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REMARKS ON WEIGHTED ORLICZ SPACES ON LOCALLY COMPACT GROUPS

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Abstract. In this paper, we give some equivalent condition for a weighted Orlicz space $L^\Phi_w(G)$ on a locally compact group G to be a convolution Banach algebra, and by Jensen's inequality we study a hereditary property for weighted Orlicz algebras on quotient spaces. In addition, we characterize compact convolution operators from $L^1_w(G)$ into $L^\Phi_w(G)$.

1. Introduction

If $1<p<\infty$ and G be a locally compact group, it is well-known that the Lebesgue space $L^p(G)$ is a convolution Banach algebra if and only if G is compact. The first results related to this fact is due to [19, 18]. This problem has been studied for Orlicz spaces, as a generalization of Lebesgue spaces. For any Young function Φ satisfying Δ_2-condition, H. Hudzik, A. Kamiska and J. Musielak in [9] prove that the Orlicz space $L^\Phi(G)$ is a Banach algebra under convolution if and only if $L^\Phi(G)\subseteq L^1(G)$. In [17], it is proved that if Φ satisfies a given sequence condition, then $L^\Phi(G)$ is a Banach algebra if and only if $f*g$ exists for all $f,g\in L^\Phi(G)$. Similar prorblems about the weighted Lebesgue spaces have been studied in several papers. For instance, Yu. N. Kuznetsova in [10, 11] gives some conditions under which the weighted Lebesgue space $L^p_w(G)$ is a Banach algebra under the convolution. Recently, A. Osançlıol and S. Öztop in [12] studied the weighted Orlicz algebras under the convolution and proved that if the inclusion $L^\Phi_w(G)\subseteq L^1_w(G)$ holds, then $L^\Phi_w(G)$ is a convolution Banach algebra. In this paper, we study a hereditary property for weighted Orlicz algebras, and prove that if H is a compact normal subgroup of a locally compact group G and $L^\Phi_w(G)$ is a convolution Banach algebra, then $L^\Phi_{\tilde{w}}(G/H)$ is a Banach algebra under a product $\circledast$ given by the formula (11) induced by the usual convolution product, where $\tilde{w}(xH):=\inf_{y\in H} w(xy)$ for all $x\in G$. In section 4, we look at $L^\Phi_w(G)$ as an $L^1_w(G)$-module, and prove that a convolution operator from the weighted group algebra $L^1_w(G)$ into a weighted Orlicz space $L^\Phi_w(G)$ is compact if and only if a related function (given by the formula (14)) vanishes at infinity. The main motivation for this study is the

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characterization of compact elements of $L_w^1(G)$ by F. Ghahramani [5, Theorem 1]. One can find similar results about weakly compact elements of $L_w^1(G)$ in [6], and an extension of them on locally compact hypergroups in [7, 8]. The obtained results in section 4 can be considered as improvements of well-known results about compact elements from S. Sakai, C. Akemann and F. Ghahramani in [16, 2, 5]. In particular, some results for compact convolution operators into a weighted Lebesgue space $L_w^p(G)$ are provided.

In next section, we recall some basic definitions and notations about Orlicz spaces as an important extension of Lebesgue spaces; see the monograph [13].

2. Preliminaries

Let G be a locally compact group. The set of all bounded Radon measures on G is denoted by $M(G)$. Throughout, G is a locally compact group, and the integrals without any specified measure are considered with a given left Haar measure. Also, all (weighted) Lebesgue spaces on G are given by a left Haar measure. For any $\mu \in M(G)$ and measurable functions f and g on G denote

$$(f*g)(x) := \int_G f(y)g(y^{-1}x)\,dy, \quad (\mu*g)(x) := \int_G g(y^{-1}x)\,d\mu(y),$$

for all $x \in G$, while these integrals exist.

Now, we recall some basic definitions and notations about Orlicz spaces. A convex even mapping $\Phi : \mathbb{R} \to [0,\infty]$ satisfying $\Phi(0) = \lim_{x\to 0}\Phi(x) = 0$ and $\lim_{x\to\infty}\Phi(x) = \infty$, is called a *Young function*. The *complementary* of a Young function Φ is given by

$$\Psi(x) := \sup\{y|x| - \Phi(y) : y \geqslant 0\}, \quad (x \in \mathbb{R}).$$

In this case, (Φ,Ψ) is called a *Young pair*.

We say that a Young function Φ satisfies Δ_2-*condition* (and write $\Phi \in \Delta_2$) if for some constants $c > 0$ and $x_0 \geqslant 0$,

$$\Phi(2x) \leqslant c\Phi(x), \quad (x \geqslant x_0).$$

In sequel, (Φ,Ψ) is a Young pair and $\Phi \in \Delta_2$. A Borel measurable function f belongs to $L^\Phi(G)$ if there exists a number $\alpha > 0$ such that

$$\int_G \Phi(\alpha|f(x)|)\,dx < \infty.$$

Two elements $f, g \in L^\Phi(G)$ are considered the same if $f = g$ a.e. For every $f \in L^\Phi(G)$ we put

$$\|f\|_\Phi := \sup\left\{\int_G |f(x)g(x)|\,dx : \int_G \Psi(|g(x)|)\,dx \leqslant 1\right\}.$$

The complete normed space $(L^\Phi(G), \|\cdot\|_\Phi)$ is called an *Orlicz space*. In particular, if $p \geqslant 1$ and the Young function Φ is defined by $\Phi(x) := |x|^p$ for all $x \in \mathbb{R}$, then $L^\Phi(G)$ is same as the Lebesgue space $L^p(G)$.

Set

$$\|f\|_{\Phi}^{\circ} := \inf\left\{\lambda > 0 : \int_G \Phi(\frac{1}{\lambda}|f(x)|)\,dx \leqslant 1\right\}, \quad (f \in L^{\Phi}(G)).$$

Then, $\|\cdot\|_{\Phi}^{\circ}$ is also a norm on $L^{\Phi}(G)$ and for each $f \in L^{\Phi}(G)$,

$$\|f\|_{\Phi}^{\circ} \leqslant \|f\|_{\Phi} \leqslant 2\|f\|_{\Phi}^{\circ}.$$

If $f \in L^{\Phi}(G)$ and $g \in L^{\Psi}(G)$, then by [13, Page 58] we have

$$\int_G |f(x)g(x)|\,dx \leqslant 2\|f\|_{\Phi}\,\|g\|_{\Psi}, \tag{1}$$

which is the *Hölder's inequality* for Orlicz spaces. If H is a compact group with a normalized Haar measure, and f is a real-valued measurable function on H such that $\int_H f(x)\,dx$ and $\int_H \Phi(f(x))\,dx$ exist, then by the Jensen's inequality [13, Proposition 5, Chapter III] we have

$$\Phi\left(\int_H f(x)\,dx\right) \leqslant \int_H \Phi(f(x))\,dx. \tag{2}$$

In this paper, w is a continuous positive function on G (called a *weight*). We write $w^{-1} := \frac{1}{w}$. The *weighted Orlicz space* $L_w^{\Phi}(G)$ consists all measurable functions f on G such that $wf \in L^{\Phi}(G)$. It is known that $(L_w^{\Phi}(G), \|\cdot\|_{\Phi,w})$ is a Banach space, where $\|f\|_{\Phi,w} := \|wf\|_{\Phi}$ for all $f \in L_w^{\Phi}(G)$. The set of all elements $\mu \in M(G)$ such that $w\mu \in M(G)$ is denoted by $M_w(G)$, and for each $\mu \in M_w(G)$ we put $\|\mu\|_w := \|w\mu\|$. Easily one can see that for each $f \in L_w^{\Phi}(G)$ and $\mu \in M_w(G)$,

$$\|\mu * f\|_{\Phi,w} \leqslant \|\mu\|_w\,\|f\|_{\Phi,w}.$$

The set of all functions $f : G \to \mathbb{C}$ such that $\frac{f}{w} \in C_0(G)$ is denoted by $C_0^w(G)$, where $C_0(G)$ is the space of all complex-valued continuous functions on G vanishing at infinity. For each $f \in C_0^w(G)$ we put $\|f\|_{\infty,w} := \|\frac{f}{w}\|_{\infty}$. In general, we have $C_0^w(G)^* \cong M_w(G)$. The weighted Orlicz space $L_w^{\Phi}(G)$ is called a *convolution Banach algebra* if there exists a constant $c > 0$ such that $f * g \in L_w^{\Phi}(G)$ and

$$\|f * g\|_{\Phi,w} \leqslant c\,\|f\|_{\Phi,w}\,\|g\|_{\Phi,w},$$

for all $f, g \in L_w^{\Phi}(G)$. In sequel, we assume that for each $x, y \in G$, $w(xy) \leqslant w(x)w(y)$.

3. Weighted Orlicz convolution algebras

In this section, we give some sufficient and necessary condition for a weighted Orlicz space on a locally compact group to be a convolution Banach algebra.

Since $\Phi \in \Delta_2$, same as non-weighted case [13, Page 111] (see also [12]) we have $\left(L_w^{\Phi}(G)\right)^* \cong L_{w^{-1}}^{\Psi}(G)$ with the duality formula

$$\langle f, g\rangle = \int_G f(x)g(x)\,dx. \tag{3}$$

Let $y \in G$. The right translation of a function $g : G \to \mathbb{C}$ is defined by

$$\mathrm{R}_y g : G \to \mathbb{C}, \quad \mathrm{R}_y g(x) := g(xy)$$

for all $x \in G$. Also, for each $x, y \in G$ we define

$$\Omega(x,y) := \frac{w(xy)}{w(x)w(y)}.$$

The following result is an extension of Proposition 2.1 in [1].

PROPOSITION 1. *Let (Φ, Ψ) be an Orlicz pair with $\Phi \in \Delta_2$. Then, $L^{\Phi}_w(G)$ is a convolution Banach algebra if and only if there is a constant $k > 0$ such that for each $f \in L^{\Phi}(G)$ and $g \in L^{\Psi}(G)$,*

$$\left\| \int_G f(y)\, \mathrm{R}_y g\, \Omega(\cdot, y)\, dy \right\|_{\Psi} \leqslant k \|f\|_{\Phi} \|g\|_{\Psi}. \tag{4}$$

Proof. Note that the mapping

$$L^{\Phi}_w(G) \to L^{\Phi}(G), \quad f \mapsto fw$$

is an isometric isomorphism. The statement can be concluded from the well known fact that if there is an associative multiplication on a Banach space A, then it makes A a Banach algebra if and only if the dual space A^* is a Banach module over A by the natural module action.

Now, we intend to study a hereditary property for weighted Orlicz algebras. For this, let H be a compact normal subgroup of a locally compact group G with a normalized Haar measure dy, and let $L^{\Phi}_w(G)$ be a Banach algebra under the convolution product. For each $x \in G$ we denote $\dot{x} := xH$. By [14, Theorem 3.4.6], there is a left-invariant Radon measure $d\dot{x}$ on the quotient space G/H satisfying

$$\int_G f(x)\, dx = \int_{G/H} \int_H f(xy)\, dy\, d\dot{x}, \tag{5}$$

for all $f \in L^1(G)$. This relation is called *Weil's formula* which plays a key role in the sequel.

For each $f \in C_c(G)$ we define

$$P_f(xH) := \int_H f(xy)\, dy, \quad (x \in G).$$

By [14, Theorem 3.5.4], for each $f, g \in C_c(G)$ we have

$$P_f * P_g = P_{f*g}.$$

Also, $P : f \mapsto P_f$ is a surjective function from $C_c(G)$ to $C_c(G/H)$ [4, Proposition 2.48]. If $\tilde{w}$ is defined by

$$\tilde{w}(xH) := \inf_{y \in H} w(xy), \quad (x \in G),$$

then, $\tilde{w}$ is a weight on G/H and $\tilde{w}(\dot{x}\dot{y}) \leqslant \tilde{w}(\dot{x})\tilde{w}(\dot{y})$ for all $x, y \in G$.

Now, since Φ is a convex (and so an increasing) function, for each $\alpha > 0$ we have

$$\begin{aligned}\int_{G/H} \Phi\left(\alpha P_f(xH)\tilde{w}(xH)\right) d\dot{x} &= \int_{G/H} \Phi\left(\int_H \alpha f(xy) \inf_{t\in H} w(xt)\, dy\right) d\dot{x} \\ &\leqslant \int_{G/H} \Phi\left(\int_H \alpha f(xy) w(xy)\, dy\right) d\dot{x} \\ &\leqslant \int_{G/H}\int_H \Phi(\alpha f(xy)w(xy))\, dy\, d\dot{x} \\ &= \int_G \Phi(\alpha f(x)w(x))\, dx,\end{aligned}$$

thanks to the Jensen's inequality (2) and the Weil's formula (5). This implies that

$$\|P_f\|_{\Phi,\tilde{w}} \leqslant 2\|Pf\|^{\circ}_{\Phi,\tilde{w}} \leqslant 2\|f\|^{\circ}_{\Phi,w} \leqslant 2\|f\|_{\Phi,w}. \tag{6}$$

This inequality shows that $\mathscr{I} := \ker(P)$ is closed in $C_c(G)$. Also,

$$C_c(G/H) \cong \frac{C_c(G)}{\mathscr{I}},$$

where $\cong$ is a linear isomorphism. By [14, Lemma 3.4.4] we have

$$\mathrm{cl}_{\|\cdot\|'}(C_c(G/H)) \cong \frac{L^{\Phi}_w(G)}{\mathscr{J}}, \tag{7}$$

where $\mathscr{J}$ is the closure of $\mathscr{I}$ in $L^{\Phi}_w(G)$ and

$$\|P_f\|' := \inf\{\|f-g\|_{\Phi,w} : g \in \mathscr{I}\}$$

for all $f \in C_c(G)$. The relation $\cong$ in (7) is an isometrically isomorphism.

Easily, one can see that

$$\|P_f\|_{\Phi,\tilde{w}} \leqslant 2\|P_f\|' \tag{8}$$

for all $f \in C_c(G)$. For each $f \in \mathscr{J}$, the equality in (8) holds. If $f \notin \mathscr{J}$, then by [3, corollary 6.8, Chapter III], there is an element $g \in L^{\Psi}_{w^{-1}}(G)$ orthogonal to $\mathscr{J}$ such that

$$\langle f, g\rangle = 1, \quad \|g\|_{\Psi,w^{-1}} = \frac{1}{\|P_f\|'}. \tag{9}$$

Since g is orthogonal to $\mathscr{J}$, for each $x \in H$ we have $g(xy) = g(y)$ for locally almost every $y \in G$. So, by [14, Proposition 3.6.13], there is a measurable function $h : G/H \to \mathbb{C}$ such that $g(x) = h(\dot{x})$, for all $x \in G$. For each $\alpha > 0$ we have

$$\begin{aligned}\int_{G/H} \Psi\left(\frac{\alpha}{M\tilde{w}(\dot{x})}|h(\dot{x})|\right) d\dot{x} &= \int_{G/H}\int_H \Psi\left(\frac{\alpha}{M\tilde{w}(\dot{x})}|h(\dot{x})|\right) dy d\dot{x} \\ &\leqslant \int_{G/H}\int_H \Psi\left(\frac{\alpha M}{M w(xy)}|g(xy)|\right) dy d\dot{x} \\ &= \int_G \Psi\left(\frac{\alpha|g(x)|}{w(x)}\right) dx,\end{aligned}$$

where $M := \sup_{y\in H} w(y)$. This shows that

$$\frac{M}{2}\|h\|_{\Psi,\tilde{w}^{-1}} \leqslant M\|h\|^{\circ}_{\Psi,\tilde{w}^{-1}} \leqslant \|g\|_{\Psi,w^{-1}}.$$

Hence,

$$1 = |\langle f,g\rangle| = |\langle P_f,h\rangle| \leqslant \|P_f\|_{\Phi,\tilde{w}}\,\|h\|_{\Psi,\tilde{w}^{-1}} \leqslant \frac{2}{M}\|P_f\|_{\Phi,\tilde{w}}\,\|g\|_{\Psi,w^{-1}} = \frac{2\|P_f\|_{\Phi,\tilde{w}}}{M\|P_f\|'},$$

and so

$$\frac{M}{2}\|P_f\|' \leqslant \|P_f\|_{\Phi,\tilde{w}}, \tag{10}$$

for all $f \in C_c(G)$. Then, by inequalities (8) and (10), the norms $\|\cdot\|_{\Phi,\tilde{w}}$ and $\|\cdot\|'$ are equivalent on $C_c(G/H)$, and so by (7) we have

$$L^{\Phi}_{\tilde{w}}(G/H) \cong \frac{L^{\Phi}_{w}(G)}{\mathscr{J}},$$

via the mapping

$$\tilde{P} : \frac{L^{\Phi}_{w}(G)}{\mathscr{J}} \to L^{\Phi}_{\tilde{w}}(G), \quad \tilde{P}(f+\mathscr{J}) := \lim_{n\to\infty} P_{f_n},$$

where $f \in L^{\Phi}_{w}(G)$ and $\{f_n\}$ is a sequence in $C_c(G)$ that converges to f in $L^{\Phi}_{w}(G)$. Now, if we define a product $\circledast$ on $L^{\Phi}_{\tilde{w}}(G/H)$ by

$$\tilde{P}(f+\mathscr{J}) \circledast \tilde{P}(g+\mathscr{J}) := \tilde{P}((f*g)+\mathscr{J}), \quad (f,g \in L^{\Phi}_{w}(G)), \tag{11}$$

then $(L^{\Phi}_{\tilde{w}}(G/H), \circledast)$ is a Banach algebra. In general, the product $\circledast$ on $L^{\Phi}_{\tilde{w}}(G/H)$ is different from the usual convolution product on this space. Although, for each $f,g \in C_c(G)$ we have

$$\tilde{P}(f+\mathscr{J}) \circledast \tilde{P}(g+\mathscr{J}) = P_f * P_g.$$

Now, we can write the following result:

THEOREM 1. *Let H be a compact normal subgroup of a locally compact group G. If $L^{\Phi}_{w}(G)$ is a convolution Banach algebra, then $L^{\Phi}_{\tilde{w}}(G/H)$ is a Banach algebra under the product $\circledast$ induced by the usual convolution given via the formula* (11).

Compared the conclusion in [1, Proposition 3.1], we have the following corollary.

COROLLARY 1. *Let $1 < p < \infty$ and H be a compact normal subgroup of a locally compact group G. If $L^{p}_{w}(G)$ is a convolution Banach algebra, then $L^{p}_{\tilde{w}}(G/H)$ is a Banach algebra under the product $\circledast$ given by the formula* (11)*, setting $\Phi(x) := |x|^p$.*

4. Compact convolution operators

In this section, we give an equivalent condition for compactness of a convolution operator from the weighted group algebra $L^1_w(G)$ into a weighted Orlicz space $L^\Phi_w(G)$. Here, $L^\Phi_w(G)$ is not necessarily an algebra, rather it is considered as an $L^1_w(G)$-module. The main idea of the proof comes from [5, Theorem 1], but the details are different. For this, we need the following theorem which is a new version of [5, Lemma 2].

THEOREM 2. *Let $g \in L^\Phi_w(G)$, and suppose that the bounded linear operator $T_g : L^1_w(G) \to L^\Phi_w(G)$ is defined by*

$$T_g(f) := f * g, \quad (f \in L^1_w(G)).$$

Then, T_g is compact if and only if the mapping $\tilde{T}_g : M_w(G) \to L^\Phi_w(G)$ defined by

$$\tilde{T}_g(\mu) := \mu * g, \quad (\mu \in M_w(G)), \tag{12}$$

is a compact operator.

Proof. Let T_g be compact. There is a net $\{e_\alpha\}_{\alpha\in I}$ which is the bounded (left) approximate identity of $L^1_w(G)$ and the (left) approximate identity of $L^\Phi_w(G)$ (see [12, Theorem 4.2] and its proof). Then,

$$\left\{\tilde{T}_g(\mu) : \|\mu\|_w \leqslant 1\right\} \subseteq \mathrm{cl}_{\Phi,w}\left(\left\{T_g(\mu * e_\alpha) : \alpha \in I, \mu \in M_w(G), \|\mu\|_w \leqslant 1\right\}\right), \tag{13}$$

where $\mathrm{cl}_{\Phi,w}(E)$ means the $\|\cdot\|_{\Phi,w}$-closure of a set $E \subseteq L^\Phi_w(G)$. Indeed, for each $\mu \in M_w(G)$ we have

$$\left\|\tilde{T}_g(\mu) - T_g(\mu * e_\alpha)\right\|_{\Phi,w} = \|\mu * g - \mu * (e_\alpha * g)\|_{\Phi,w} \leqslant \|\mu\|_w \|g - (e_\alpha * g)\|_{\Phi,w},$$

and this implies that

$$\tilde{T}_g(\mu) = \lim_\alpha T_g(\mu * e_\alpha),$$

in $L^\Phi_w(G)$. So, the inclusion (13) holds. The right side of (13) is a compact subset of $L^\Phi_w(G)$ because T_g is a compact operator and the set

$$\{\mu * e_\alpha : \alpha \in I, \mu \in M_w(G), \|\mu\|_w \leqslant 1\}$$

is bounded in $L^1_w(G)$. So, $\tilde{T}_g$ is a compact operator. Conversely, let the operator $\tilde{T}_g$ be compact. Then, easily its restriction $\tilde{T}_g|_{L^1_w(G)} = T_g$ is also compact.

The following result is a generalization of a similar one from F. Ghahramani [5, Theorem 1].

For each $x, y \in G$ and $g : G \to \mathbb{C}$ we denote $\mathrm{L}_x g(y) := (\delta_x * g)(y) = g(x^{-1}y)$.

THEOREM 3. *Let (Φ,Ψ) be a Young pair with $\Phi,\Psi \in \Delta_2$, and $g \in L^\Phi_w(G)$. Define the operator $T_g : L^1_w(G) \to L^\Phi_w(G)$ by*

$$T_g(f) := f * g, \quad (f \in L^1_w(G)).$$

Then, T_g is compact if and only if the function F_g defined by

$$F_g : G \to \mathbb{R}, \quad F_g(x) := \frac{1}{w(x)} \|\mathrm{L}_x g\|_{\Phi,w} \tag{14}$$

for all $x \in G$, belongs to $C_0(G)$.

Proof. Let T_g be a compact operator. By [12, Lemma 2.3(ii)], the function F_g is continuous. In contrast, suppose that $F_g \notin C_0(G)$. So, there is a number $\varepsilon > 0$ such that for each compact set $F \subseteq G$, there exists an element $x_F \in G \setminus F$ such that

$$\left\| \tilde{T}_g \left(\frac{1}{w(x_F)} \delta_{x_F} \right) \right\|_{\Phi,w} = \frac{1}{w(x_F)} \|L_{x_F} g\|_{\Phi,w} > \varepsilon, \tag{15}$$

where $\tilde{T}_g$ is the operator defined by (12). By Theorem 2, the operator $\tilde{T}_g$ is also compact. Then, by boundedness of the set

$$\left\{ \frac{1}{w(x_F)} \delta_{x_F} : F \subseteq G \text{ is compact} \right\}$$

in $M_w(G)$, there exists a subnet $\{x_{F_i}\}$ of $\{x_F\}$ and a function $h \in L^\Phi_w(G)$ such that

$$\lim_i \tilde{T}_g \left(\frac{1}{w(x_{F_i})} \delta_{x_{F_i}} \right) = h \tag{16}$$

in $L^\Phi_w(G)$. By (15), we have $\|h\|_{\Phi,w} \geqslant \varepsilon$. So, since

$$\|h\|_{\Phi,w} = \sup \left\{ |\langle h, f \rangle| : f \in L^\Psi_{w^{-1}}(G), \|f\|_{\Psi,w^{-1}} = 1 \right\},$$

there is a function $\eta \in L^\Psi_{w^{-1}}(G)$ with $\|\eta\|_{\Psi,w^{-1}} = 1$ such that $|\langle h, \eta \rangle| > \frac{\varepsilon}{2}$.

Since $C_c(G)$ is dense in $L^\Psi_{w^{-1}}(G)$ (note that $\Psi \in \Delta_2$), there is a function $\psi \in C_c(G)$ such that $\|\psi\|_{\Psi,w^{-1}} < \frac{3}{2}$ and

$$|\langle h, \psi \rangle| > \frac{\varepsilon}{2}.$$

So, thanks to (16), there exists an index i_0 such that for each index i, if $F_{i_0} \subseteq F_i$, then

$$\left| \left\langle \tilde{T}_g \left(\frac{1}{w(x_{F_i})} \delta_{x_{F_i}} \right), \psi \right\rangle \right| > \frac{\varepsilon}{2}. \tag{17}$$

But, since $\Phi \in \Delta_2$, there is a function $\gamma \in C_c(G)$ such that $\|g-\gamma\|_{\Phi,w} < \frac{\varepsilon}{8}$. Because of [12, Lemma 2.3(i)] and the Hölder's inequality (1) we have

$$\begin{aligned}
&\left|\left\langle \tilde{T}_g\left(\frac{1}{w(x_{F_i})}\delta_{x_{F_i}}\right), \psi\right\rangle - \left\langle \tilde{T}_\gamma\left(\frac{1}{w(x_{F_i})}\delta_{x_{F_i}}\right), \psi\right\rangle\right| \\
\leqslant &2\left\|\tilde{T}_g\left(\frac{1}{w(x_{F_i})}\delta_{x_{F_i}}\right) - \tilde{T}_\gamma\left(\frac{1}{w(x_{F_i})}\delta_{x_{F_i}}\right)\right\|_{\Phi,w} \|\psi\|_{\Psi,w^{-1}} \\
= &2\left\|\tilde{T}_{g-\gamma}\left(\frac{1}{w(x_{F_i})}\delta_{x_{F_i}}\right)\right\|_{\Phi,w} \|\psi\|_{\Psi,w^{-1}} \\
= &\frac{2}{w(x_{F_i})}\left\|L_{x_{F_i}}(g-\gamma)\right\|_{\Phi,w} \|\psi\|_{\Psi,w^{-1}} \\
\leqslant &\frac{2}{w(x_{F_i})} w(x_{F_i}) \|g-\gamma\|_{\Phi,w} \|\psi\|_{\Psi,w^{-1}} \\
< &\frac{\varepsilon}{4}\|\psi\|_{\Psi,w^{-1}}.
\end{aligned}$$

So,

$$\begin{aligned}
\left|\left\langle \tilde{T}_\gamma\left(\frac{1}{w(x_{F_i})}\delta_{x_{F_i}}\right), \psi\right\rangle\right| &> \left|\left\langle \tilde{T}_g\left(\frac{1}{w(x_{F_i})}\delta_{x_{F_i}}\right), \psi\right\rangle\right| - \frac{\varepsilon}{4}\|\psi\|_{\Psi,w^{-1}} \\
&\geqslant \frac{\varepsilon}{2} - \frac{\varepsilon}{4}\|\psi\|_{\Psi,w^{-1}} > \frac{\varepsilon}{8}.
\end{aligned}$$

Put $A_0 := \operatorname{supp}(\psi)$ and $A_1 := \operatorname{supp}(\gamma)$. For some index i we have

$$F_{i_0} \cup (A_0 A_1^{-1}) \subseteq F_i,$$

and so,

$$\left\langle \tilde{T}_\gamma\left(\frac{1}{w(x_{F_i})}\delta_{x_{F_i}}\right), \psi\right\rangle = \frac{1}{w(x_{F_i})}\int_{A_1} \gamma(x)\psi(x_{F_i}x)\,dx = 0,$$

a contradiction.

Conversely, let $0 \neq g \in L^\Phi_w(G)$ and $F_g \in C_0(G)$. The mappings

$$S_1 : L^\Psi(G) \to L^\Psi_{w^{-1}}(G), \quad S_1(f) := fw, \quad (f \in L^\Psi(G)), \tag{18}$$

and

$$S_2 : C_0^w(G) \to C_0(G), \quad S_2(f) := \frac{f}{w}, \quad (f \in C_0^w(G)), \tag{19}$$

are isometrically isomorphisms. Also, $\tilde{T}_g$ is the adjoint of the operator

$$S_3 : L^\Psi_{w^{-1}}(G) \to C_0^w(G), \quad S_3(f) := \langle g, \mathrm{L}_{(\cdot)^{-1}} f\rangle, \quad (f \in L^\Psi_{w^{-1}}(G)). \tag{20}$$

Then, because of Theorem 2 and the Schauder's Theorem [3, Chapter VI], it would be sufficient to prove that the operator

$$S_g : L^\Psi(G) \to C_0(G), \quad S_g := S_2 S_3 S_1$$

is compact. For this, let $\{f_n\}$ be a bounded sequence in $L^{\Psi}(G)$. For each $n \in \mathbb{N}$ we put

$$K_n := \mathrm{cl}\left(\left\{x \in G : |F_g(x)| \geqslant \frac{1}{n}\right\}\right).$$

Then, for each n we have $K_n \subseteq K_{n+1}$, and since F_g vanishes at infinity, K_n's are compact subsets of G. Also, for each $n \in \mathbb{N}$ and $x \in G \setminus K_n$,

$$\begin{aligned}
\left|S_g(f_n)(x)\right| &= \frac{1}{w(x)} \left|\langle g, \mathrm{L}_{x^{-1}}(wf_n)\rangle\right| \\
&= \frac{1}{w(x)} \left|\langle w\mathrm{L}_x g, f_n\rangle\right| \\
&\leqslant \frac{2}{w(x)} \|w\mathrm{L}_x g\|_{\Phi} \|f_n\|_{\Psi} \\
&= 2\left|F_g(x)\right| \, \|f_n\|_{\Psi} \leqslant \frac{2}{n} \sup_m \|f_m\|_{\Psi}.
\end{aligned}$$

So, similar to the proof of second part of [5, Theorem 1] (see also [15, Theorem 7.23]), by the diagonal method, there is a subsequence of $\{S_g(f_n)\}$ which converges in $C_0(G)$, and this completes the proof.

REMARK 1. In [13, Chapter II], one can find several sufficient conditions for that the hypothesis $\Phi, \Psi \in \Delta_2$ in the above theorem holds.

Now, as a direct conclusion one can see an extension of both [2, Theorem 4] and [5, Corollary 1].

COROLLARY 2. *If G is a compact group, then for each $g \in L^{\Phi}(G)$, the operator $T_g : L^1(G) \to L^{\Phi}_w(G)$ given by*

$$T_g(f) := f * g, \quad (f \in L^1(G))$$

is compact.

Setting $w \equiv 1$, we conclude the following result which is an extension of a well-known one from S. Sakai [16, Theorem 1] (see also [5, Corollary 3]).

COROLLARY 3. *Let G be a locally compact non-compact group and $g \in L^{\Phi}(G)$. If the bounded linear operator $T_g : L^1(G) \to L^{\Phi}(G)$ defined by*

$$T_g(f) := f * g, \quad (f \in L^1(G))$$

is compact, then $g = 0$.

Proof. If $g \neq 0$, then for each compact set $E \subset G$ and $x \in G \setminus E$ we have $|F_g(x)| = \|L_x g\|_{\Phi} = \|g\|_{\Phi} > \frac{1}{2}\|g\|_{\Phi}$, where F_g is defined by (14) with $w \equiv 1$. This implies that $F_g \notin C_0(G)$ and so T_g is not compact, thanks to Theorem 3.

REMARK 2. If $p > 1$, putting $\Phi(x) := |x|^p$ in the above resluts, one can conclude some similar facts for the weighted Lebesgue space $L^p_w(G)$.

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