m=2}^{k_\infty} \sum_{k=m}^{k_\infty} \left(\sum_{i=j_{m-2}}^{j_{m-1}} f_i \right)^\theta \sum_{n=j_{k-1}}^{j_k-1} \omega_n^\theta \left(\sum_{s=j_{m-1}}^{\min(n,j_m-1)} \varphi_s^q \right)^{\frac{\theta}{q}} \\ &\leqslant 4^\theta \sum_{m=2}^{k_\infty} \left(\sum_{i=j_{m-2}}^{j_{m-1}} f_i \right)^\theta \sum_{n=j_{m-1}}^{\infty} \omega_n^\theta \left(\sum_{s=j_{m-1}}^{n} \varphi_s^q \right)^{\frac{\theta}{q}}. \end{aligned}$$

Therefore, by using Hölder's inequality and (1.12), we obtain that

$$\begin{aligned} I^\theta(f) &\leqslant 4^\theta \sum_{m=2}^{k_\infty} \left(\sum_{i=j_{m-2}}^{j_{m-1}} |f_i \cdot u_i|^p \right)^{\frac{\theta}{p}} \left(\sum_{i=j_{m-2}}^{j_{m-1}} u_i^{-p'} \right)^{\frac{\theta}{p'}} \sum_{n=j_{m-1}}^{\infty} \omega_n^\theta \left(\sum_{s=j_{m-1}}^{n} \varphi_s^q \right)^{\frac{\theta}{q}} \\ &\leqslant 4^\theta \left(\sum_{m=2}^{k_\infty} \sum_{i=j_{m-2}}^{j_{m-1}} |f_i \cdot u_i|^p \right)^{\frac{\theta}{p}} \left[\sup_{m \geqslant 2} \left(\sum_{n=j_{m-1}}^{\infty} \omega_n^\theta \left(\sum_{s=j_{m-1}}^{n} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \left(\sum_{i=1}^{j_{m-1}} u_i^{-p'} \right)^{\frac{1}{p'}} \right]^\theta \\ &\leqslant 4^\theta \left(2 \sum_{i=1}^{\infty} |f_i \cdot u_i|^p \right)^{\frac{\theta}{p}} \left[\sup_{r \geqslant 1} \left(\sum_{n=r}^{\infty} \omega_n^\theta \left(\sum_{s=r}^{n} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \left(\sum_{i=1}^{r} u_i^{-p'} \right)^{\frac{1}{p'}} \right]^\theta \end{aligned}$$

$$= \left(2^{2+\frac{1}{p}} B_1 \|f\|_{p,u}\right)^{\theta} \ll \left(B_1 \|f\|_{p,u}\right)^{\theta}.$$

Hence,

$$I^{\theta}(f) \ll \left(B_1 \|f\|_{p,u}\right)^{\theta},$$

so that

$$I(f) \ll B_1 \|f\|_{p,u}, \ \ if \ \ \theta \leqslant q. \tag{2.11}$$

2.2. The case $\theta > q$

We start with the inequality (2.10):

$$I^{\theta}(f) \leqslant 4^{\theta} \sum_{k=2}^{k_{\infty}} \sum_{n=j_{k-1}}^{j_k-1} \omega_n^{\theta} \left(\sum_{m=2}^{k} \sum_{s=j_{m-1}}^{\min(n,j_m-1)} \varphi_s^q \left(\sum_{i=j_{m-2}}^{j_{m-1}} f_i \right)^q \right)^{\frac{\theta}{q}}.$$

First we raise both sides in (2.10) to power $\frac{q}{\theta} \leqslant 1$

$$I^q(f) \leqslant 4^q \left[\sum_{k=2}^{k_{\infty}} \sum_{n=j_{k-1}}^{j_k-1} \left(\sum_{m=2}^{k} \omega_n^q \sum_{s=j_{m-1}}^{\min(n,j_m-1)} \varphi_s^q \left(\sum_{i=j_{m-2}}^{j_{m-1}} f_i \right)^q \right)^{\frac{\theta}{q}} \right]^{\frac{q}{\theta}} \leqslant$$

Next we apply (1.9) in the inner sum with $\sigma = \frac{\theta}{q}$ and obtain that

$$I^q(f) \leqslant 4^q \left[\sum_{k=2}^{k_{\infty}} \left\{ \sum_{m=2}^{k} \left(\sum_{n=j_{k-1}}^{j_k-1} \omega_n^{\theta} \left(\sum_{s=j_{m-1}}^{\min(n,j_m-1)} \varphi_s^q \right)^{\frac{\theta}{q}} \left(\sum_{i=j_{m-2}}^{j_{m-1}} f_i \right)^{\theta} \right)^{\frac{q}{\theta}} \right\}^{\frac{\theta}{q}} \right]^{\frac{q}{\theta}}.$$

Using (1.10), we get

$$I^q(f) \leqslant 4^q \sum_{m=2}^{k_{\infty}} \left[\sum_{k=m}^{k_{\infty}} \sum_{n=j_{k-1}}^{j_k-1} \omega_n^{\theta} \left(\sum_{s=j_{m-1}}^{\min(n,j_m-1)} \varphi_s^q \right)^{\frac{\theta}{q}} \left(\sum_{i=j_{m-2}}^{j_{m-1}} f_i \right)^{\theta} \right]^{\frac{q}{\theta}}$$

$$= 4^q \sum_{m=2}^{k_{\infty}} \left(\sum_{i=j_{m-2}}^{j_{m-1}} f_i \right)^q \left[\sum_{k=m}^{k_{\infty}} \sum_{n=j_{k-1}}^{j_k-1} \omega_n^{\theta} \left(\sum_{s=j_{m-1}}^{\min(n,j_m-1)} \varphi_s^q \right)^{\frac{\theta}{q}} \right]^{\frac{q}{\theta}}$$

$$\leqslant 4^q \sum_{m=2}^{k_{\infty}} \left(\sum_{i=j_{m-2}}^{j_{m-1}} f_i \right)^q \left[\sum_{n=j_{m-1}}^{j_m-1} \omega_n^{\theta} \left(\sum_{s=j_{m-1}}^{n} \varphi_s^q \right)^{\frac{\theta}{q}} + \sum_{k=m+1}^{k_{\infty}} \sum_{n=j_{k-1}}^{j_k-1} \omega_n^{\theta} \left(\sum_{s=j_{m-1}}^{j_m-1} \varphi_s^q \right)^{\frac{\theta}{q}} \right]^{\frac{q}{\theta}}.$$

Thus, we get that

$$I^q(f) \leqslant \sum_{m=2}^{k_{\infty}} \left(\sum_{i=j_{m-2}}^{j_{m-1}} f_i \right)^q \left[\sum_{n=j_{m-1}}^{j_m-1} \omega_n^{\theta} \left(\sum_{s=j_{m-1}}^{n} \varphi_s^q \right)^{\frac{\theta}{q}} + \sum_{n=j_m}^{\infty} \omega_n^{\theta} \left(\sum_{s=j_{m-1}}^{n} \varphi_s^q \right)^{\frac{\theta}{q}} \right]^{\frac{q}{\theta}}$$

$$\leqslant 4^q \sum_{m=2}^{k_\infty} \left(\sum_{i=j_{m-2}}^{j_{m-1}} f_i \right)^q \left[\sum_{n=j_{m-1}}^{\infty} \omega_n^\theta \left(\sum_{s=j_{m-1}}^{n} \varphi_s^q \right)^{\frac{\theta}{q}} \right]^{\frac{q}{\theta}}.$$

Hence, by using Hölder's inequality,

$$I^q(f) \leqslant 4^q \sum_{m=2}^{k_\infty} \left(\sum_{i=j_{m-2}}^{j_{m-1}} |f_i u_i|^p \right)^{\frac{q}{p}} \left(\sum_{i=j_{m-2}}^{j_{m-1}} u_i^{-p'} \right)^{\frac{q}{p'}} \left[\sum_{n=j_{m-1}}^{\infty} \omega_n^\theta \left(\sum_{s=j_{m-1}}^{n} \varphi_s^q \right)^{\frac{\theta}{q}} \right]^{\frac{q}{\theta}}.$$

Therefore, by applying (1.12) with $\alpha = \frac{q}{p}$, we obtain that

$$I^q(f) \leqslant 4^q \left(\sum_{m=2}^{k_\infty} \sum_{i=j_{m-2}}^{j_{m-1}} |f_i u_i|^p \right)^{\frac{q}{p}} \left[\sup_{m\geqslant 2} \left(\sum_{n=j_{m-1}}^{\infty} \omega_n^\theta \left(\sum_{s=j_{m-1}}^{n} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \left(\sum_{i=1}^{j_{m-1}} u_i^{-p'} \right)^{\frac{1}{p'}} \right]^q$$

$$\leqslant (4B_1)^q \left(2 \sum_{i=1}^{\infty} |f_i u_i|^p \right)^{\frac{q}{p}} \leqslant (2^{2+\frac{1}{p}} B_1 ||f||_{p,u})^q \ll (B_1 ||f||_{p,u})^q,$$

so that

$$I(f) \ll B_1 \|f\|_{p,u}, \tag{2.12}$$

also for the case $\theta > q$. From the inequalities (2.11) and (2.12), we have that

$$\left(\sum_{n=1}^{\infty} \omega_n^\theta \left(\sum_{k=1}^{n} \left| \varphi_k \sum_{i=1}^{k} f_i \right|^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \ll B_1 \|f\|_{p,u}, \tag{2.13}$$

and $C \ll B_1$, where C is the best constant in (1.4).

From the inequalities (2.4) and (2.13), we get $C \approx B_1$. The proof is complete.

REMARK 2. Theorem 1 means that the inequality (1.4) holds for both cases $1 < p \leqslant \theta \leqslant q < \infty$ and $1 < p \leqslant q < \theta < \infty$ whenever (2.1) is satisfied.

3. The case $0 < q < p \leqslant \theta < \infty$, $p > 1$

For this case the main result is the following:

THEOREM 2. *Let* $0 < q < p \leqslant \theta < \infty$, $p > 1$. *Then the inequality* (1.4) *holds for some* $C < \infty$ *if and only if* $\max\{B_1, B_2\} < \infty$, *where* B_1 *is defined by* (2.1) *and*

$$B_2 := \sup_{k\geqslant 1} \left(\sum_{r=1}^{k} \left(\sum_{s=r}^{k} \varphi_s^q \right)^{\frac{q}{p-q}} \varphi_r^q \left(\sum_{n=1}^{r} u_n^{-p'} \right)^{\frac{q(p-1)}{p-q}} \right)^{\frac{p-q}{pq}} \left(\sum_{i=k}^{\infty} \omega_i^\theta \right)^{\frac{1}{\theta}}.$$

Moreover, $C \approx \max\{B_1, B_2\}$ *with the equivalency constants depending only on* k, p, q *and* θ, *where* C *is the best constant in* (1.4).

Proof. Let $0<q<p\leqslant\theta<\infty$, $p>1$ and $g=\{g_i\}_{i=1}^{\infty}$, $g\geqslant 0 \Leftrightarrow g_i\geqslant 0$, $\forall i\geqslant 1$. From (1.4) we have that

$$C=\sup_{g\geqslant 0}\frac{\left(\sum\limits_{n=1}^{\infty}\left(\omega_n^q\cdot\sum\limits_{k=1}^{n}\left|\varphi_k\sum\limits_{j=1}^{k}g_j\right|^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}}}{\left(\sum\limits_{j=1}^{\infty}|u_jg_j|^p\right)^{\frac{1}{p}}}<\infty, \tag{3.1}$$

where C is the best constant in (1.4).

We raise both sides of (3.1) to power q and get that

$$C^q=\sup_{g\geqslant 0}\frac{\left(\sum\limits_{n=1}^{\infty}\left(\omega_n^q\cdot\sum\limits_{k=1}^{n}\left|\varphi_k\sum\limits_{j=1}^{k}g_j\right|^q\right)^{\frac{\theta}{q}}\right)^{\frac{q}{\theta}}}{\left(\sum\limits_{j=1}^{\infty}|u_jg_j|^p\right)^{\frac{q}{p}}}, \tag{3.2}$$

We define $r:=\frac{\theta}{q}$, $w_n^r:=(\omega_n^q)^{\frac{\theta}{q}}$ and $\Phi_n:=\sum\limits_{k=1}^{n}\left(\varphi_k\sum\limits_{j=1}^{k}g_j\right)^q$. Then

$$\left(\sum_{n=1}^{\infty}\left(\omega_n^q\cdot\sum_{k=1}^{n}\left|\varphi_k\sum_{j=1}^{k}g_j\right|^q\right)^{\frac{\theta}{q}}\right)^{\frac{q}{\theta}}=\left(\sum_{n=1}^{\infty}(w_n\Phi_n)^r\right)^{\frac{1}{r}}.$$

Let $h=\{h_i\}_{i=1}^{\infty}$, $h_i\geqslant 0$, $\forall i\geqslant 1$. Then, by the Hölder inequality,

$$\sum_{n=1}^{\infty}h_n\Phi_n\leqslant\left(\sum_{n=1}^{\infty}|w_n\Phi_n|^r\right)^{\frac{1}{r}}\left(\sum_{k=1}^{\infty}|h_k\cdot w_k^{-1}|^{r'}\right)^{\frac{1}{r'}},\quad \frac{1}{r}+\frac{1}{r'}=1.$$

The sharpness of this inequality implies that

$$\left(\sum_{n=1}^{\infty}(w_n\Phi_n)^r\right)^{\frac{1}{r}}=\sup_{h\geqslant 0}\frac{\sum\limits_{n=1}^{\infty}h_n\Phi_n}{\left(\sum\limits_{k=1}^{\infty}h_k^{r'}w_k^{-r'}\right)^{\frac{1}{r'}}}=\sup_{h\geqslant 0}\frac{\sum\limits_{n=1}^{\infty}h_n\sum\limits_{k=1}^{n}\left(\varphi_k\sum\limits_{j=1}^{k}g_j\right)^q}{\left(\sum\limits_{k=1}^{\infty}(\omega_k^{-q}h_k)^{\frac{\theta}{\theta-q}}\right)^{\frac{\theta-q}{\theta}}}, \tag{3.3}$$

where $r'=\frac{r}{r-1}=\frac{\theta}{\theta-q}$.

Now, we put (3.3) into (3.2) and find that

$$C^q=\sup_{g\geqslant 0}\sup_{h\geqslant 0}\frac{\sum\limits_{n=1}^{\infty}h_n\sum\limits_{k=1}^{n}\left(\varphi_k\sum\limits_{j=1}^{k}g_j\right)^q}{\left(\sum\limits_{j=1}^{\infty}|u_jg_j|^p\right)^{\frac{q}{p}}\left(\sum\limits_{k=1}^{\infty}(\omega_k^{-q}h_k)^{\frac{\theta}{\theta-q}}\right)^{\frac{\theta-q}{\theta}}}$$

$$= \sup_{g\geqslant 0}\sup_{h\geqslant 0} \frac{\sum_{k=1}^{\infty}\left(\varphi_k \sum_{j=1}^{k} g_j\right)^q \sum_{n=k}^{\infty} h_n}{\left(\sum_{j=1}^{\infty}|u_j g_j|^p\right)^{\frac{q}{p}} \left(\sum_{k=1}^{\infty}(\omega_k^{-q}h_k)^{\frac{\theta}{\theta-q}}\right)^{\frac{\theta-q}{\theta}}}$$

$$= \sup_{h\geqslant 0} \frac{1}{\left(\sum_{k=1}^{\infty}(\omega_k^{-q}h_k)^{\frac{\theta}{\theta-q}}\right)^{\frac{\theta-q}{\theta}}} \sup_{g\geqslant 0} \frac{\sum_{k=1}^{\infty}\left(\varphi_k \sum_{j=1}^{k} g_j\right)^q \sum_{n=k}^{\infty} h_n}{\left(\sum_{j=1}^{\infty}|u_j g_j|^p\right)^{\frac{q}{p}}}. \tag{3.4}$$

Let $H_k := \sum_{n=k}^{\infty} h_n$. We calculate the second supremum connected to g separately. By using Theorem A we obtain that

$$\sup_{g\geqslant 0} \frac{\left(\sum_{k=1}^{\infty}\left(\sum_{j=1}^{k} g_j\right)^q \varphi_k^q H_k\right)^{\frac{1}{q}}}{\left(\sum_{j=1}^{\infty}|u_j g_j|^p\right)^{\frac{1}{p}}} \approx \left(\sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty}\varphi_k^q H_k\right)^{\frac{p}{p-q}} \left(\sum_{j=1}^{n} u_j^{-p'}\right)^{\frac{p(q-1)}{p-q}} u_n^{-p'}\right)^{\frac{p-q}{pq}}. \tag{3.5}$$

By inserting (3.5) into (3.4) we find that

$$C^q \approx \sup_{h\geqslant 0} \frac{\left(\sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty}\varphi_k^q H_k\right)^{\frac{p}{p-q}} \left(\sum_{j=1}^{n} u_j^{-p'}\right)^{\frac{p(q-1)}{p-q}} u_n^{-p'}\right)^{\frac{p-q}{p}}}{\left(\sum_{k=1}^{\infty}(\omega_k^{-q}h_k)^{\frac{\theta}{\theta-q}}\right)^{\frac{\theta-q}{\theta}}}.$$

Next we note that

$$\sum_{k=n}^{\infty}\varphi_k^q H_k = \sum_{k=n}^{\infty}\varphi_k^q \sum_{i=k}^{\infty} h_i = \sum_{i=n}^{\infty} h_i \sum_{k=n}^{i}\varphi_k^q, \tag{3.6}$$

and define

$$U_n := \left(\left(\sum_{j=1}^{n} u_j^{-p'}\right)^{\frac{p(q-1)}{p-q}} u_n^{-p'}\right)^{\frac{p-q}{p}} \quad \text{and} \quad W_k := \omega_k^{-q}. \tag{3.7}$$

Accordingly, we get that

$$C^q \approx \sup_{h\geqslant 0} \frac{\left(\sum_{n=1}^{\infty}\left(\sum_{i=n}^{\infty} h_i \sum_{k=n}^{i}\varphi_k^q\right)^{\frac{p}{p-q}} U_n^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}}}{\left(\sum_{k=1}^{\infty}(W_k h_k)^{\frac{\theta}{\theta-q}}\right)^{\frac{\theta-q}{\theta}}} = \sup_{h\geqslant 0} \frac{\left(\sum_{n=1}^{\infty}\left(U_n \sum_{i=n}^{\infty} a_{i,n} h_i\right)^{\tilde{q}}\right)^{\frac{1}{\tilde{q}}}}{\left(\sum_{k=1}^{\infty}(W_k h_k)^{\tilde{p}}\right)^{\frac{1}{\tilde{p}}}}, \tag{3.8}$$

where $a_{i,n} = \sum_{k=n}^{i}\varphi_k^q$, $i \geqslant n$, $n \geqslant 1$, $\tilde{q} := \frac{p}{p-q}$ and $\tilde{p} := \frac{\theta}{\theta-q}$. We conclude that

$$\left(\sum_{n=1}^{\infty}\left(U_n \sum_{i=n}^{\infty} a_{i,n} h_i\right)^{\tilde{q}}\right)^{\frac{1}{\tilde{q}}} \leqslant C^q \left(\sum_{k=1}^{\infty}(W_k h_k)^{\tilde{p}}\right)^{\frac{1}{\tilde{p}}}, \ \ h \geqslant 0. \tag{3.9}$$

Here

$$a_{i,n} = \sum_{k=n}^{i} \varphi_k^q \leqslant \sum_{k=n}^{j} \varphi_k^q + \sum_{k=j}^{i} \varphi_k^q = a_{i,j} + a_{j,n},$$

and

$$a_{i,n} \geqslant \sum_{k=n}^{j} \varphi_k^q, \;\; a_{i,n} \geqslant \sum_{k=j}^{i} \varphi_k^q \;\text{ so that } a_{i,n} \geqslant \frac{1}{2}(a_{i,j} + a_{j,n}).$$

We conclude that

$$\frac{1}{2}(a_{i,j} + a_{j,n}) \leqslant a_{i,n} \leqslant a_{i,j} + a_{j,n} \;\Leftrightarrow\; a_{i,n} \approx a_{i,j} + a_{j,n}, \;\; i \geqslant j \geqslant n, \tag{3.10}$$

which means that $(a_{i,n})$ satisfies the discrete Oinarov condition and, moreover, $1 < \tilde{p} \leqslant \tilde{q} < \infty$, and for the operator A defined by $(Ah)_n = \sum_{i=n}^{\infty} a_{i,n} h_i, \; n \geqslant 1$, we have that

$$\left(\sum_{n=1}^{\infty} \Big(U_n (Ah)_n \Big)^{\tilde{q}} \right)^{\frac{1}{\tilde{q}}} \leqslant C^q \left(\sum_{k=1}^{\infty} (W_k h_k)^{\tilde{p}} \right)^{\frac{1}{\tilde{p}}}, \;\; h \geqslant 0. \tag{3.11}$$

Accordingly, if C^q is the best constant in (3.9), by Theorem B it yields that $C^q \approx \max\{\tilde{B}_1, \tilde{B}_2\}$, where

$$\tilde{B}_1 = \sup_{k \geqslant 1} \left(\sum_{n=1}^{k} U_n^{\tilde{q}} \right)^{\frac{1}{\tilde{q}}} \left(\sum_{i=k}^{\infty} a_{i,k}^{\tilde{p}'} W_i^{-\tilde{p}'} \right)^{\frac{1}{\tilde{p}'}},$$

and

$$\tilde{B}_2 = \sup_{k \geqslant 1} \left(\sum_{n=1}^{k} a_{k,n}^{\tilde{q}} U_n^{\tilde{q}} \right)^{\frac{1}{\tilde{q}}} \left(\sum_{i=k}^{\infty} W_i^{-\tilde{p}'} \right)^{\frac{1}{\tilde{p}'}},$$

Next we calculate the values of $\tilde{B}_1$ and $\tilde{B}_2$ using Lemma 1. In fact,

$$\begin{aligned}
\tilde{B}_1 &= \sup_{k \geqslant 1} \left(\sum_{n=1}^{k} U_n^{\tilde{q}} \right)^{\frac{1}{\tilde{q}}} \left(\sum_{i=k}^{\infty} a_{i,k}^{\tilde{p}'} W_i^{-\tilde{p}'} \right)^{\frac{1}{\tilde{p}'}} = \sup_{k \geqslant 1} \left(\sum_{n=1}^{k} U_n^{\frac{p}{p-q}} \right)^{\frac{p-q}{p}} \left(\sum_{i=k}^{\infty} a_{i,k}^{\frac{\theta}{q}} W_i^{-\frac{\theta}{q}} \right)^{\frac{q}{\theta}} \\
&= \sup_{k \geqslant 1} \left(\sum_{n=1}^{k} \left(\sum_{j=1}^{n} u_j^{-p'} \right)^{\frac{p(q-1)}{p-q}} u_n^{-p'} \right)^{\frac{p-q}{p}} \left(\sum_{i=k}^{\infty} \left(\sum_{j=k}^{i} \varphi_j^q \right)^{\frac{\theta}{q}} (\omega_i^{-q})^{-\frac{\theta}{q}} \right)^{\frac{q}{\theta}} \\
&\approx \sup_{k \geqslant 1} \left(\sum_{i=k}^{\infty} \omega_i^{\theta} \left(\sum_{j=k}^{i} \varphi_j^q \right)^{\frac{\theta}{q}} \right)^{\frac{q}{\theta}} \left(\sum_{n=1}^{k} u_n^{-p'} \right)^{\frac{q}{p'}} = B_1^q,
\end{aligned}$$

so that

$$\tilde{B}_1 \approx B_1^q. \tag{3.12}$$

Moreover,

$$\tilde{B}_2 = \sup_{k \geqslant 1} \left(\sum_{n=1}^{k} a_{k,n}^{\tilde{q}} U_n^{\tilde{q}} \right)^{\frac{1}{\tilde{q}}} \left(\sum_{i=k}^{\infty} W_i^{-\tilde{p}'} \right)^{\frac{1}{\tilde{p}'}}$$

$$= \sup_{k\geqslant 1}\left(\sum_{n=1}^{k}\left(\sum_{r=n}^{k}\varphi_r^q\right)^{\frac{p}{p-q}}\left(\sum_{j=1}^{n}u_j^{-p'}\right)^{\frac{p(q-1)}{p-q}}u_n^{-p'}\right)^{\frac{p-q}{p}}\left(\sum_{i=k}^{\infty}\omega_i^{\theta}\right)^{\frac{q}{\theta}}.$$

Hence, by applying Lemma 1, we obtain that

$$\tilde{B_2}\approx \sup_{k\geqslant 1}\left(\sum_{n=1}^{k}\sum_{r=n}^{k}\varphi_r^q\left(\sum_{s=r}^{k}\varphi_s^q\right)^{\frac{q}{p-q}}\left(\sum_{j=1}^{n}u_j^{-p'}\right)^{\frac{p(q-1)}{p-q}}u_n^{-p'}\right)^{\frac{p-q}{p}}\left(\sum_{i=k}^{\infty}\omega_i^{\theta}\right)^{\frac{q}{\theta}},$$

so that, by changing the orders of sums and by using Lemma 1 again, we have that

$$\begin{aligned}\tilde{B_2} &= \sup_{k\geqslant 1}\left(\sum_{r=1}^{k}\sum_{n=1}^{r}\varphi_r^q\left(\sum_{s=r}^{k}\varphi_s^q\right)^{\frac{q}{p-q}}\left(\sum_{j=1}^{n}u_j^{-p'}\right)^{\frac{p(q-1)}{p-q}}u_n^{-p'}\right)^{\frac{p-q}{p}}\left(\sum_{i=k}^{\infty}\omega_i^{\theta}\right)^{\frac{q}{\theta}}\\ &= \sup_{k\geqslant 1}\left(\sum_{r=1}^{k}\varphi_r^q\left(\sum_{s=r}^{k}\varphi_s^q\right)^{\frac{q}{p-q}}\sum_{n=1}^{r}u_n^{-p'}\left(\sum_{j=1}^{n}u_j^{-p'}\right)^{\frac{p(q-1)}{p-q}}\right)^{\frac{p-q}{p}}\left(\sum_{i=k}^{\infty}\omega_i^{\theta}\right)^{\frac{q}{\theta}}\\ &\approx \sup_{k\geqslant 1}\left(\sum_{r=1}^{k}\left(\sum_{s=r}^{k}\varphi_s^q\right)^{\frac{q}{p-q}}\left(\sum_{n=1}^{r}u_n^{-p'}\right)^{\frac{q(p-1)}{p-q}}\varphi_r^q\right)^{\frac{p-q}{pq}\cdot q}\left(\sum_{i=k}^{\infty}\omega_i^{\theta}\right)^{\frac{q}{\theta}} = B_2^q.\end{aligned}$$

Therefore $\tilde{B_2}\approx B_2^q$ so that, by also using (3.12), we find that $C^q\approx \max\{\tilde{B_1},\tilde{B_2}\}\approx \{B_1^q,B_2^q\}$. We conclude that $C\approx \{B_1,B_2\}$ and the equivalence constants depend only on p,q and θ. The proof is complete.

4. The cases $0<q<\theta<p<\infty$, $p>1$

The main result in this section reads:

THEOREM 3. *Let $0<q<\theta<p<\infty$, $p>1$. Then the inequality (1.4) holds for some $C<\infty$ if and only if $\max\{D_1,D_2\}<\infty$, where*

$$D_1 := \left(\sum_{j=1}^{\infty}\left(\sum_{i=j}^{\infty}\left(\sum_{s=j}^{i}\varphi_s^q\right)^{\frac{\theta}{q}}\omega_i^{\theta}\right)^{\frac{p}{p-\theta}}\left(\sum_{i=1}^{j}u_i^{-p'}\right)^{\frac{p(\theta-1)}{p-\theta}}u_j^{-p'}\right)^{\frac{p-\theta}{p\theta}},$$

and

$$D_2 := \left(\sum_{j=1}^{\infty}\left(\sum_{s=1}^{j}\left(\sum_{r=s}^{j}\varphi_r^q\right)^{\frac{q}{p-q}}\varphi_s^q\left(\sum_{k=1}^{s}u_k^{-p'}\right)^{\frac{q(p-1)}{p-q}}\right)^{\frac{\theta(p-q)}{q(p-\theta)}}\left(\sum_{i=j}^{\infty}\omega_i^{\theta}\right)^{\frac{\theta}{p-\theta}}\omega_j^{\theta}\right)^{\frac{p-\theta}{p\theta}}.$$

Moreover, $C\approx\max\{D_1,D_2\}$ with the equivalency constants depending only on k,p,q and θ, where C is the best constant in (1.4).

Proof. The proof of Theorem 3 is the same as the proof of Theorem 2 up to (3.9). But in Theorem 2, we proved that the inequality (3.9) holds for the case $1 < \frac{\theta}{\theta-q} = \tilde{p} \leqslant \tilde{q} = \frac{p}{p-q} < \infty$, from which it follows that $0 < q < p \leqslant \theta < \infty$.

Note that, $\frac{\theta}{\theta-q} > \frac{p}{p-q} \Rightarrow p\theta - \theta q - p\theta + pq > 0 \Rightarrow pq > q\theta \Rightarrow p > \theta,\ p > q,\ \theta > q$. Then the case $1 < \frac{p}{p-q} = \tilde{q} < \tilde{p} = \frac{\theta}{\theta-q} < \infty$ follows from the case $0 < q < \theta < p < \infty,\ p > 1$. Therefore we will consider for the inequality (3.9) for the following case: $1 < \frac{p}{p-q} = \tilde{q} < \tilde{p} = \frac{\theta}{\theta-q} < \infty$.

Let $1 < \frac{p}{p-q} = \tilde{q} < \tilde{p} = \frac{\theta}{\theta-q} < \infty$. We can estimate the value of best constant C^q in the inequality (3.9), where $(a_{i,j})$ satisfies the discrete Oinarov condition. In fact, by using Theorem C, we have that $C^q \approx \max\{\tilde{D_1}, \tilde{D_2}\}$, where

$$\tilde{D_1} = \left(\sum_{j=1}^{\infty} \left(\sum_{i=j}^{\infty} a_{i,j}^{\tilde{p}'} W_i^{-\tilde{p}'} \right)^{\frac{\tilde{q}(\tilde{p}-1)}{\tilde{p}-\tilde{q}}} \left(\sum_{i=1}^{j} U_i^{\tilde{q}} \right)^{\frac{\tilde{q}}{\tilde{p}-\tilde{q}}} U_j^{\tilde{q}} \right)^{\frac{\tilde{p}-\tilde{q}}{\tilde{p}\tilde{q}}},$$

$$\tilde{D_2} = \left(\sum_{j=1}^{\infty} \left(\sum_{k=1}^{j} a_{j,k}^{\tilde{q}} U_k^{\tilde{q}} \right)^{\frac{\tilde{p}}{\tilde{p}-\tilde{q}}} \left(\sum_{i=j}^{\infty} W_i^{-\tilde{p}'} \right)^{\frac{\tilde{p}(\tilde{q}-1)}{\tilde{p}-\tilde{q}}} W_j^{-\tilde{p}'} \right)^{\frac{\tilde{p}-\tilde{q}}{\tilde{p}\tilde{q}}}.$$

Next we rewrite the values of $\tilde{D_1}$ and $\tilde{D_2}$ by using (3.6) and (3.7). In fact,

$$\begin{aligned}
\tilde{D_1} &= \left(\sum_{j=1}^{\infty} \left(\sum_{i=j}^{\infty} \left(\sum_{s=j}^{i} \varphi_s^q \right)^{\frac{\theta}{q}} W_i^{-\frac{\theta}{q}} \right)^{\frac{p}{p-\theta}} \left(\sum_{i=1}^{j} U_i^{\frac{p}{p-q}} \right)^{\frac{p(\theta-q)}{q(p-\theta)}} U_j^{\frac{p}{p-q}} \right)^{\frac{q(p-\theta)}{p\theta}} \\
&= \left(\sum_{j=1}^{\infty} \left(\sum_{i=j}^{\infty} \left(\sum_{s=j}^{i} \varphi_s^q \right)^{\frac{\theta}{q}} \omega_i^{\theta} \right)^{\frac{p}{p-\theta}} \left(\sum_{i=1}^{j} \left(\sum_{n=1}^{i} u_n^{-p'} \right)^{\frac{p(q-1)}{p-q}} u_i^{-p'} \right)^{\frac{p(\theta-q)}{q(p-\theta)}} \right. \\
&\qquad \left. \times \left(\sum_{n=1}^{j} u_n^{-p'} \right)^{\frac{p(q-1)}{p-q}} u_j^{-p'} \right)^{\frac{q(p-\theta)}{p\theta}},
\end{aligned}$$

so that, by applying Lemma 1, we obtain that

$$\begin{aligned}
\tilde{D_1} &\approx \left(\sum_{j=1}^{\infty} \left(\sum_{i=j}^{\infty} \left(\sum_{s=j}^{i} \varphi_s^q \right)^{\frac{\theta}{q}} \omega_i^{\theta} \right)^{\frac{p}{p-\theta}} \left(\sum_{i=1}^{j} u_i^{-p'} \right)^{\frac{q(p-1)}{p-q} \cdot \frac{p(\theta-q)}{q(p-\theta)} + \frac{p(q-1)}{p-q}} u_j^{-p'} \right)^{\frac{q(p-\theta)}{p\theta}} \\
&= \left(\sum_{j=1}^{\infty} \left(\sum_{i=j}^{\infty} \left(\sum_{s=j}^{i} \varphi_s^q \right)^{\frac{\theta}{q}} \omega_i^{\theta} \right)^{\frac{p}{p-\theta}} \left(\sum_{i=1}^{j} u_i^{-p'} \right)^{\frac{p(\theta-1)}{p-\theta}} u_j^{-p'} \right)^{\frac{q(p-\theta)}{p\theta}} = D_1^q.
\end{aligned}$$

Moreover,

$$\tilde{D_2} = \left(\sum_{j=1}^{\infty} \left(\sum_{k=1}^{j} \left(\sum_{s=k}^{j} \varphi_s^q \right)^{\frac{p}{p-q}} U_k^{\frac{p}{p-q}} \right)^{\frac{\theta(p-q)}{q(p-\theta)}} \left(\sum_{i=j}^{\infty} W_i^{-\frac{\theta}{q}} \right)^{\frac{\theta}{p-\theta}} W_j^{-\frac{\theta}{q}} \right)^{\frac{q(p-\theta)}{p\theta}}$$

$$= \left(\sum_{j=1}^{\infty} \left(\sum_{k=1}^{j} \left(\sum_{s=k}^{j} \varphi_s^q \right)^{\frac{p}{p-q}} \left(\sum_{n=1}^{k} u_n^{-p'} \right)^{\frac{p(q-1)}{p-q}} u_k^{-p'} \right)^{\frac{\theta(p-q)}{q(p-\theta)}} \left(\sum_{i=j}^{\infty} \omega_i^{\theta} \right)^{\frac{\theta}{p-\theta}} \omega_j^{\theta} \right)^{\frac{q(p-\theta)}{p\theta}},$$

so that, by applying Lemma 1, we get that

$$\tilde{D_2} \approx \left(\sum_{j=1}^{\infty} \left(\sum_{k=1}^{j} \sum_{s=k}^{j} \left(\sum_{r=s}^{j} \varphi_r^q \right)^{\frac{q}{p-q}} \varphi_s^q \left(\sum_{n=1}^{k} u_n^{-p'} \right)^{\frac{p(q-1)}{p-q}} u_k^{-p'} \right)^{\frac{\theta(p-q)}{q(p-\theta)}} \left(\sum_{i=j}^{\infty} \omega_i^{\theta} \right)^{\frac{\theta}{p-\theta}} \omega_j^{\theta} \right)^{\frac{q(p-\theta)}{p\theta}}.$$

Therefore, by changing the orders of sums and by using Lemma 1 again, we have that

$$\tilde{D_2} = \left(\sum_{j=1}^{\infty} \left(\sum_{s=1}^{j} \left(\sum_{r=s}^{j} \varphi_r^q \right)^{\frac{q}{p-q}} \varphi_s^q \sum_{k=1}^{s} \left(\sum_{n=1}^{k} u_n^{-p'} \right)^{\frac{p(q-1)}{p-q}} u_k^{-p'} \right)^{\frac{\theta(p-q)}{q(p-\theta)}} \left(\sum_{i=j}^{\infty} \omega_i^{\theta} \right)^{\frac{\theta}{p-\theta}} \omega_j^{\theta} \right)^{\frac{q(p-\theta)}{p\theta}}$$

$$\approx \left(\sum_{j=1}^{\infty} \left(\sum_{s=1}^{j} \left(\sum_{r=s}^{j} \varphi_r^q \right)^{\frac{q}{p-q}} \varphi_s^q \left(\sum_{k=1}^{s} u_k^{-p'} \right)^{\frac{q(p-1)}{p-q}} \right)^{\frac{\theta(p-q)}{q(p-\theta)}} \left(\sum_{i=j}^{\infty} \omega_i^{\theta} \right)^{\frac{\theta}{p-\theta}} \omega_j^{\theta} \right)^{\frac{q(p-\theta)}{p\theta}} = D_2^q.$$

Hence $C^q \approx \max\{\tilde{D_1}, \tilde{D_2}\} \approx \{D_1^q, D_2^q\}$ so that $C \approx \{D_1, D_2\}$ and the equivalency constants depend only on p, q and θ, where C is the best constant in (1.4).

The proof is complete.

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