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# ON HARDY TYPE INEQUALITIES FOR WEIGHTED QUASIDEVIATION MEANS

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*Abstract.* Using recent results concerning the homogenization and the Hardy property of weighted means, we establish sharp Hardy constants for concave and monotone weighted quasideviation means and for a few particular subclasses of this broad family. More precisely, for a mean $\mathscr{D}$ like above and a sequence $(\lambda_n)$ of positive weights such that $\lambda_n/(\lambda_1+\ldots+\lambda_n)$ is nondecreasing, we determine the smallest number $H\in(1,+\infty]$ such that

$$\sum_{n=1}^{\infty}\lambda_n\mathscr{D}\big((x_1,\ldots,x_n),(\lambda_1,\ldots,\lambda_n)\big)\leqslant H\cdot\sum_{n=1}^{\infty}\lambda_n x_n \text{ for all } x\in\ell_1(\lambda).$$

It turns out that $H$ depends only on the limit of the sequence $(\lambda_n/(\lambda_1+\ldots+\lambda_n))$ and the behaviour of the mean $\mathscr{D}$ near zero.

## 1. Introduction

In 1920's several authors, motivated by a conjecture of Hilbert, proved that

$$\sum_{n=1}^{\infty}\mathscr{P}_p(x_1,\ldots,x_n)\leqslant C(p)\sum_{n=1}^{\infty}x_n \tag{1.1}$$

for every sequences $(x_n)_{n=1}^{\infty}$ with positive terms, where $\mathscr{P}_p$ denotes the $p$-th *power mean* (extended to the limiting cases $p=\pm\infty$),

$$C(p):=\begin{cases}1 & p=-\infty,\\ (1-p)^{-1/p} & p\in(-\infty,0)\cup(0,1),\\ e & p=0,\\ \infty & p\in[1,\infty],\end{cases}$$

and this constant is sharp, i.e., it cannot be diminished.

The first result of this type with a nonoptimal constant was established by Hardy in [13]. Later this result was improved and extended by Landau [17], Knopp [15], and

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Carleman [3] whose results are summarized in the inequality (1.1). Meanwhile, Copson [4] adopted Elliott's [11] proof of the Hardy inequality and showed (in an equivalent form) that if $\mathscr{P}_p(x,\lambda)$ denotes the $p$-th $\lambda$-*weighted power mean* of the vector $x$, then

$$\sum_{n=1}^{\infty} \lambda_n \mathscr{P}_p\big((x_1,\dots,x_n),(\lambda_1,\dots,\lambda_n)\big) \leqslant C(p) \sum_{n=1}^{\infty} \lambda_n x_n \tag{1.2}$$

for all $p \in (0,1)$, and sequences $(x_n)_{n=1}^{\infty}$ and $(\lambda_n)_{n=1}^{\infty}$ with positive terms. For more details about the history of the developments related to Hardy type inequalities, see papers Pečarić–Stolarsky [26], Duncan–McGregor [10], and the book of Kufner–Maligranda–Persson [16].

Obviously, the constant $C(p)$ is sharp if we require the inequality to be valid for all positive sequences $\lambda$ and $x$. One of the main goal of this presentation is to determine the best possible constant $C_\lambda(p)$ such that the inequality (1.2) be valid with $C(p)$ replaced by $C_\lambda(p)$ for all positive sequences $x$. Moreover, we will extend this result also for the case $p \leqslant 0$. In fact, under some additional assumptions, we will show that $C_\lambda(p)$ is function of $p$ and the limit of the sequence $\big(\frac{\lambda_n}{\lambda_1+\cdots+\lambda_n}\big)$. On the other hand, our results will be developed not only for power means, but in a much larger class of weighted means, in the class of weighted quasideviation means which includes quasiarithmetic and also Gini means. The motivation for this paper originates from the paper [38] related to the nonweighted and homogeneous case.

## 2. Weighted means

For $n \in \mathbb{N}$, define the set of $n$-dimensional real weight vectors $W_n$ by

$$W_n := \{(\lambda_1,\dots,\lambda_n) \in \mathbb{R}^n \mid \lambda_1,\dots,\lambda_n \geqslant 0,\ \lambda_1+\ldots+\lambda_n > 0\}$$

and let

$$W_0 := \{(\lambda_n)_{n\in\mathbb{N}} \mid \lambda_1 > 0 \text{ and } \lambda_2,\dots,\lambda_n,\ldots \geqslant 0\}.$$

Now we recall the concept of a weighted mean as it was introduced in the paper [37].

For a given subinterval $I \subset \mathbb{R}$, *a weighted mean on* $I$ is a function

$$\mathscr{M}\colon \bigcup_{n=1}^{\infty} I^n \times W_n \to I$$

which is nullhomogeneous in the weights, admits the reduction principle, the mean value property, and the elimination principle (see [37] for the details). For $n \in \mathbb{N}$ and $(x,\lambda) \in I^n \times W_n$, we will frequently use the sum type abbreviation:

$$\mathop{\mathscr{M}}_{i=1}^{n}(x_i,\lambda_i) := \mathscr{M}\big((x_1,\dots,x_n),(\lambda_1,\dots,\lambda_n)\big).$$

Let us now introduce some important properties of weighted means. A weighted mean $\mathscr{M}$ is said to be *symmetric*, if for all $n \in \mathbb{N}$, $(x,\lambda) \in I^n \times W_n$, and $\sigma \in S_n$,

$$\mathscr{M}(x,\lambda) = \mathscr{M}(x\circ\sigma,\lambda\circ\sigma).$$

We will call a weighted mean $\mathscr{M}$ *Jensen concave* if, for all $n\in\mathbb{N}$, $x,y\in I^n$ and $\lambda\in W_n$,

$$\mathscr{M}\Big(\frac{x+y}{2},\lambda\Big)\geqslant\frac{1}{2}\big(\mathscr{M}(x,\lambda)+\mathscr{M}(y,\lambda)\big). \tag{2.1}$$

If the above inequality holds with reversed inequality sign, then we speak about the *Jensen convexity* of $\mathscr{M}$. Using that the mapping $x\mapsto\mathscr{M}(x,\lambda)$ is locally bounded, the Bernstein–Doetsch Theorem [2] implies that $\mathscr{M}$ is in fact concave or convex, respectively.

A weighted mean $\mathscr{M}$ is said to be *monotone* (or *nondecreasing*) if, for all $n\in\mathbb{N}$ and $\lambda\in W_n$, the mapping $x_i\mapsto\mathscr{M}(x,\lambda)$ is nondecreasing for all $i\in\{1,\dots,n\}$.

Assuming that $I$ is a subinterval of $\mathbb{R}_+$, we call a weighted mean $\mathscr{M}$ *homogeneous*, if for all $t>0$, $n\in\mathbb{N}$ and $(x,\lambda)\in\big(I\cap\frac{1}{t}I\big)^n\times W_n$,

$$\mathscr{M}(tx,\lambda)=t\mathscr{M}(x,\lambda).$$

For a given subinterval $I$ of $\mathbb{R}_+$ with $\inf I=0$ and a weighted mean $\mathscr{M}$ on $I$, we define two functions $\mathscr{M}_\#,\mathscr{M}^\#\colon\bigcup_{n=1}^\infty\mathbb{R}_+^n\times W_n\to\mathbb{R}_+$ by

$$\mathscr{M}_\#(x,\lambda):=\liminf_{t\to0^+}\tfrac{1}{t}\mathscr{M}(tx,\lambda)\qquad\text{and}\qquad\mathscr{M}^\#(x,\lambda):=\limsup_{t\to0^+}\tfrac{1}{t}\mathscr{M}(tx,\lambda).$$

We call $\mathscr{M}_\#$ and $\mathscr{M}^\#$ the *lower and upper homogenization* of the weighted mean $\mathscr{M}$, respectively. It is obvious that $\mathscr{M}_\#$ and $\mathscr{M}^\#$ are homogeneous weighted means on $\mathbb{R}_+$, furthermore, we have the inequality $\mathscr{M}_\#\leqslant\mathscr{M}^\#$ on $\bigcup_{n=1}^\infty\mathbb{R}_+^n\times W_n$. It is also easy to see that if $\mathscr{M}$ is symmetric (monotone), then also $\mathscr{M}_\#$ and $\mathscr{M}^\#$ are symmetric (monotone). Moreover, in the case when $\mathscr{M}$ is concave, we have a few additional properties.

LEMMA 2.1. ([35], Theorem 2.1) *Let $I$ be a subinterval of $\mathbb{R}^+$ with $\inf I=0$ and $\mathscr{M}$ be a Jensen concave weighted mean on $I$. Then $\mathscr{M}_\#=\mathscr{M}^\#$ and these means are also Jensen concave. In addition, $\mathscr{M}\leqslant\mathscr{M}_\#=\mathscr{M}^\#$ on the domain of $\mathscr{M}$.*

In what follows, we recall several particular classes of weighted means. For a parameter $p\in\mathbb{R}$, define the *weighted power mean* $\mathscr{P}_p\colon\bigcup_{n=1}^\infty\mathbb{R}_+^n\times W_n\to\mathbb{R}_+$ by

$$\mathscr{P}_p(x,\lambda):=\begin{cases}\left(\dfrac{\lambda_1x_1^p+\cdots+\lambda_nx_n^p}{\lambda_1+\cdots+\lambda_n}\right)^{1/p} & \text{if } p\neq0,\\[2ex] \left(x_1^{\lambda_1}\cdots x_n^{\lambda_n}\right)^{1/(\lambda_1+\cdots+\lambda_n)} & \text{if } p=0.\end{cases}$$

In a more general setting, we can define weighted quasiarithmetic means in the spirit of [14]. Given an interval $I$ and a continuous strictly monotone function $f\colon I\to\mathbb{R}$, the *weighted quasiarithmetic mean* $\mathscr{A}_f\colon\bigcup_{n=1}^\infty I^n\times W_n\to I$ is defined by

$$\mathscr{A}_f(x,\lambda):=f^{-1}\left(\frac{\lambda_1f(x_1)+\cdots+\lambda_nf(x_n)}{\lambda_1+\cdots+\lambda_n}\right). \tag{2.2}$$

Another important generalization of power means was introduced in the paper [12]. For two real parameters $p,q$, the Gini mean $\mathscr{G}_{p,q}\colon \bigcup_{n=1}^{\infty}\mathbb{R}_+^n\times W_n\to\mathbb{R}_+$ is defined by

$$\mathscr{G}_{p,q}(x,\lambda):=\begin{cases}\left(\dfrac{\lambda_1x_1^p+\cdots+\lambda_nx_n^p}{\lambda_1x_1^q+\cdots+\lambda_nx_n^q}\right)^{\frac{1}{p-q}} & \text{if } p\neq q,\\[2ex] \exp\left(\dfrac{\lambda_1x_1^p\log x_1+\cdots+\lambda_nx_n^p\log x_n}{\lambda_1x_1^p+\cdots+\lambda_nx_n^p}\right) & \text{if } p=q.\end{cases}$$

In a sequence papers, further generalizations were obtained: *Bajraktarević means* [1], *deviation (or Daróczy) means* [5] and *quasideviation means* [27]. For more details, we just refer the reader to a series of papers by Losonczi [18, 19, 21, 20, 22, 23] (for Bajraktarević means), Daróczy [5, 6], Daróczy–Losonczi [7], Daróczy–Páles [8, 9] (for deviation means), Páles [27, 28, 29, 30, 31, 32, 33] (for deviation and quasideviation means) and Páles–Pasteczka [35] (for semideviation means).

In what follows, we recall the notions of a quasideviation and the related weighted quasideviation mean (cf. [27], [34] and [35]).

DEFINITION 2.2. A function $E\colon I\times I\to\mathbb{R}$ is said to be a *quasideviation* if

(a) for all elements $x,y\in I$, the sign of $E(x,y)$ coincides with that of $x-y$,

(b) for all $x\in I$, the map $y\mapsto E(x,y)$ is continuous and,

(c) for all $x<y$ in $I$, the mapping $(x,y)\ni t\mapsto \frac{E(y,t)}{E(x,t)}$ is strictly increasing.

By the results of the paper [27], for all $n\in\mathbb{N}$ and $(x,\lambda)\in I^n\times W_n$, the equation

$$\lambda_1E(x_1,y)+\cdots+\lambda_nE(x_n,y)=0 \qquad (2.3)$$

has a unique solution $y$, which will be called the $E$*-quasideviation mean* of $(x,\lambda)$ and denoted by $\mathscr{D}_E(x,\lambda)$.

One can easily notice that power means, quasiarithmetic means, Gini means are quasideviation means.

We say that a quasideviation $E\colon I\times I\to\mathbb{R}$ is *normalizable* if, for all $x\in I$, the function $y\mapsto E(x,y)$ is differentiable at $x$ and the mapping $x\mapsto\partial_2E(x,x)$ is strictly negative and continuous on $I$. The normalization $E^*\colon I\times I\to\mathbb{R}$ of $E$ is defined by

$$E^*(x,y):=\frac{E(x,y)}{-\partial_2E(y,y)}\qquad (x,y\in I).$$

The quasideviation means generated by $E$ and $E^*$ are identical. In [35, Lemma 5.1] we proved that, for a normalized quasideviation $E$, the partial derivative $\partial_2E$ is identically equal to $-1$ on the diagonal of $I\times I$, hence $E^*$ is also a normalizable quasideviation and $(E^*)^*=E^*$ holds.

The following two results of the papers [38] and [35] are instrumental for us.

LEMMA 2.3. ([38], Theorem 2.3) *Let* $f\colon \mathbb{R}_+ \to \mathbb{R}$ *be a concave function such that* $\operatorname{sign}(f(x)) = \operatorname{sign}(x-1)$ *for all* $x \in \mathbb{R}_+$. *Then the function* $E\colon \mathbb{R}_+^2 \to \mathbb{R}$ *defined by* $E(x,y) := f\left(\frac{x}{y}\right)$ *is a quasideviation and the weighted quasideviation mean* $\mathscr{E}_f := \mathscr{D}_E$ *is homogeneous, continuous, nondecreasing and concave.*

LEMMA 2.4. ([35], Theorem 6.3) *Let* $E : I \times I \to \mathbb{R}$ *be a normalizable quasideviation such that* $E^*$ *is concave. Assume that* $\lim_{t\to 0^+} E^*(xt,t) = 0$ *for all* $x \in \mathbb{R}_+$. *Then, for all* $x \in \mathbb{R}_+$, *the limit*

$$h_E(x) := \lim_{t\to 0^+} t^{-1} E^*(xt,t) \tag{2.4}$$

*exists,* $\operatorname{sign}(h_E(x)) = \operatorname{sign}(x-1)$, *and the function* $h_E : \mathbb{R}_+ \to \mathbb{R}$ *so defined is concave and nondecreasing on* $\mathbb{R}_+$, *and is strictly increasing on* $(0,1)$. *Furthermore, the weighted quasideviation mean* $\mathscr{D}_E$ *is Jensen concave, monotone, and*

$$\mathscr{E}_{h_E} = (\mathscr{D}_E)_\# = (\mathscr{D}_E)^\#.$$

## 3. Hardy type inequalities for general weighted means

We recall several definitions and results of the papers [39] and [35]. Throughout the rest of the paper, let $I$ be an interval with $\inf I = 0$.

DEFINITION 3.1. (Weighted Hardy property) For a weighted mean $\mathscr{M}$ on $I$ and a weight sequence $\lambda \in W_0$, let $C$ be the smallest extended real number such that

$$\sum_{n=1}^{\infty} \lambda_n \cdot \mathop{\mathscr{M}}_{i=1}^{n} (x_i, \lambda_i) \leqslant C \cdot \sum_{n=1}^{\infty} \lambda_n x_n \qquad \text{for all sequences } (x_n) \text{ in } I.$$

We call $C$ to be the $\lambda$*-weighted Hardy constant of* $\mathscr{M}$ or the $\lambda$*-Hardy constant of* $\mathscr{M}$ and denote it by $\mathscr{H}_\lambda(\mathscr{M})$. Whenever this constant is finite, then $\mathscr{M}$ is called a $\lambda$*-weighted Hardy mean* or simply a $\lambda$*-Hardy mean*.

Extending some previous results by Elliott [11] and Copson [4], we have obtained in [39] that, in a large class of weighted means, the Hardy constant corresponding to the weight sequence $\mathbf{1} := (1,1,\ldots)$ is the maximal one.

THEOREM 3.2. *For every symmetric and monotone weighted mean* $\mathscr{M}$ *on* $I$, *we have*

$$\mathscr{H}_{\mathbf{1}}(\mathscr{M}) = \sup_{\lambda \in W_0} \mathscr{H}_\lambda(\mathscr{M}).$$

The following lemma from [39] will be used.

LEMMA 3.3. *Let* $\mathscr{M}$ *be a weighted mean on* $I$ *and* $\lambda \in W_0$. *Then, for all* $n \in \mathbb{N}$ *and* $x \in I^n$,

$$\sum_{i=1}^{n} \lambda_i \cdot \mathop{\mathscr{M}}_{j=1}^{i} (x_j, \lambda_j) \leqslant \mathscr{H}_\lambda(\mathscr{M}) \sum_{i=1}^{n} \lambda_i x_i. \tag{3.1}$$

Based on this lemma and Lemma 2.1, we can compare the $\lambda$-Hardy constant of the weighted mean $\mathscr{M}$ and its lower homogenization $\mathscr{M}_\#$.

THEOREM 3.4. *Let $\mathscr{M}$ be a weighted mean on $I$. Then, for all $\lambda \in W_0$,*

$$\mathscr{H}_\lambda(\mathscr{M}_\#) \leqslant \mathscr{H}_\lambda(\mathscr{M}). \tag{3.2}$$

*If, in addition, $\mathscr{M}$ is Jensen concave, then* (3.2) *holds with equality.*

*Proof.* Let $(x_m)$ be a sequence in $\mathbb{R}_+$. For any fixed $n \in \mathbb{N}$, there exists a positive number $\tau_n$ such that $t(x_1,\dots,x_n) \in I^n$ for $t \in (0,\tau_n]$. Using Lemma 3.3, it follows that

$$\sum_{i=1}^{n} \lambda_i \cdot \mathop{\mathscr{M}}_{j=1}^{i} (tx_j,\lambda_j) \leqslant \mathscr{H}_\lambda(\mathscr{M}) \sum_{i=1}^{n} \lambda_i t x_i.$$

Dividing by $t \in (0,\tau_n]$, and then taking the liminf of the left hand side of the inequality so obtained as $t \to 0^+$, (by the superadditivity of the liminf operation), we arrive at

$$\sum_{i=1}^{n} \lambda_i \cdot \liminf_{t\to 0^+} \frac{1}{t} \mathop{\mathscr{M}}_{j=1}^{i} (tx_j,\lambda_j) \leqslant \mathscr{H}_\lambda(\mathscr{M}) \sum_{i=1}^{n} \lambda_i x_i.$$

This inequality is equivalent to

$$\sum_{i=1}^{n} \lambda_i \cdot \mathop{\mathscr{M}_\#}_{j=1}^{i} (x_j,\lambda_j) \leqslant \mathscr{H}_\lambda(\mathscr{M}) \sum_{i=1}^{n} \lambda_i x_i.$$

Finally, passing the limit $n \to \infty$ in the above inequality, we get

$$\sum_{n=1}^{\infty} \lambda_n \cdot \mathop{\mathscr{M}_\#}_{i=1}^{n} (x_i,\lambda_i) \leqslant \mathscr{H}_\lambda(\mathscr{M}) \cdot \sum_{n=1}^{\infty} \lambda_n x_n,$$

which proves that $\mathscr{H}_\lambda(\mathscr{M}_\#) \leqslant \mathscr{H}_\lambda(\mathscr{M})$.

If, additionally, $\mathscr{M}$ is Jensen concave, then, by Lemma 2.1, the comparison inequality $\mathscr{M} \leqslant \mathscr{M}_\#$ is valid, and hence, (3.2) must hold with equality, indeed.

The following result of the paper [39], which is a weighted analogue of [36, Thm 3.3], provides a lower bound for the Hardy constant $\mathscr{H}_\lambda(\mathscr{M})$.

LEMMA 3.5. *Let $\mathscr{M}$ be a weighted mean on $I$, $\lambda \in W_0$, and $(x_n)_{n=1}^{\infty}$ be a sequence of elements in $I$. If $\sum_{n=1}^{\infty} \lambda_n x_n = \infty$, then*

$$\mathscr{H}_\lambda(\mathscr{M}) \geqslant \liminf_{n\to\infty} \frac{1}{x_n} \mathop{\mathscr{M}}_{i=1}^{n} (x_i,\lambda_i).$$

By taking $x_n := \frac{y}{\Lambda_n}$ for a fixed $y \in \lambda_1 I$ in the above theorem, the first inequality of the following consequence was deduced in [39]. The second inequality is an application of the first one to the mean $\mathscr{M}_\#$ and Theorem 3.4.

COROLLARY 3.6. *Let $\mathscr{M}$ be a weighted mean on $I$ and $\lambda \in W_0$ be a weight sequence with $\sum_{n=1}^{\infty} \lambda_n = \infty$. Then we have the following two lower estimates for the $\lambda$-Hardy constant $\mathscr{H}_\lambda(\mathscr{M})$:*

$$\mathscr{H}_\lambda(\mathscr{M}) \geqslant \sup_{y \in \lambda_1 I} \liminf_{n\to\infty} \frac{\Lambda_n}{y} \cdot \underset{k=1}{\overset{n}{\mathscr{M}}} \Big(\frac{y}{\Lambda_k}, \lambda_k\Big) =: \mathscr{C}_\lambda(\mathscr{M})$$

*and*

$$\mathscr{H}_\lambda(\mathscr{M}) \geqslant \liminf_{n\to\infty} \underset{k=1}{\overset{n}{\mathscr{M}_\#}} \Big(\frac{\Lambda_n}{\Lambda_k}, \lambda_k\Big) = \mathscr{C}_\lambda(\mathscr{M}_\#),$$

*where $\Lambda_n := \lambda_1 + \cdots + \lambda_n$ for $n \in \mathbf{N}$.*

Finally let us recall one of key results from [39].

PROPOSITION 3.7. ([39], Corollary 4.3) *Let $\mathscr{M}$ be a symmetric, monotone and Jensen-concave weighted mean which is continuous in the weights and $\lambda \in W_0$ such that $\big(\frac{\lambda_n}{\lambda_1+\ldots+\lambda_n}\big)_{n=1}^{\infty}$ is nonincreasing. Then $\mathscr{H}_\lambda(\mathscr{M}) \leqslant \mathscr{C}_\lambda(\mathscr{M})$. Furthermore, if $\sum_{n=1}^{\infty} \lambda_n = \infty$, then $\mathscr{H}_\lambda(\mathscr{M}) = \mathscr{C}_\lambda(\mathscr{M})$.*

## 4. Auxiliary results

In this section we prove a number of results which will be instrumental in the forthcoming sections. Throughout this section, let $\lambda \in W_0$ be a fixed weight sequence and $\Lambda_n := \lambda_1 + \cdots + \lambda_n$ for $n \in \mathbb{N}$.

LEMMA 4.1. *The sequence $(\Lambda_n)$ and the series $\sum \lambda_n/\Lambda_n$ are equi-convergent (either both of them are convergent or both of them are divergent).*

*Proof.* If $\Lambda_\infty := \sum_{n=1}^{\infty} \lambda_n < \infty$, then

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{\Lambda_n} \leqslant \sum_{n=1}^{\infty} \frac{\lambda_n}{\Lambda_1} = \frac{\Lambda_\infty}{\Lambda_1} < \infty.$$

Conversely, if $\sum_{n=1}^{\infty} \frac{\lambda_n}{\Lambda_n} < \infty$, then there exists $n_0 \in \mathbb{N}$ such that for $n \geqslant n_0$, $\lambda_n/\Lambda_n < \frac{1}{2}$. Equivalently, $\Lambda_{n-1}/\Lambda_n = 1 - \lambda_n/\Lambda_n > \frac{1}{2}$ for $n \geqslant n_0$. Thus,

$$\infty > \sum_{n=n_0}^{\infty} \frac{\lambda_n}{\Lambda_n} \geqslant \frac{1}{2} \sum_{n=n_0}^{\infty} \frac{\lambda_n}{\Lambda_{n-1}} \geqslant \frac{1}{2} \sum_{n=n_0}^{\infty} \int_{\Lambda_{n-1}}^{\Lambda_n} \tfrac{1}{x}\, dx = \frac{1}{2} \int_{\Lambda_{n_0-1}}^{\Lambda_\infty} \tfrac{1}{x}\, dx.$$

As this integral is finite, we obtain $\Lambda_\infty < \infty$.

LEMMA 4.2. *If $\lambda_n/\Lambda_n \to 0$ and $\Lambda_n \to \infty$, then*

$$\lim_{n\to\infty} \frac{\max(\lambda_1, \ldots, \lambda_n)}{\Lambda_n} = 0. \tag{4.1}$$

*Proof.* Fix $\varepsilon > 0$. There exists $k_0 \in \mathbb{N}$ such that $\lambda_k/\Lambda_k \leqslant \varepsilon$ for all $k > k_0$. Take $n_0 \geqslant k_0$ such that $\Lambda_{n_0} \geqslant \frac{1}{\varepsilon} \cdot \max(\lambda_1, \dots, \lambda_{k_0})$. Fix $n > n_0$ arbitrarily. Then

$$\frac{\lambda_k}{\Lambda_n} \leqslant \frac{\max(\lambda_1, \dots, \lambda_{k_0})}{\Lambda_{n_0}} \leqslant \varepsilon, \qquad k \in \{1, \dots, k_0\},$$
$$\frac{\lambda_k}{\Lambda_n} = \frac{\lambda_k}{\Lambda_k} \cdot \frac{\Lambda_k}{\Lambda_n} \leqslant \varepsilon \cdot 1 = \varepsilon, \qquad k \in \{k_0+1, \dots, n\}.$$

Therefore,

$$\frac{\max(\lambda_1, \dots, \lambda_n)}{\Lambda_n} \leqslant \varepsilon \quad \text{for every} \quad n \geqslant n_0,$$

which completes the proof of the statement.

LEMMA 4.3. *Let $\varphi \colon (0,1] \to \mathbb{R}$ be a continuous and nonincreasing function and $q \in (0,1)$. Then the integral $\int_0^1 \varphi$ and the series $\sum_{k=1}^{\infty} q^k \varphi(q^k)$ are equiconvergent. Furthermore,*

$$\frac{q}{1-q} \int_0^1 \varphi \leqslant \sum_{k=1}^{\infty} q^k \varphi(q^k) \leqslant \frac{1}{1-q} \int_0^q \varphi. \tag{4.2}$$

*Proof.* If $\varphi$ is constant then both the integral and the series are convergent. Therefore, replacing $\varphi$ by $\varphi - \varphi(1)$ if necessary, we may assume that $\varphi(1) = 0$. Using the nonincerasingness of $\varphi$, for all $k \in \mathbb{N}$, we obtain

$$q \int_{q^k}^{q^{k-1}} \varphi \leqslant q \int_{q^k}^{q^{k-1}} \varphi(q^k) = (1-q) q^k \varphi(q^k) = \int_{q^{k+1}}^{q^k} \varphi(q^k) \leqslant \int_{q^{k+1}}^{q^k} \varphi.$$

Summing up these inequalities side by side, the inequality (4.2) follows. which proves the integrability of $\varphi$ over $(0,1]$. This inequality also shows the equiconvergence of the integral and the series.

PROPOSITION 4.4. *Let $\varphi \colon (0,1] \to \mathbb{R}$ be a continuous and monotone function. If $\Lambda_n \to \infty$ and the sequence $\left(\frac{\lambda_n}{\Lambda_n}\right)$ is convergent with a limit $\eta$ belonging to $[0,1)$, then*

$$\lim_{n \to \infty} \sum_{k=1}^{n} \frac{\lambda_k}{\Lambda_n} \cdot \varphi\left(\frac{\Lambda_k}{\Lambda_n}\right) = \begin{cases} \int_0^1 \varphi(x)dx & \text{if } \eta = 0, \\ \sum_{k=0}^{\infty} \eta (1-\eta)^k \varphi\big((1-\eta)^k\big) & \text{if } \eta \in (0,1). \end{cases} \tag{4.3}$$

*Proof.* We may suppose without loss of generality that $\varphi$ is nonincreasing. The equality (4.3) is obvious if $\varphi$ is a constant function. Therefore, replacing $\varphi$ by $\varphi - \varphi(1)$, we also can assume that $\varphi(1) = 0$ and then $\varphi$ is nonnegative. Thus the integral and the sum of the series on the right hand side of formula (4.3) are well-defined, however, their value could be equal to $+\infty$.

Assume first that $\eta = 0$. For $n \in \mathbb{N}$, consider the partition $0 < \frac{\Lambda_1}{\Lambda_n} < \cdots < \frac{\Lambda_n}{\Lambda_n} = 1$ of the interval $[0,1]$. By Lemma 4.2, the mesh size of this partition tends to zero as $n \to \infty$. The sum on the left hand side of (4.3) is the Lebesgue integral of the step function $\varphi_n$ defined as $\varphi_n(t) = \varphi(\Lambda_k/\Lambda_n)$ for $t \in (\Lambda_{k-1}/\Lambda_n, \Lambda_k/\Lambda_n]$. Due to the inequality $\varphi_n \leqslant \varphi$, we have that the left hand side of (4.3) is smaller than or equal to the right side. To prove the reversed inequality, let $c < \int_0^1 \varphi(x)dx$. Then, there exists $0 < \alpha < 1$ such that $c < \int_\alpha^1 \varphi(x)dx$. By the continuity of $\varphi$ and (4.1), the sequence of functions $\varphi_n$ pointwise converges to $\varphi$ and the convergence is uniform on the interval $[\alpha, 1]$. Therefore the sequence of integrals $\int_\alpha^1 \varphi_n(x)dx$ converges to $\int_\alpha^1 \varphi(x)dx$. Thus, for large $n$, we have that

$$c < \int_\alpha^1 \varphi_n(x)dx \leqslant \int_0^1 \varphi_n(x)dx = \sum_{k=1}^{n} \frac{\lambda_k}{\Lambda_n} \cdot \varphi\left(\frac{\Lambda_k}{\Lambda_n}\right).$$

This proves the reversed inequality in (4.3) in the case $\eta = 0$.

From now on let us assume that $\eta > 0$. We know that

$$\lim_{n\to\infty} \frac{\Lambda_{n-1}}{\Lambda_n} = 1 - \eta$$

and therefore, by simple induction,

$$\lim_{n\to\infty} \frac{\Lambda_{n-k}}{\Lambda_n} = (1-\eta)^k, \quad k \in \mathbb{N}.$$

In particular,

$$\lim_{n\to\infty} \frac{\lambda_{n-k}}{\Lambda_n} = \lim_{n\to\infty} \frac{\Lambda_{n-k}}{\Lambda_n} - \frac{\Lambda_{n-k-1}}{\Lambda_n} = (1-\eta)^k - (1-\eta)^{k+1} = \eta \cdot (1-\eta)^k.$$

Therefore, for all $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \frac{\lambda_{n-k}}{\Lambda_n} \cdot \varphi\left(\frac{\Lambda_{n-k}}{\Lambda_n}\right) = \eta \cdot (1-\eta)^k \cdot \varphi\left((1-\eta)^k\right). \tag{4.4}$$

For all $n > m \geqslant 1$, we have

$$\sum_{k=1}^{n} \frac{\lambda_k}{\Lambda_n} \varphi\left(\frac{\Lambda_k}{\Lambda_n}\right) = \sum_{k=0}^{n-1} \frac{\lambda_{n-k}}{\Lambda_n} \varphi\left(\frac{\Lambda_{n-k}}{\Lambda_n}\right) \geqslant \sum_{k=0}^{m} \frac{\lambda_{n-k}}{\Lambda_n} \varphi\left(\frac{\Lambda_{n-k}}{\Lambda_n}\right).$$

Thus, using (4.4), the above inequality implies

$$\liminf_{n\to\infty} \sum_{k=0}^{n} \frac{\lambda_k}{\Lambda_n} \varphi\left(\frac{\Lambda_k}{\Lambda_n}\right) \geqslant \lim_{n\to\infty} \sum_{k=0}^{m} \frac{\lambda_{n-k}}{\Lambda_n} \varphi\left(\frac{\Lambda_{n-k}}{\Lambda_n}\right) = \sum_{k=0}^{m} \eta(1-\eta)^k \cdot \varphi\big((1-\eta)^k\big).$$

Upon taking the limit $m \to \infty$, it follows that

$$\liminf_{n\to\infty} \sum_{k=0}^{n} \frac{\lambda_k}{\Lambda_n} \varphi\left(\frac{\Lambda_k}{\Lambda_n}\right) \geqslant \sum_{k=0}^{\infty} \eta(1-\eta)^k \cdot \varphi\big((1-\eta)^k\big).$$

This implies also the equality in (4.3) if the right-hand-side series is divergent. Therefore, in the rest of the proof, we can assume that this series is convergent.

Fix $\varepsilon > 0$ and choose $k_0$ such that

$$\sum_{k=k_0}^{\infty} \eta \cdot (1-\eta)^k \cdot \varphi\left((1-\eta)^k\right) \leqslant \frac{\varepsilon}{4} \quad \text{and} \quad \int_0^{(1-\eta)^{k_0}} \varphi(x)dx \leqslant \frac{\varepsilon}{4}. \tag{4.5}$$

Moreover, by (4.4), there exists $n_0$ such that for all $k \in \{0,1,\dots,k_0-1\}$ and $n \geqslant n_0$

$$\left|\frac{\lambda_{n-k}}{\Lambda_n} \cdot \varphi\left(\frac{\Lambda_{n-k}}{\Lambda_n}\right) - \eta \cdot (1-\eta)^k \cdot \varphi((1-\eta)^k)\right| \leqslant \frac{\varepsilon}{4k_0}. \tag{4.6}$$

Now, applying the nonincreasingness of $\varphi$ again, for all $n \geqslant k_0$,

$$\sum_{k=k_0}^{n-1} \frac{\lambda_{n-k}}{\Lambda_n} \cdot \varphi\left(\frac{\Lambda_{n-k}}{\Lambda_n}\right) = \sum_{k=1}^{n-k_0} \frac{\lambda_k}{\Lambda_n} \cdot \varphi\left(\frac{\Lambda_k}{\Lambda_n}\right) \leqslant \int_0^{\frac{\Lambda_{n-k_0}}{\Lambda_n}} \varphi(x)dx. \tag{4.7}$$

But

$$\lim_{n\to\infty} \int_0^{\frac{\Lambda_{n-k_0}}{\Lambda_n}} \varphi(x)dx = \int_0^{(1-\eta)^{k_0}} \varphi(x)dx \leqslant \frac{\varepsilon}{4},$$

so there exists $n_1 \geqslant \max(n_0,k_0)$ such that

$$\int_0^{\frac{\Lambda_{n-k_0}}{\Lambda_n}} \varphi(x)dx \leqslant \frac{\varepsilon}{2} \qquad \text{for all} \qquad n \geqslant n_1.$$

Thus, by (4.7),

$$\sum_{k=k_0}^{n-1} \frac{\lambda_{n-k}}{\Lambda_n} \cdot \varphi\left(\frac{\Lambda_{n-k}}{\Lambda_n}\right) \leqslant \frac{\varepsilon}{2} \qquad \text{for all} \qquad n \geqslant n_1. \tag{4.8}$$

Finally, applying (4.6), (4.8), and (4.5), for all $n \geqslant n_1$,

$$\begin{aligned}
&\left|\sum_{k=1}^{n} \frac{\lambda_k}{\Lambda_n} \cdot \varphi\left(\frac{\Lambda_k}{\Lambda_n}\right) - \sum_{k=0}^{\infty} \eta(1-\eta)^k\varphi((1-\eta)^k)\right| \\
&= \left|\sum_{k=0}^{n-1} \frac{\lambda_{n-k}}{\Lambda_n} \cdot \varphi\left(\frac{\Lambda_{n-k}}{\Lambda_n}\right) - \sum_{k=0}^{\infty} \eta(1-\eta)^k\varphi((1-\eta)^k)\right| \\
&\leqslant \sum_{k=0}^{k_0-1} \left|\frac{\lambda_{n-k}}{\Lambda_n} \cdot \varphi\left(\frac{\Lambda_{n-k}}{\Lambda_n}\right) - \eta(1-\eta)^k\varphi((1-\eta)^k)\right| \\
&\quad + \sum_{k=k_0}^{n-1} \frac{\lambda_{n-k}}{\Lambda_n} \cdot \varphi\left(\frac{\Lambda_{n-k}}{\Lambda_n}\right) + \sum_{k=k_0}^{\infty} \eta(1-\eta)^k\varphi((1-\eta)^k) \leqslant k_0 \cdot \frac{\varepsilon}{4k_0} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}$$

This completes the proof in the case $\eta \in (0,1)$.

It is worth mentioning that (4.2) with $q = 1-\eta$ follows that

$$\lim_{\eta\to 0^+} \sum_{k=0}^{\infty} \eta(1-\eta)^k\varphi\left((1-\eta)^k\right) = \int_0^1 \varphi(x)dx,$$

which means, that the right hand side of (4.3) is a continuous function of $\eta$. Applying the above result to the power function $\varphi(x) = x^{-p}$ (where $p < 1$), we immediately get

COROLLARY 4.5. *Let $p < 1$. If $\Lambda_n \to \infty$ and $\frac{\lambda_n}{\Lambda_n} \to \eta \in [0,1)$, then*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\lambda_k}{\Lambda_n} \cdot \left(\frac{\Lambda_k}{\Lambda_n}\right)^{-p} = \begin{cases} \dfrac{1}{1-p} & \eta = 0, \\ \dfrac{\eta}{1-(1-\eta)^{1-p}} & \eta \in (0,1). \end{cases}$$

LEMMA 4.6. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a concave function such that $\operatorname{sign}(f(x)) = \operatorname{sign}(x-1)$ holds for all $x \in \mathbb{R}_+$ and the function $x \mapsto f(1/x)$ is integrable over $(0,1]$. Then the function $F$ given by*

$$F(x,q) := \sum_{k=0}^{\infty} q^k f\big(q^{-k}x\big) \qquad \big((x,q) \in \mathbb{R}_+ \times (0,1)\big)$$

*is well-defined, continuous and nondecreasing in its first variable. Furthermore, for all fixed $q \in (0,1)$, the equation $F(x,q) = 0$ has a unique solution $x(q) \in (0,1)$. The mapping $x(\cdot)$ so defined is continuous, and we have the following estimates:*

$$\frac{q}{1-q}\int_0^{1/q} f\Big(\frac{x}{t}\Big)dt \leqslant F(x,q) \leqslant \frac{1}{1-q}\int_0^{1} f\Big(\frac{x}{t}\Big)dt \qquad (x,q) \in \mathbb{R}_+ \times (0,1). \tag{4.9}$$

*Proof.* By elementary considerations, it follows from the concavity and the sign properties that $f$ is nondecreasing on $\mathbb{R}_+$ and is strictly increasing on $(0,1)$, furthermore, it is also continuous. It also follows from the concavity that the map

$$u \mapsto \frac{f(u)-f(1)}{u-1} = \frac{f(u)}{u-1}$$

is nonincreasing on $\mathbb{R}_+ \setminus \{1\}$. Therefore, if $1 < u_0 \leqslant u$, then

$$\frac{f(u)}{u} \leqslant \frac{f(u)}{u-1} \leqslant \frac{f(u_0)}{u_0-1}. \tag{4.10}$$

To show that $F$ is continuous, let $(x_0,q_0) \in \mathbb{R}_+ \times (0,1)$ be fixed. The product $q_0^{-k}x$ is bigger than $1$ for $k \geqslant k_0 := 1 + \lceil \log(x_0)/\log(q_0) \rceil$. Therefore, there exist $0 < x_* < x_0 < x^*$ and $0 < q_* < q_0 < q^* < 1$ such that $q^{-k}x > 1$ for all $(x,q) \in V := [x_*,x^*] \times [q_*,q^*]$ and $k \geqslant k_0$. The expression $\sum_{k=0}^{k_0-1} q^k f(q^{-k}x)$ being a finite sum of continuous functions is obviously continuous at $(x_0,q_0)$. Therefore, it suffices to show that tail sum

$$F_{k_0}(x,q) := \sum_{k=k_0}^{\infty} q^k f(q^{-k}x)$$

is also continuous at $(x_0,q_0)$. By the choice of $k_0$, each term is positive for $(x,q)\in V$. By the nondecreasingness of $f$, for all $k\geqslant 0$ and $(x,q)\in\mathbb{R}_+\times(0,1)$, we clearly have

$$\begin{aligned} q^k f(q^{-k}x) &= \frac{1}{1-q}\int_{q^{k+1}}^{q^k} f\Big(\frac{x}{q^k}\Big)dt \leqslant \frac{1}{1-q}\int_{q^{k+1}}^{q^k} f\Big(\frac{x}{t}\Big)dt,\\ q^k f(q^{-k}x) &= \frac{q}{1-q}\int_{q^k}^{q^{k-1}} f\Big(\frac{x}{q^k}\Big)dt \geqslant \frac{q}{1-q}\int_{q^k}^{q^{k-1}} f\Big(\frac{x}{t}\Big)dt. \end{aligned} \tag{4.11}$$

Summarizing the first inequality for $k\geqslant k_0$, it follows that

$$\sum_{k=k_0}^{\infty} q^k f(q^{-k}x) \leqslant \frac{1}{1-q}\int_0^{q^{k_0}} f\Big(\frac{x}{t}\Big)dt < +\infty \qquad (x,q)\in V,$$

which implies that the series on the left hand side is convergent. To prove that the sum of this series (i.e., $F_{k_0}(x,q)$) is a continuous function of $(x,q)$ at $(x_0,q_0)$, it suffices to show that the convergence is uniform over $V$.

Observe that, for $k\geqslant k_0$ and $(x,q)\in V$, we have

$$u_0 := \frac{x_*}{(q^*)^k} \leqslant \frac{x}{q^k} := u.$$

Now using the inequality (4.10) for the above $u_0$ and $u$, it follows that

$$\begin{aligned} q^k f(q^{-k}x) = x\frac{f(u)}{u} &\leqslant x^*\frac{f(u_0)}{u_0-1} = \frac{x^*}{x_*-(q^*)^k}\cdot (q^*)^k f\big((q^*)^{-k}x_*\big)\\ &\leqslant \frac{x^*}{x_*-(q^*)^{k_0}}\cdot (q^*)^k f\big((q^*)^{-k}x_*\big). \end{aligned}$$

This inequality shows that, for $(x,q)\in V$ and $k\geqslant k_0$, the $k$th term of the series corresponding to $F_{k_0}(x,q)$ is majorized by a constant multiple of the corresponding term of the series for $F_{k_0}(x_*,q^*)$. Thus, in view of the Weierstrass $M$-test, the convergence of the series corresponding to $F_{k_0}(x,q)$ is uniform. By the continuity of each term of this series, it follows that the sum function is also continuous at $(x_0,q_0)$.

Thus, we have proved that $F$ is a well-defined continuous function on $\mathbb{R}_+\times(0,1)$. Moreover, as $f$ is nondecreasing on $(1,\infty)$ and strictly increasing on $(0,1)$, we obtain that $F(\cdot,q)$ is nondecreasing on $(1,\infty)$ and strictly increasing on $(0,1)$ (as all terms of the sum are nondecreasing and the very first of them is strictly increasing on $(0,1)$).

Finally, in order to prove that $F(\cdot,q)$ has a unique zero note that $f(q^n)<f(q)$, and $f(q^k)<0$ for all $k\in\{1,\dots,n-1\}$. Thus we get

$$F(q^n,q) = f(q^n)+\dots+q^{n-1}f(q)+q^nF(1,q) < f(q)+q^nF(1,q).$$

Therefore, for large $n$, we have that $F(q^n,q)<0$. This, with the easy-to-see inequality $F(1,q)>0$, implies that for all $q\in(0,1)$, the equality $F(x,q)=0$ has a solution $x=x(q)\in(0,1)$.

To show that $x(q)$ depends continuously on $q$, let $q_0 \in (0,1)$ be fixed and $0 < \varepsilon < \min(x(q_0), 1-x(q_0))$. Since $F(x(q_0),q_0) = 0$, we have that

$$F(x(q_0)-\varepsilon, q_0) < 0 < F(x(q_0)+\varepsilon, q_0).$$

By the continuity of $F$, there exists $0 < \delta < \min(q_0, 1-q_0)$ such that, for all $q \in (q_0-\delta, q_0+\delta)$,

$$F(x(q_0)-\varepsilon, q) < 0 < F(x(q_0)+\varepsilon, q).$$

Therefore, the uniquely defined value $x(q)$ must be between $x(q_0)-\varepsilon$ and $x(q_0)+\varepsilon$, that is, $|x(q)-x(q_0)| < \varepsilon$ for all $q \in (q_0-\delta, q_0+\delta)$.

Finally, if we sum up (both) inequalities side by side in (4.11) for all $k \in \{0,1,\dots\}$, we easily obtain (4.9).

PROPOSITION 4.7. *Let* $\lambda \in W_0$ *with* $\Lambda_n \to \infty$, $\lambda_n/\Lambda_n \to \eta \in [0,1)$ *and let* $f : \mathbb{R}_+ \to \mathbb{R}$ *be a concave function such that* $\operatorname{sign}(f(x)) = \operatorname{sign}(x-1)$ *holds for all* $x \in \mathbb{R}_+$ *and the function* $x \mapsto f(1/x)$ *is integrable over* $(0,1]$. *Then* $c := \mathscr{C}_\lambda(\mathscr{E}_f)$ *is the unique solution of the equation*

$$\begin{aligned} \int_0^1 f\Big(\frac{1}{cx}\Big)\,dx &= 0 \qquad \text{for } \eta = 0, \\ \sum_{k=0}^{\infty}(1-\eta)^k f\Big(\frac{1}{c(1-\eta)^k}\Big) &= 0 \qquad \text{for } \eta > 0. \end{aligned} \tag{4.12}$$

*Proof.* The first equation in (4.12) is equivalent to

$$\int_0^c f\Big(\frac{1}{x}\Big)\,dx = 0,$$

which, by [38, Theorem 3.4] has a unique solution $c$ in the interval $(1,\infty)$. On the other hand, putting $q := 1-\eta$, the second equation in (4.12) is equivalent to the $F(1/c, 1-\eta) = 0$, which, according to Lemma 4.6, also has a unique solution in the interval $(1,\infty)$.

Fix any $K \in (0,c)$. Then there exists $n_K \in \mathbb{N}$ such that

$$K < \mathop{\mathscr{E}_f}_{k=1}^{n}\Big(\frac{\Lambda_n}{\Lambda_k}, \lambda_k\Big) \qquad \text{for all } n > n_K.$$

Equivalently,

$$0 < \sum_{k=1}^{n}\frac{\lambda_k}{\Lambda_n} f\Big(\frac{\Lambda_n}{K\Lambda_k}\Big) \qquad \text{for all } n > n_K.$$

Then, with $\varphi_K(x) := f(\frac{1}{Kx})$, we have

$$0 < \sum_{k=1}^{n}\frac{\lambda_k}{\Lambda_n}\varphi_K\Big(\frac{\Lambda_k}{\Lambda_n}\Big) \qquad \text{for all } K \in (0,c) \text{ and } n \geqslant n_K.$$

Observe that $\varphi_K$ is a nonincreasing, continuous and integrable function on $(0,1]$, therefore upon taking the limit $n \to \infty$ and using Proposition 4.4, it follows that

$$0 \leqslant \begin{cases} \int_0^1 \varphi_K(x)dx & \text{if } \eta = 0, \\ \sum_{k=0}^{\infty} \eta(1-\eta)^k \varphi_K\big((1-\eta)^k\big) & \text{if } \eta \in (0,1). \end{cases} \tag{4.13}$$

Similarly, for all $L \in (c, +\infty)$, there exist a sequence of integers $n_i \to \infty$, such that, for all $i \in \mathbb{N}$,

$$0 > \sum_{k=1}^{n_i} \frac{\lambda_k}{\Lambda_{n_i}} \varphi_L\Big(\frac{\Lambda_k}{\Lambda_{n_i}}\Big).$$

Upon taking the limit $i \to \infty$ and again using Proposition 4.4, we obtain that

$$0 \geqslant \begin{cases} \int_0^1 \varphi_L(x)dx & \text{if } \eta = 0, \\ \sum_{k=0}^{\infty} \eta(1-\eta)^k \varphi_L\big((1-\eta)^k\big) & \text{if } \eta \in (0,1). \end{cases} \tag{4.14}$$

Combining the first inequalities from (4.13) and (4.14), in the case $\eta = 0$, we get

$$\int_0^L f\Big(\frac{1}{x}\Big)dx = \int_0^1 f\Big(\frac{1}{Lx}\Big)dx \leqslant 0 \leqslant \int_0^1 f\Big(\frac{1}{Kx}\Big)dx = \int_0^K f\Big(\frac{1}{x}\Big)dx,$$

while, for $\eta \in (0,1)$, we obtain

$$\begin{aligned} \eta F(L^{-1}, 1-\eta) &= \sum_{k=0}^{\infty} \eta(1-\eta)^k f\Big(\frac{1}{L(1-\eta)^k}\Big) \leqslant 0 \leqslant \sum_{k=0}^{\infty} \eta(1-\eta)^k f\Big(\frac{1}{K(1-\eta)^k}\Big) \\ &= \eta F(K^{-1}, 1-\eta). \end{aligned}$$

If we now take the common limits $K \nearrow c$ and $L \searrow c$, and we use the continuity of $F$ established in Lemma 4.6, we get (4.12).

## 5. Applications

Now we are going to present some weighted Hardy constants for quasiarithmetic means. It is well known that for $\pi_p(x) := x^p$ if $p \neq 0$ and $\pi_0(x) := \ln x$ equality $\mathscr{A}_{\pi_p} = \mathscr{P}_p$ holds. Furthermore, the comparability problem within this family can be (under natural smoothness assumptions) boiled down to pointwise comparability of the mapping $f \mapsto \frac{f''}{f'}$ (cf. [14]). More precisely, we have

PROPOSITION 5.1. *Let $I \subset \mathbb{R}$ be an interval, $f, g \colon I \to \mathbb{R}$ be twice differentiable functions having nowhere vanishing first derivatives. Then the following two conditions are equivalent:*

(i) $\mathscr{A}_f(x,\lambda) \leqslant \mathscr{A}_g(x,\lambda)$ *for all* $n \in \mathbb{N}$ *and* $(x,\lambda) \in I^n \times W_n$*;*

(ii) $\frac{f''(x)}{f'(x)} \leqslant \frac{g''(x)}{g'(x)}$ *for all* $x \in I$*.*

In a special case $I \subseteq \mathbb{R}_+$ condition (ii) can be equivalently written as

$$\chi_f(x) := \frac{xf''(x)}{f'(x)} + 1 \leqslant \frac{xg''(x)}{g'(x)} + 1 =: \chi_g(x) \qquad (x \in I).$$

It is easy to verify that the equality $\chi_{\pi_p} \equiv p$ holds for all $p \in \mathbb{R}$. Therefore, in view of Proposition 5.1, we have

$$\mathscr{P}_q = \mathscr{A}_{\pi_q} \leqslant \mathscr{A}_f \leqslant \mathscr{A}_{\pi_p} = \mathscr{P}_p,$$

where $q := \inf_I \chi_f$ and $p := \sup_I \chi_f$, moreover these parameters are sharp. In other words, the operator $\chi_{(\cdot)}$ could be applied to embed quasiarithmetic means into the scale of power means (cf. [25]).

This fact will be used to establish some weighted Hardy constants for quasiarithmetic means. Our main idea is to compare a quasiarithmetic mean with a suitable power mean. As a matter of fact, this is not so restrictive as it seams to be at first glance. Namely, Mulholland [24] proved that a quasiarithmetic mean is Hardy if and only if it is majorized up to a constant number by some power mean with parameter strictly smaller than one. Throughout this section, we will use the already introduced notation $\Lambda_n := \lambda_1 + \cdots + \lambda_n$.

PROPOSITION 5.2. *Let* $(\lambda_n) \in W_0$ *such that* $\Lambda_n \to \infty$ *and* $\lim_{n\to\infty} \lambda_n/\Lambda_n =: \eta$ *exists. Let* $f\colon \mathbb{R}_+ \to \mathbb{R}$ *be a twice continuously differentiable function with a nonvanishing first derivative and define*

$$q := \liminf_{x\to 0^+} \chi_f(x) \leqslant \limsup_{x\to 0^+} \chi_f(x) =: p.$$

*Assume that* $p < 1$*. Then, for all* $x \in \mathbb{R}_+$*,*

$$\begin{aligned} C(q,\eta) &\leqslant \liminf_{n\to\infty} \frac{\Lambda_n}{x} \mathscr{A}_f\left(\left(\frac{x}{\Lambda_1}, \frac{x}{\Lambda_2}, \dots, \frac{x}{\Lambda_n}\right), (\lambda_1, \dots, \lambda_n)\right) \\ &\leqslant \limsup_{n\to\infty} \frac{\Lambda_n}{x} \mathscr{A}_f\left(\left(\frac{x}{\Lambda_1}, \frac{x}{\Lambda_2}, \dots, \frac{x}{\Lambda_n}\right), (\lambda_1, \dots, \lambda_n)\right) \leqslant C(p,\eta), \end{aligned} \tag{5.1}$$

*where the function* $C\colon (-\infty,1) \times [0,1) \to \mathbb{R}$ *is defined by*

$$C(r,\eta) := \begin{cases} \left(\dfrac{\eta}{1-(1-\eta)^{1-r}}\right)^{1/r} & \eta \in (0,1) \text{ and } r \neq 0; \\ (1-\eta)^{1-1/\eta} & \eta \in (0,1) \text{ and } r = 0; \\ (1-r)^{-1/r} & \eta = 0 \text{ and } r \neq 0; \\ e & \eta = 0 \text{ and } r = 0. \end{cases} \tag{5.2}$$

*Proof.* It is elementary to see that $C$ is a continuous function which is strictly increasing in its first variable.

Following the lines of proof of [39, Theorem 3.1] we get that for all $r \in (p,1) \setminus \{0\}$,

$$U := \limsup_{n\to\infty} \frac{\Lambda_n}{x} \cdot \mathscr{A}_f\Big(\Big(\frac{x}{\Lambda_1}, \frac{x}{\Lambda_2}, \dots, \frac{x}{\Lambda_n}\Big), (\lambda_1, \dots, \lambda_n)\Big)$$
$$\leqslant \limsup_{n\to\infty} \frac{\Lambda_n}{x} \cdot \mathscr{P}_r\Big(\Big(\frac{x}{\Lambda_1}, \frac{x}{\Lambda_2}, \dots, \frac{x}{\Lambda_n}\Big), (\lambda_1, \dots, \lambda_n)\Big).$$

Therefore, as $\mathscr{P}_r$ is homogeneous, we obtain

$$U \leqslant \limsup_{n\to\infty} \mathscr{P}_r\Big(\Big(\frac{\Lambda_n}{\Lambda_1}, \frac{\Lambda_n}{\Lambda_2}, \dots, \frac{\Lambda_n}{\Lambda_n}\Big), (\lambda_1, \dots, \lambda_n)\Big) = \limsup_{n\to\infty} \left(\sum_{k=1}^{n} \frac{\lambda_k}{\Lambda_n} \cdot \Big(\frac{\Lambda_k}{\Lambda_n}\Big)^{-r}\right)^{1/r}.$$

Thus, due to Corollary 4.5, we obtain $U \leqslant C(r,\eta)$ for $r \in (p,1) \setminus \{0\}$. Now, as the function $C$ is continuous, we can pass the limit $r \searrow p$ and obtain $U \leqslant C(p,\eta)$. The verification of the left hand side inequality in (5.1) is completely analogous.

We will now establish some $\lambda$-Hardy constants in a family of quasiarithmetic means.

COROLLARY 5.3. *Let $(\lambda_n) \in W_0$ such that $\Lambda_n \to \infty$ and $\big(\frac{\lambda_n}{\Lambda_n}\big)_{n=1}^{\infty}$ is nonincreasing with a limit $\eta \in [0,1)$. Let $f\colon \mathbb{R}_+ \to \mathbb{R}$ be a twice continuously differentiable function with a nonvanishing first derivative, such that the limit*

$$p := \lim_{x\to 0^+} \chi_f(x)$$

*exists, is smaller than 1, and $\chi_f(x) \leqslant p$ for all $x \in \mathbb{R}_+$. Then $\mathscr{H}_\lambda(\mathscr{A}_f) = C(p,\eta)$, where the function $C$ was defined by (5.2).*

*Proof.* By Corollary 3.6 and Proposition 5.2 we have

$$\mathscr{H}_\lambda(\mathscr{A}_f) \geqslant \sup_{x>0} \liminf_{n\to\infty} \frac{\Lambda_n}{x} \mathscr{A}_f\Big(\Big(\frac{x}{\Lambda_1}, \dots, \frac{x}{\Lambda_n}\Big), (\lambda_1, \dots, \lambda_n)\Big) = C(p,\eta).$$

Furthermore, by $\chi_f(x) \leqslant p$ we get $\mathscr{A}_f \leqslant \mathscr{P}_p$ so

$$\mathscr{H}_\lambda(\mathscr{A}_f) \leqslant \mathscr{H}_\lambda(\mathscr{P}_p).$$

But $\mathscr{P}_p$ is repetition invariant and concave, thus it is a $\lambda$-Kedlaya mean (in the sense of our paper [37]). Thus, by Proposition 3.7 and Proposition 5.2,

$$\mathscr{H}_\lambda(\mathscr{P}_p) \leqslant \liminf_{n\to\infty} \frac{\Lambda_n}{x} \mathscr{P}_p\Big(\Big(\frac{x}{\Lambda_1}, \dots, \frac{x}{\Lambda_n}\Big), (\lambda_1, \dots, \lambda_n)\Big) = C(p,\eta).$$

Binding all these inequalities, we get

$$C(p,\eta) \leqslant \mathscr{H}_\lambda(\mathscr{A}_f) \leqslant \mathscr{H}_\lambda(\mathscr{P}_p) \leqslant C(p,\eta),$$

which implies $\mathscr{H}_\lambda(\mathscr{A}_f) = C(p,\eta)$.

THEOREM 5.4. *Let $(\lambda_n) \in W_0$ such that $\Lambda_n \to \infty$ and $\left(\frac{\lambda_n}{\Lambda_n}\right)_{n=1}^{\infty}$ is nonincreasing with limit $\eta \in [0,1)$. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a concave function such that $\operatorname{sign}(f(x)) = \operatorname{sign}(x-1)$ holds for all $x \in \mathbb{R}_+$. Then the homogeneous quasideviation mean $\mathscr{E}_f$ is $\lambda$-Hardy if and only if function $x \mapsto f(1/x)$ is integrable over $(0,1]$. In the latter case, $c := \mathscr{H}_\lambda(\mathscr{E}_f)$ is the unique solution of the equation (4.12).*

*Proof.* Assume that $\mathscr{E}_f$ is $\lambda$-Hardy. Then, by Corollary 3.6,

$$\liminf_{n\to\infty} \mathop{\mathscr{E}_f}_{k=1}^{n}\left(\frac{\Lambda_n}{\Lambda_k}, \lambda_k\right) = \mathscr{C}_\lambda(\mathscr{E}_f) \leqslant \mathscr{H}_\lambda(\mathscr{E}_f) < \infty.$$

Then, there exists a strictly increasing sequence of integers $(n_i)$ such that

$$\mathop{\mathscr{E}_f}_{k=1}^{n_i}\left(\frac{\Lambda_{n_i}}{\Lambda_k}, \lambda_k\right) < \mathscr{H}_\lambda(\mathscr{E}_f) + 1 =: K \qquad (i \in \mathbb{N}),$$

which is equivalent to the inequality

$$\sum_{k=1}^{n_i} \frac{\lambda_k}{\Lambda_{n_i}} f\left(\frac{\Lambda_{n_i}}{K\Lambda_k}\right) < 0 \qquad (i \in \mathbb{N}).$$

Applying now Proposition 4.4 for the nonincreasing function $\varphi(x) := f\left(\frac{1}{Kx}\right)$, upon taking the limit $i \to \infty$, in the case when $\eta = 0$, it follows that

$$\int_0^1 f\left(\frac{1}{Kx}\right)dx \leqslant 0,$$

while in the case $\eta \in (0,1)$, we get that

$$\sum_{k=0}^{\infty} \eta(1-\eta)^k f\left(\frac{1}{K(1-\eta)^k}\right) \leqslant 0.$$

The first inequality implies that $\varphi$ is integrable over $(0,1]$, hence the mapping $x \mapsto f(1/x)$ is also integrable on $(0,1]$. In view of Lemma 4.3, the same conclusion is derived from the second inequality.

In the rest of the proof, assume that the mapping $x \mapsto f(1/x)$ is also integrable on $(0,1]$. Obviously $\mathscr{E}_f$ is a homogeneous, symmetric and continuously weighted mean. Moreover, in view of Lemma 2.3, $\mathscr{E}_f$ is monotone and Jensen concave. Thus, by Proposition 3.7, $\mathscr{H}_\lambda(\mathscr{E}_f) = \mathscr{C}_\lambda(\mathscr{E}_f)$. Consequently, applying Proposition 4.7, one obtains that $c = \mathscr{H}_\lambda(\mathscr{E}_f)$ is a unique and finite solution of equation (4.12), indeed. In particular, this yields, that $\mathscr{E}_f$ is a $\lambda$-Hardy mean.

An interesting consequence of the previous result is that a homogeneous quasideviation mean $\mathscr{E}_f$ is $\lambda$-Hardy (where $\lambda$ is like above) if and only if it is $\mathbf{1}$-Hardy.

One of our main results is stated in the subsequent theorem.

THEOREM 5.5. *Let $(\lambda_n) \in W_0$ such that $\Lambda_n \to \infty$ and $\left(\frac{\lambda_n}{\Lambda_n}\right)_{n=1}^{\infty}$ is nonincreasing with limit $\eta \in [0,1)$. Let $E : I \times I \to \mathbb{R}$ be a normalizable quasideviation such that $E^*$ is concave. Assume that, for all $x \in \mathbb{R}_+$, $\lim_{t\to 0} E^*(xt,t) = 0$ and define $h_E : \mathbb{R}_+ \to \mathbb{R}$ by (2.4). Then the quasideviation mean $\mathscr{D}_E$ is $\lambda$-Hardy if and only if the mapping $x \mapsto h_E(1/x)$ is integrable over $(0,1]$ and in this case, $c := \mathscr{H}_\lambda(\mathscr{D}_E)$ is a unique solution of (4.12) with $f := h_E$.*

*Proof.* First, by Lemma 2.4 we know that $f := h_E$ is correctly defined. Furthermore it is nondecreasing on $(0,\infty)$, strictly increasing on $(0,1)$, and admits the sign property $\operatorname{sign}(f(x)) = \operatorname{sign}(x-1)$ and $\mathscr{E}_f$ is a homogeneous quasideviation mean.

First assume that $\mathscr{D}_E$ is a $\lambda$-Hardy mean. Then, by Theorem 3.4, $(\mathscr{D}_E)_\#$ is also a $\lambda$-Hardy mean. On the other hand, Lemma 2.4 implies that $(\mathscr{D}_E)_\# = \mathscr{E}_f$, hence, we get that $\mathscr{E}_f$ is a $\lambda$-Hardy mean, too. By the previous theorem, this implies that the mapping $x \mapsto f(1/x)$ is integrable over $(0,1]$.

In the rest of the proof, assume that the mapping $x \mapsto f(1/x)$ is integrable over $(0,1]$. In view of Proposition 4.7, we have that $c := \mathscr{H}_\lambda(\mathscr{E}_f)$ is a unique solution of (4.12). On the other hand, by Theorem 3.4, we have that $\mathscr{H}_\lambda(\mathscr{D}_E) = \mathscr{H}_\lambda(\mathscr{E}_f)$, which yields that $\mathscr{H}_\lambda(\mathscr{D}_E) = c$.

COROLLARY 5.6. *Let $(\lambda_n) \in W_0$ such that $\Lambda_n \to \infty$ and $\left(\frac{\lambda_n}{\Lambda_n}\right)_{n=1}^{\infty}$ is nonincreasing with a limit $\eta \in [0,1)$. Let $p,q \in \mathbb{R}$, $\min(p,q) \leqslant 0 \leqslant \max(p,q) < 1$. Then*

$$\mathscr{H}_\lambda(\mathscr{G}_{p,q}) = \begin{cases} \left(\dfrac{1-(1-\eta)^{1-q}}{1-(1-\eta)^{1-p}}\right)^{\frac{1}{p-q}} & \eta \in (0,1) \text{ and } p \neq q; \\ \left(\dfrac{1-q}{1-p}\right)^{\frac{1}{p-q}} & \eta = 0 \text{ and } p \neq q; \\ (1-\eta)^{1-1/\eta} & \eta \in (0,1) \text{ and } p = q = 0; \\ e & \eta = 0 \text{ and } p = q = 0. \end{cases}$$

*Proof.* Fix $p,q$ like above. In the case $p = 0$ (resp. $q = 0$), we have $\mathscr{G}_{p,q} = \mathscr{P}_q$ (resp. $\mathscr{G}_{p,q} = \mathscr{P}_p$) and the assertion is implied by Corollary 5.3. As $\mathscr{G}_{p,q} = \mathscr{G}_{q,p}$ and the right hand side is symmetric, we can assume that $p < 0 < q < 1$.

Observe that Gini means are homogeneous deviation means – more precisely $\mathscr{G}_{p,q} = \mathscr{E}_f$ with $f(x) = \frac{x^p - x^q}{p-q}$. The condition $p < 0 < q < 1$ implies that $f$ is concave, satisfies the sign condition and the mapping $x \mapsto f(1/x)$ is integrable. Therefore, Theorem 5.4 yields that $\mathscr{H}_\lambda(\mathscr{G}_{p,q})$ is the unique solution $c$ of equation (4.12).

Let us now split our considerations into two parts. For $\eta = 0$, we have

$$0 = \int_0^1 f\left(\frac{1}{cx}\right)dx = \int_0^1 \frac{c^{-p}}{p-q}x^{-p} - \frac{c^{-q}}{p-q}x^{-q}dx = \frac{1}{p-q}\cdot\left(\frac{c^{-p}}{1-p} - \frac{c^{-q}}{1-q}\right),$$

which, after an easy transformation, is equivalent to $c = \left(\frac{1-q}{1-p}\right)^{1/(p-q)}$.

For $\eta > 0$, we need to solve the second equation of (4.12), which in our setting states

$$\frac{1}{p-q} \cdot \sum_{k=0}^{\infty} (1-\eta)^k \big(c^{-p}(1-\eta)^{-kp} - c^{-q}(1-\eta)^{-kq}\big) = 0.$$

As $\eta \in (0,1)$, we can calculate the sums of the geometric series to obtain

$$\frac{1}{p-q} \cdot \Big(\frac{c^{-p}}{1-(1-\eta)^{1-p}} - \frac{c^{-q}}{1-(1-\eta)^{1-q}}\Big) = 0.$$

As $p \neq q$ and $c \neq 0$, it implies

$$c^{p-q} = \frac{1-(1-\eta)^{1-q}}{1-(1-\eta)^{1-p}},$$

and yields the assertion in the last case.

REMARK. *It is worth mentioning that (exept the case $p = q = 0$) we have the equality $\mathscr{H}_\lambda(\mathscr{G}_{p,q}) = C(p,\eta)^{p/(p-q)} C(q,\eta)^{q/(q-p)}$. As a matter of fact, this assertion could be obtained using a similar identity: $\mathscr{G}_{p,q}(x,\lambda) = \mathscr{P}_p(x,\lambda)^{p/(p-q)} \mathscr{P}_q(x,\lambda)^{q/(q-p)}$, which is valid for all $p,q \in \mathbb{R}$, $p \neq q$ and all admissible pairs $(x,\lambda)$.*

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