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FUNDAMENTAL HLAWKA–LIKE INEQUALITIES FOR THREE AND FOUR VECTORS

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Abstract. We investigate Hlawka-like inequalities for three vectors and determine necessary and sufficient conditions such that

$$a_1\sum_{i=1}^{3}\|x_i\|+a_2\sum_{1\leqslant i<j\leqslant 3}\left\|x_i+x_j\right\|+a_3\|x_1+x_2+x_3\|\geqslant 0$$

is satisfied for all x_1,x_2,x_3 in a Hlawka space. In addition, we show that any such inequality can be obtained as a linear combination with nonnegative coefficients of three fundamental inequalities, one of which is Hlawka's inequality.

In the case of four vectors in an inner product space, we prove that any (valid) inequality of the form

$$a_1\sum_{i=1}^{4}\|x_i\|+a_2\sum_{1\leqslant i<j\leqslant 4}\left\|x_i+x_j\right\|+a_3\sum_{1\leqslant i<j<k\leqslant 4}\left\|x_i+x_j+x_k\right\|+a_4\left\|\sum_{i=1}^{4}x_i\right\|\geqslant 0$$

can be written as a linear combination with nonnegative coefficients of six fundamental inequalities.

1. Introduction

Hlawka's inequality (see [7], [8], or [9] for various proofs) asserts that for any x_1,x_2,x_3 in an inner product space E, the following inequality is satisfied:

$$\sum_{i=1}^{3}\|x_i\|-\sum_{1\leqslant i<j\leqslant 3}\left\|x_i+x_j\right\|+\|x_1+x_2+x_3\|\geqslant 0.$$

Equality holds iff

(i) $x_i=\alpha_i u, 1\leqslant i\leqslant 3$, for some $u\in E$ and $\alpha_i\geqslant 0$ or

(ii) $x_i=\alpha_i u, 1\leqslant i\leqslant 3$, for some $u\in E$, two of the scalars α_i are positive, and $\alpha_1+\alpha_2+\alpha_3\leqslant 0$ or

(iii) $x_1+x_2+x_3=0$.

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Normed linear spaces satisfying Hlawka's inequality are called *Hlawka spaces* according to [10] or *quadrilateral spaces* according to [8]. Clearly, any inner product space is a Hlawka space. However, the class of Hlawka spaces is significantly larger than that of inner product spaces. For example, $L^p([0,1])$ is a Hlawka space for $1 \leqslant p \leqslant 2$ (see [11]) and so is any two-dimensional normed linear space (see [5]). The normed space of smallest dimension that is not a Hlawka space is $\mathbb{R}^3$ with $\|(x,y,z)\| = \max(|x|,|y|,|z|)$. The typical counterexample is given by the vectors $(1,1,-1),(1,-1,1)$, and $(-1,1,1)$.

Hlawka's inequality has been generalized in several directions by many authors (see, e.g., [1], [2], [3], or [4]). Our approach in extending Hlawka's inequality is somewhat modest in that we consider inequalities of the form

$$a_1 \sum_{i=1}^{3} \|x_i\| + a_2 \sum_{1 \leqslant i<j \leqslant 3} \|x_i + x_j\| + a_3 \|x_1 + x_2 + x_3\| \geqslant 0$$

and

$$a_1 \sum_{1 \leqslant i \leqslant 4} \|x_i\| + a_2 \sum_{1 \leqslant i<j \leqslant 4} \|x_i + x_j\| + a_3 \sum_{1 \leqslant i<j<k \leqslant 4} \|x_i + x_j + x_k\| + a_4 \left\| \sum_{i=1}^{4} x_i \right\| \geqslant 0$$

with $a_i \in \mathbb{R}$. We refer to these inequalities as *Hlawka-like* inequalities. We began our study of these inequalities in [6] where we proved the following result

THEOREM 1. *The inequality*

$$a_1 \sum_{1 \leqslant i \leqslant 4} \|x_i\| + a_2 \sum_{1 \leqslant i<j \leqslant 4} \|x_i + x_j\| + a_3 \sum_{1 \leqslant i<j<k \leqslant 4} \|x_i + x_j + x_k\| + a_4 \left\| \sum_{i=1}^{4} x_i \right\| \geqslant 0$$

is satisfied for all $x_i, 1 \leqslant i \leqslant 4$, *in an inner product space iff the following inequalities hold*

$$\begin{aligned} a_1 + a_2 + a_3 &\geqslant 0, \\ a_1 + 2a_2 + a_3 &\geqslant 0, \\ a_1 + 3a_2 + 3a_3 + a_4 &\geqslant 0, \\ 2a_1 + 3a_2 + 3a_3 + a_4 &\geqslant 0, \\ 5a_1 + 9a_2 + 3a_3 + a_4 &\geqslant 0. \end{aligned}$$

In Section 2 of this paper, we obtain a similar characterization for inequalities involving three vectors, with the significant difference that we only assume that the three vectors x_1, x_2, x_3 are in a Hlawka space. As a consequence, we obtain that any Hlawka-like inequality involving three vectors can be obtained as a linear combination with nonnegative coefficients of three fundamental inequalities, one of which is Hlawka's inequality. In Section 3, we prove that any Hlawka-like inequality for four vectors in an inner product space can be obtained as a linear combination with nonnegative coefficients of six fundamental inequalities. We also show that at least five of these inequalities are valid in any Hlawka space, thus leaving open the question of whether or not all Hlawka-like inequalities for four vectors are valid in a Hlawka space.

2. Hlawka-like inequalities for three vectors

LEMMA 1. *If E is a Hlawka space, then the following inequality*

$$-\sum_{i=1}^{3}\|x_i\|+\sum_{1\leqslant i<j\leqslant 3}\left\|x_i+x_j\right\|+\|x_1+x_2+x_3\|\geqslant 0$$

is satisfied for all $x_1,x_2,x_3\in E$. Equality holds if and only if

(i) $x_1+x_2+x_3=0$ *or*

(ii) $x_i=\alpha_i u, 1\leqslant i\leqslant 3$, *for some* $u\in E$, $\alpha_1+\alpha_2+\alpha_3\leqslant 0$, *and (exactly) two of the numbers* $\alpha_1+\alpha_2,\alpha_1+\alpha_3,\alpha_2+\alpha_3$ *are nonnegative.*

Proof. By applying Hlawka's inequality to $\frac{1}{2}a,\frac{1}{2}b,\frac{1}{2}c$ we obtain

$$\frac{1}{2}(\|a\|+\|b\|+\|c\|)-\frac{1}{2}(\|a+b\|+\|a+c\|+\|b+c\|)+\frac{1}{2}\|a+b+c\|\geqslant 0.$$

The triangle inequality applied three times yields

$$\frac{1}{2}(\|a\|+\|b\|+\|c\|)+\frac{1}{2}(\|b+c\|+\|a+c\|+\|a+b\|) \\ -\frac{1}{2}(\|b+c-a\|+\|a+c-b\|+\|a+b-c\|)\geqslant 0.$$

By adding the two inequalities above, we have

$$\|a\|+\|b\|+\|c\|-\frac{1}{2}\left(\|a+b-c\|+\|a+c-b\|+\|b+c-a\|\right)+\frac{1}{2}\|a+b+c\|\geqslant 0.$$

Finally, by relabeling $x_1=1/2(a+b-c),x_2=1/2(b+c-a),x_3=1/2(a+c-b)$, we have $x_1+x_2=b,x_1+x_3=a,x_2+x_3=c,x_1+x_2+x_3=1/2(a+b+c)$, we obtain the conclusion of the lemma.

The equality case can be ascertained by combining the equality cases in Hlawka's inequality and the triangle inequality.

Next, we consider real numbers a_1,a_2,a_3 with the goal of finding necessary and sufficient conditions for a_1,a_2,a_3 ensuring that the expression

$$H_a(x):=a_1\sum_{i=1}^{3}\|x_i\|+a_2\sum_{1\leqslant i<j\leqslant 3}\left\|x_i+x_j\right\|+a_3\|x_1+x_2+x_3\|$$

is nonnegative for all $x_1,x_2,x_3\in E$, where E is a Hlawka space, $a=(a_1,a_2,a_3)$, and $x=(x_1,x_2,x_3)$. Note that $H_{(1,-1,1)}(x)$ is the left side in Hlawka's inequality and $H_{(-1,1,1)}(x)$ is the left sides in the inequality in the previous lemma.

PROPOSITION 1. *Let E be a Hlawka space and let a_1, a_2, a_3 be real numbers. The inequality*

$$H_a(x) \geqslant 0$$

is satisfied for all $x \in E^3$ if and only if the following inequalities are satisfied

$$\begin{aligned} a_1 + a_2 &\geqslant 0, \\ a_1 + 2a_2 + a_3 &\geqslant 0, \\ 3a_1 + 2a_2 + a_3 &\geqslant 0. \end{aligned}$$

Proof. If $H_a(x) \geqslant 0$ for all $x = (x_1, x_2, x_3) \in E^3$, then we can consider the following three special cases:

(i) $x_2 = x_3 = 0, x_1 \neq 0$,

(ii) $x_1 = -x_2, x_1 \neq 0, x_3 = 0$,

(iii) $x_1 = -x_2 = -x_3, x_1 \neq 0$.

The corresponding values for $H_a(x)$ are $(a_1 + 2a_2 + a_3)\|x_1\|, 2(a_1 + a_2)\|x_1\|$, and $(3a_1 + 2a_2 + a_3)\|x_1\|$, respectively. Thus, the three inequalities in a_1, a_2, a_3 in the lemma must be satisfied.

Conversely, let us show that $H_a(x) \geqslant 0$ as long as a_1, a_2, a_3 satisfy the inequalities above. First, let us note that

$$H_a(x) = \frac{3a_1 + 2a_2 + a_3}{2} H_{(1,-1,1)}(x) + \frac{a_1 + 2a_2 + a_3}{2} H_{(-1,1,1)}(x) + (a_1 + a_2) H_{(0,1,-2)}(x). \tag{1}$$

Observe that $H_{(1,-1,1)}(x) \geqslant 0$ is Hlawka's inequality and $H_{(-1,1,1)}(x) \geqslant 0$ by lemma 1. Since

$$H_{(0,1,-2)}(x) = \sum_{1 \leqslant i < j \leqslant 3} \|x_i + x_j\| - 2\|x_1 + x_2 + x_3\| \geqslant 0$$

by the triangle inequality, the conclusion of the lemma follows.

OBSERVATION 1. Based on relation 1, any (valid) Hlawka-like inequality in a Hlawka space can be written as a linear combination with nonnegative coefficients of $H_{(1,-1,1)} \geqslant 0, H_{(-1,1,1)} \geqslant 0$, and $H_{(0,1,-2)} \geqslant 0$. In fact, the space of Hlawka-like inequalities is isomorphic to the polyhedral cone determined by the three inequalities in a_1, a_2, a_3 from the proposition above. As it can be easily seen, the vectors $(1,-1,1), (-1,1,1)$, and $(0,1,-2)$ are the generators of this cone.

In Lemma 1 we, in fact, showed that in any normed space $H_{(1,-1,1)}(x) \geqslant 0$ implies $H_{(-1,1,1)}(x) \geqslant 0$. While we suspect that the converse is not valid, we do not have a counterexample.

QUESTION 1. In a normed space, does the inequality $H_{(-1,1,1)}(x) \geqslant 0$ for all x imply that $H_{(1,-1,1)}(x) \geqslant 0$ for all x?

By relation 1, studying the equality case in $H_{(a_1,a_2,a_3)}(x) \geqslant 0$ amounts to identifying when $H_{(1,-1,0)}(x) = 0, H_{(-1,1,1)}(x) = 0$, and $H_{(0,1,-2)}(x) = 0$. As we have already discussed when the first two equalities hold, we would like to note that since $H_{(0,1,-2)} \geqslant 0$ can be obtained from the triangle inequality, it can be easily seen that $H_{(0,1,-2)}(x) \geqslant 0$ iff there exists $u \in E$ and real numbers $\alpha_i, 1 \leqslant i \leqslant 3$ such that $\alpha_i + \alpha_j \geqslant 0$ for all $1 \leqslant i < j \leqslant 3$. Consequently, by combining the various equality cases, we obtain the following.

OBSERVATION 2. If a_1, a_2, a_3 satisfy the three conditions in proposition 1 and at least one of them is different from zero, then $H_a(x) = 0$ if and only if we have one of the following cases:

(i) If $a_1 + 2a_2 + a_3 = 0, 3a_1 + 2a_2 + a_3 > 0, a_1 + a_2 > 0$ then $x_i = \alpha_i u$, for some $u \in E$ and $\alpha_i \geqslant 0$.

(ii) If $a_1 + a_2 = 0, 3a_1 + 2a_2 + a_3 > 0, a_1 + a_2 + a_3 > 0$ then $x_1 + x_2 + x_3 = 0$.

(iii) If two of the expressions $3a_1 + 2a_2 + a_3, a_1 + 2a_2 + a_3, a_1 + a_2$ are zero, then, by relation 1, $H_{(a_1,a_2,a_3)}(x)$ is a positive multiple of either $H_{(1,-1,1)}(x), H_{(-1,1,1)}(x)$, or $H_{(0,1,-2)}(x)$. Thus, $H_{(a_1,a_2,a_3)}(x) = 0$ iff $H_{1,-1,1}(x) = 0, H_{(-1,1,1)}(x) = 0$, or $H_{(0,1,-2)}(x) = 0$.

3. Hlawka-like inequalities for four vectors

In this section we extend our investigations to Hlawka-like inequalities involving four vectors and obtain a result similar to Proposition 1 which can be seen as a minor improvement of Theorem 1. In the process, we show that any such inequality can be obtained as a linear combination of six fundamental inequalities. We begin by proving several lemmas that will show that at least five of the six fundamental inequalities are valid not only on inner product spaces but also on Hlawka spaces.

LEMMA 2. *In any Hlawka space E, the following inequality is satisfied*

$$b_1 \sum_{i=1}^{4} \|x_i\| + b_2 \sum_{1 \leqslant i < j \leqslant 4} \|x_i + x_j\| + (3b_2 - b_1) \sum_{1 \leqslant i < j < k \leqslant 4} \|x_i + x_j + x_k\| + b_3 \left\| \sum_{i=1}^{4} x_i \right\| \geqslant 0. \tag{2}$$

for all $x_1, x_2, x_3, x_4 \in E$ and all b_1, b_2, b_3 satisfying

$$\begin{aligned} b_2 &\geqslant 0, \\ -2b_1 + 12b_2 + b_3 &\geqslant 0, \\ 2b_1 + 12b_2 + b_3 &\geqslant 0. \end{aligned}$$

Proof. For any permutation $\{i,j,k,l\}$ of $\{1,2,3,4\}$, Proposition 1 applied to the vectors $x_i, x_j, x_k + x_l$ yields

$$a_1(\|x_i\| + \|x_j\| + \|x_k + x_l\|) + a_2(\|x_i + x_j\| + \|x_i + x_j + x_k\| + \|x_i + x_j + x_l\|) + a_3 \|x_i + x_j + x_k + x_l\| \geqslant 0.$$

By adding all these inequalities, we obtain

$$3a_1\sum_{i=1}^{4}\|x_i\|+(a_1+a_2)\sum_{1\leqslant i<j\leqslant 4}\|x_i+x_j\|+3a_2\sum_{1\leqslant i<j<k\leqslant 4}\|x_i+x_j+x_k\|$$
$$+6a_3\|x_1+x_2+x_3+x_4\|\geqslant 0.$$

After relabeling $b_1=3a_1, b_2=a_1+a_2, b_3=6a_3$ and rewriting the inequalities involving a_1,a_2,a_3 in Proposition 1 in terms of $b_1,b_2,b_3,$ the conclusion of the lemma follows.

LEMMA 3. *In any Hlawka space E, the following inequality is satisfied*

$$b_1\sum_{1\leqslant i\leqslant 4}\|x_i\|+b_2\sum_{1\leqslant i<j\leqslant 4}\|x_i+x_j\|+b_3\sum_{1\leqslant i<j<k\leqslant 4}\|x_i+x_j+x_k\|\geqslant 0$$

for all $x_1,x_2,x_3,x_4\in E$ and any b_1,b_2,b_3 satisfying

$$\begin{aligned}2b_1+3b_2&\geqslant 0,\\ b_1+3b_2+3b_3&\geqslant 0,\\ b_1+b_2+b_3&\geqslant 0.\end{aligned}$$

Proof. If we apply lemma 1 for all triples $x_i,x_j,x_k, 1\leqslant i<j<k\leqslant 4$ and add the resulting inequalities, we obtain

$$3a_1\sum_{i=1}^{4}\|x_i\|+2a_2\sum_{1\leqslant i<j\leqslant 4}\|x_i+x_j\|+a_3\sum_{1\leqslant i<j<k\leqslant 4}\|x_i+x_j+x_k\|\geqslant 0.$$

The conclusion follows by relabeling $b_1=3a_1, b_2=2a_2, b_3=a_3$.

As before, for any $a=(a_1,a_2,a_3,a_4)\in\mathbb{R}^4$ and any $(x_1,x_2,x_3,x_4)\in E^4$, we introduce the following notation:

$$H_a(x)=a_1\sum_{1\leqslant 1\leqslant 4}\|x_i\|+a_2\sum_{1\leqslant i<j\leqslant 4}\|x_i+x_j\|+a_3\sum_{1\leqslant i<j<k\leqslant 4}\|x_i+x_j+x_k\|+a_4\left\|\sum_{i=1}^{4}x_i\right\|.$$

LEMMA 4. *In any Hlawka space E, for all $x_1,x_2,x_3,x_4\in E$ we have*

$$H_{(1,0,-1,2)}(x)=\sum_{i=1}^{4}\|x_i\|-\sum_{1\leqslant i<j<k\leqslant 4}\|x_i+x_j+x_k\|+2\|x_1+x_2+x_3+x_4\|\geqslant 0,$$

$$H_{(-1,0,1,2)}(x)=-\sum_{i=1}^{4}\|x_i\|+\sum_{1\leqslant i<j<k\leqslant 4}\|x_i+x_j+x_k\|+2\|x_1+x_2+x_3+x_4\|\geqslant 0,$$

$$H_{(0,1,-1,0)}(x)=\sum_{1\leqslant i<j\leqslant 4}\|x_i+x_j\|-\sum_{1\leqslant i<j<k\leqslant 4}\|x_i+x_j+x_k\|\geqslant 0,$$

$$H_{(3,-2,1,0)}(x)=3\sum_{1\leqslant i\leqslant 4}\|x_i\|-2\sum_{1\leqslant i<j\leqslant 4}\|x_i+x_j\|+\sum_{1\leqslant i<j<k\leqslant 4}\|x_i+x_j+x_k\|\geqslant 0,$$

$$H_{(0,0,1,-3)}(x)=\sum_{1\leqslant i<j<k\leqslant 4}\|x_i+x_j+x_k\|-3\|x_1+x_2+x_3+x_4\|\geqslant 0,$$

for all $x_1, x_2, x_3, x_4 \in E$.

Proof. To show that $H_{(1,0,-1,2)}(x) \geqslant 0$ and $H_{(1,0,-1,2)}(x) \geqslant 0$, it is enough to apply Lemma 2 for $b_1 = 1, b_2 = 0, b_3 = 2$ and $b_1 = -1, b_2 = 0, b_3 = 2$, respectively. For $H_{(0,1,-1,0)}(x) \geqslant 0$ and $H_{(3,-2,1,0)}(x) \geqslant 0$, we can use Lemma 3 for $b_1 = 3, b_2 = -2, b_3 = 1$ and $b_1 = 0, b_2 = 1, b_3 = -1$, respectively.

Finally, $H_{(0,0,1,-3)}(x) \geqslant 0$ follows by applying the triangle inequality for the four vectors $x_i + x_j + x_k, 1 \leqslant i < j < k \leqslant 4$.

OBSERVATION 3. The inequality $H_{(1,0,-1,2)}(x) \geqslant 0$ is not only implied by but actually equivalent to Hlawka's inequality. Indeed, if we choose $x_4 = 0$ in $H_{(1,0,-1,2)}(x)$ we obtain Hlawka's inequality for x_1, x_2, x_3.

THEOREM 2. *Let E be a Hlawka space on which the inequality*

$$-2 \sum_{1\leqslant i\leqslant 4} \|x_i\| + \sum_{1\leqslant i<j\leqslant 4} \left\|x_i + x_j\right\| + \sum_{1\leqslant i<j<k\leqslant 4} \left\|x_i + x_j + x_k\right\| - 2\|x_1 + x_2 + x_3 + x_4\| \geqslant 0$$

is satisfied. Given $a_i \in \mathbb{R}, 1 \leqslant i \leqslant 4$, the inequality

$$H_a(x) \geqslant 0$$

is satisfied for all $x_1, x_2, x_3, x_4 \in E$ iff the following inequalities hold

$$\begin{aligned} a_1 + a_2 + a_3 &\geqslant 0, \\ a_1 + 2a_2 + a_3 &\geqslant 0, \\ a_1 + 3a_2 + 3a_3 + a_4 &\geqslant 0, \\ 2a_1 + 3a_2 + 3a_3 + a_4 &\geqslant 0, \\ 5a_1 + 9a_2 + 3a_3 + a_4 &\geqslant 0. \end{aligned}$$

Proof. For the direct implication, we can obtain the five inequalities in a_1, a_2, a_3, a_4 by considering the following special cases:

(i) If $x_1 = -x_2 = -x_3 = x_4 \neq 0$, then $H_a(x) = \|x_1\| (a_1 + a_2 + a_3) \geqslant 0$.

(ii) If $x_1 = -x_2 \neq 0, x_3 = x_4 = 0$, then $H_a(x) = 2\|x_1\| (a_1 + 2a_2 + a_3) \geqslant 0$.

(iii) If $x_2 = x_3 = x_4 = -x_1 \neq 0$, then $H_a(x) = 2\|x_1\| (2a_1 + 3a_2 + a_3 + a_4) \geqslant 0$.

(iv) If $x_1 \neq 0, x_2 = x_3 = x_4 = 0$, then $H_a(x) = \|x_1\| (a_1 + 3a_2 + 3a_3 + a_4) \geqslant 0$.

(v) If $x_1 = x_2 = x_3 \neq 0, x_4 = -2x_1$, then $H_a(x) = \|x_1\| (5a_1 + 9a_2 + 3a_3 + a_4) \geqslant 0$.

Conversely, we will show that under the given assumptions on a_1, a_2, a_3, and $a_4, H_a(x)$ can be written as a linear combination with nonnegative coefficients of the following: $H_{(1,0,-1,2)}(x)$, $H_{(-1,0,1,2)}(x)$, $H_{(0,1,-1,0)}(x)$, $H_{(3,-2,1,0)}(x)$, $H_{(0,0,1,-3)}(x)$, and $H_{(-2,1,1,-2)}(x)$. Since the first five expressions have been shown to be nonnegative in Lemma 4 and since $H_{(-2,1,1,-2)}(x)$ is assumed to be nonnegative, the claim will follow.

To prove that $H_a(x) = H_{(a_1,a_2,a_3,a_4)}(x) \geqslant 0$, note that it can be checked that

(i) If $a_1+a_2+3a_3+a_4 \leqslant 0$, then

$$H_a(x) = \lambda_1 H_{(-2,1,1,-2)}(x) + \lambda_2 H_{1,0,-1,2}(x) + \lambda_3 H_{0,0,1,-3}(x) + \lambda_4 H_{0,-1,1,0}(x),$$

where

$$\begin{aligned}
\lambda_1 &= \frac{a_1+3a_2+3a_3+a_4}{2} \geqslant 0, \\
\lambda_2 &= 2a_1+3a_2+a_3+a_4 \geqslant 0, \\
\lambda_3 &= a_1+a_2+a_3 \geqslant 0, \\
\lambda_4 &= -\frac{a_1+a_2+3a_3+a_4}{2} \geqslant 0.
\end{aligned}$$

(ii) If $a_1+a_2+3a_3+a_4 \geqslant 0$, then

$$\begin{aligned}
H_a(x) =& \lambda_1 H_{(1,0,1,-2)}(x) + \lambda_2 H_{(-1,0,1,2)}(x) + \lambda_3 H_{(0,0,1,-3)}(x) + \lambda_4 H_{(3,-2,1,0)}(x) \\
&+ \lambda_5 H_{(-2,1,1,-2)}(x),
\end{aligned}$$

where

$$\begin{aligned}
\lambda_1 &= \frac{5a_1+9a_2+3a_3+a_4}{2} + \frac{3(a_1+3a_2+3a_3+a_4)}{2} \geqslant 0, \\
\lambda_2 &= \frac{a_1+3a_2+3a_3+a_4}{2} \geqslant 0, \\
\lambda_3 &= a_1+a_2+a_3 \geqslant 0, \\
\lambda_4 &= \frac{a_1+a_2+3a_3+a_4}{2} + a_1+3a_2+3a_3+a_4 \geqslant 0, \\
\lambda_5 &= \frac{3(a_1+3a_2+3a_3+a_4)}{2} \geqslant 0.
\end{aligned}$$

OBSERVATION 4. As with Hlawka-like inequalities in three vectors, the space of Hlawka-like inequalities in four vectors on an inner product space is isomorphic to the polyhedral cone defined by the five inequalities in a_1,a_2,a_3,a_4 from Theorem 1. By the proof of the previous theorem it can be seen that the generating vectors for this cone are $(1,0,-1,2)$, $(-1,0,1,2)$, $(0,1,-1,0)$, $(3,-2,1,0)$, $(0,0,1,-3)$, and $(-2,1,1,-2)$. Each one of these vectors corresponds to a fundamental Hlawka-like inequality. By Theorem 1, each one of these inequalities is valid in an inner product space and by Lemma 4, the first five of these inequalities are actually satisfied in a Hlawka space. While we expect that the inequality $H_{(-2,1,1,-2)}(x) \geqslant 0$ is generally not valid on a Hlawka space, we do not have a counterexample.

QUESTION 2. If E is a Hlawka space, is the inequality $H_{(-2,1,1,-2)}(x) \geqslant 0$ satisfied for all $x=(x_1,x_2,x_3,x_4) \in E^4$?

Moreover, we were also unable to determine the validity of the converse statement.

QUESTION 3. If E is a normed space such that the inequality $H_{(-2,1,1,-2)}(x) \geqslant 0$ is satisfied for all $x=(x_1,x_2,x_3,x_4) \in E^4$, does this imply that E is a Hlawka space?

OBSERVATION 5. The inequality

$$\sum_{1\leqslant i\leqslant 4} \|x_i\| - \sum_{1\leqslant i<j\leqslant 4} \left\|x_i+x_j\right\| + \sum_{1\leqslant i<j<k\leqslant 4} \left\|x_i+x_j+x_k\right\| + \|x_1+x_2+x_3+x_4\| \geqslant 0$$

obtained in [6] can be written as $H_{(1,-1,1,1)}(x) \geqslant 0$. Since

$$H_{(1,-1,1,1)}(x) = \frac{1}{2}(H_{(-1,0,1,2)}(x) + H_{(3,-2,1,0)}(x)),$$

this inequality is not optimal.

OBSERVATION 6. Djoković's inequality ([3]), for $n=4, k=3$ can be written as $H_{(1,0,-1,2)}(x) \geqslant 0$ and is optimal. On the other hand, for $n=4, k=2$, the inequality can be written as $H_{(2,-1,0,1)}(x) \geqslant 0$. This is not optimal since $H_{(2,-1,0,1)}(x) = 1/2(H_{(1,0,-1,2)}(x) + H_{(3,-2,1,0)})(x)$.

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