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SOME HARDY AND CARLESON MEASURE SPACES ESTIMATES FOR BOCHNER–RIESZ MEANS

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Abstract. In this paper, we show that the Bochner-Riesz means are bounded on weighted and variable Hardy spaces by using the finite atomic decomposition theories. The boundedness of Bochner-Riesz means on weighted and variable Carleson measure spaces is also obtained. Moreover, we also prove that the maximal Bochner-Riesz means are bounded from weighted or vaiable Hardy spaces to weighted or variable Lebesgue spaces.

1. Introduction

In this paper, we will study the Bochner-Riesz means defined in terms of Fourier transforms by

$$\widehat{B_R^\delta f}(\xi) = \left(1-\frac{|\xi|^2}{R^2}\right)_+^\delta \hat{f}(\xi),$$

where $\hat{f}$ denotes the Fourier transforms of f and $\left(1-\frac{|\xi|^2}{R^2}\right)_+^\delta$ is the positive part of $\left(1-\frac{|\xi^2|}{R^2}\right)^\delta$. They can be written as convolution operators

$$B_R^\delta f(x) = \int_{\mathbb{R}^n} B_R^\delta(x-y)f(y)dy,$$

where $B_R^\delta(x) = R^n B^\delta(Rx)$. It is well known that B^δ satisfies the inequality

$$|D^\alpha B^\delta(x)| \leqslant C(1+|x|)^{-(\delta+\frac{n+1}{2})}, \tag{1}$$

for any $x\in\mathbb{R}^n$ and any multi-index $\alpha\in\mathbb{Z}_+^n$. The maximal operator B_*^δ is defined by

$$B_*^\delta(f)(x) = \sup_{R>0}|B_R^\delta(x)|.$$

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The Bochner-Riesz means play an important role in the Fourier analysis. They were first studied by Bochner [2] in connection with summation of multiple Fourier series. Questions concerning the convergence of multiple Fourier series have led to the study of their L^p boundedness. The Hardy space $H^p(\mathbb{R}^n)$ with $0 < p \leqslant 1$, which is a suitable substitute of the Lebesgue space $L^p(\mathbb{R}^n)$, plays an important role in the study of operators and their applications to partial differential equations. Sjölin [21] and Stein, Taibleson and Weiss [22] obtained the following result:

THEOREM 1. *Suppose that* $0 < p \leqslant 1$ *and* $\delta > \frac{n}{p} - \frac{n+1}{2}$. *Then the operator* $f \mapsto B_R^\delta f$ *is bounded on* $H^p(\mathbb{R}^n)$, *and satisfies*

$$\|B_R^\delta f\|_{H^p(\mathbb{R}^n)} \leqslant C\|f\|_{H^p(\mathbb{R}^n)}.$$

Let w be a Muckenhoupt weight and $H_w^p(\mathbb{R}^n)$ be the weighted Hardy spaces. Lee [16] obtained the following $H_w^p(\mathbb{R}^n)$ boundedness for the Bochner-Riesz means $B_R^\delta f$ by using the atomic decomposition of $H_w^p(\mathbb{R}^n)$ and their molecular characterizations.

THEOREM 2. *Let* $w \in A_1$ *with critical index* r_w *for the reverse Hölder condition. Suppose that* $0 < p \leqslant 1$ *and* $\delta > \max\{\frac{n}{p} - \frac{n+1}{2}, [\frac{n}{p}]\frac{r_w}{r_w-1} - \frac{n+1}{2}\}$. *Then the operator* $f \mapsto B_R^\delta f$ *is bounded on* $H_w^p(\mathbb{R}^n)$, *and satisfies*

$$\|B_R^\delta f\|_{H_w^p(\mathbb{R}^n)} \leqslant C\|f\|_{H_w^p(\mathbb{R}^n)}.$$

Lee [16] also proved that the maximal operator B_*^δ has the following strong type boundedness.

THEOREM 3. *Let* $w \in A_1$. *Suppose that* $0 < p \leqslant 1$ *and* $\delta > \frac{n}{p} - \frac{n+1}{2}$. *Then the operator* $f \mapsto B_*^\delta f$ *is bounded from* $H_w^p(\mathbb{R}^n)$ *to* $L_w^p(\mathbb{R}^n)$, *and satisfies*

$$\|B_*^\delta f\|_{L_w^p(\mathbb{R}^n)} \leqslant C\|f\|_{H_w^p(\mathbb{R}^n)}.$$

In the paper, we aim to extend the above results to the generalized weighted Hardy spaces and variable Hardy spaces as well as their corresponding dual spaces. For a weight w let $r_w = \inf\{r \in \mathbb{N} : w \in A_r\}$, $s_w = \min\{s_0 \in \mathbb{N}\bigcup\{0\} : p(n+s_0) > nr_w\}$ and $t_w = \min\{t_0 \in \mathbb{N}\bigcup\{0\} : q(n+t_0-\alpha) > nr_{w^q}\}$. We define q' by $\frac{1}{q}+\frac{1}{q'} = 1$ and $q(\cdot)'$ by $\frac{1}{q(x)} + \frac{1}{q(x)'} = 1$ for any $x \in \mathbb{R}^n$. We also write $d = \min\{d_0 \in \mathbb{N}\bigcup\{0\} : p^-(n+d_0) > n\}$. We defer other technique definitions to Section 2.

Now we state the main results in our paper.

THEOREM 4. *Let* $0 < p_0 < 1$, $w \in A_{p/p_0}$ *and* $\delta = \frac{n}{p_0} - \frac{n+1}{2}$. *If* $p_0 < p < \infty$, *then the operator* $f \mapsto B_*^\delta f$ *is bounded from* $H_w^p(\mathbb{R}^n)$ *to* $L_w^p(\mathbb{R}^n)$, *and satisfies*

$$\|B_*^\delta f\|_{L_w^p(\mathbb{R}^n)} \leqslant C\|f\|_{H_w^p(\mathbb{R}^n)}.$$

THEOREM 5. *Let* $0 < p_0 < 1$ *and* $\delta = \frac{n}{p_0} - \frac{n+1}{2}$. *If* $p(\cdot) \in LH(\mathbb{R}^n)$, $p_0 < p^- \leqslant p^+ < \infty$, *the operator* $f \mapsto B_*^\delta f$ *is bounded from* $H^{p(\cdot)}(\mathbb{R}^n)$ *to* $L^{p(\cdot)}(\mathbb{R}^n)$, *and satisfies*

$$\|B_*^\delta f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)}.$$

THEOREM 6. *Let* $0 < p_0 < 1$, $w \in A_{p/p_0}$ *and* $\delta = \frac{n}{p_0} - \frac{n+1}{2}$. *If* $p_0 < p < \infty$, *then the operator* $f \mapsto B_R^\delta f$ *is bounded on* $H_w^p(\mathbb{R}^n)$, *and satisfies*

$$\|B_R^\delta f\|_{H_w^p(\mathbb{R}^n)} \leqslant C\|f\|_{H_w^p(\mathbb{R}^n)}.$$

Moreover, if $p_0 < p \leqslant 1$, *then the operator* $f \mapsto B_R^\delta f$ *is also bounded on* $CMO_w^p(\mathbb{R}^n)$, *and satisfies*

$$\|B_R^\delta f\|_{CMO_w^p(\mathbb{R}^n)} \leqslant C\|f\|_{CMO_w^p(\mathbb{R}^n)}.$$

THEOREM 7. *Let* $0 < p_0 < 1$ *and* $\delta = \frac{n}{p_0} - \frac{n+1}{2}$. *If* $p(\cdot) \in LH(\mathbb{R}^n)$, $p_0 < p^- \leqslant p^+ < \infty$, *the operator* $f \mapsto B_R^\delta f$ *is bounded on* $H^{p(\cdot)}(\mathbb{R}^n)$, *and satisfies*

$$\|B_R^\delta f\|_{H^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)}.$$

Moreover, if $p_0 < p^- \leqslant p^+ \leqslant 1$, *then the operator* $f \mapsto B_R^\delta f$ *is also bounded on* $CMO^{p(\cdot)}(\mathbb{R}^n)$, *and satisfies*

$$\|B_R^\delta f\|_{CMO^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\|f\|_{CMO^{p(\cdot)}(\mathbb{R}^n)}.$$

Throughout this paper, C or c denotes a positive constant that may vary at each occurrence but is independent to the main parameter, and $A \sim B$ means that there are constants $C_1 > 0$ and $C_2 > 0$ independent of the the main parameter such that $C_1 B \leqslant A \leqslant C_2 B$. Given a measurable set $S \subset \mathbb{R}^n$, $|S|$ denotes the Lebesgue measure and χ_S means the characteristic function. By a cube Q we will always mean a cube whose sides are parallel to the coordinate axes. $\ell(Q)$ will denote the length of Q and CQ will denote the cube with same center c_Q such that $\ell(CQ) = C\ell(Q)$.

2. Preliminaries

In this section, we state some definitions and known results about weighted and variable exponent function spaces. We first recall some known results about weighted function spaces. For more information, see [5, 10, 17]. Given a measurable function $w > 0$, for $1 < p < \infty$, it is said that $w \in A_p$ if

$$[w]_{A_p} = \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{1-p'}dx\right)^{p-1} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. If $1 < p < \infty$ and $w \in A_p$, then the Hardy-Littlewood maximal operator M is bounded on weighted Lebesgue space $L^p(w)$.

Define the set

$$A_\infty = \cup_{p\geqslant 1} A_p$$

and define

$$r_w = \inf\{r \geqslant 1 : w \in A_r\}.$$

A weight $w \in RH_s$ for some $s > 1$ if for every cube Q,

$$\left(\frac{1}{|Q|}\int_Q w^s dx\right)^{\frac{1}{s}} \leqslant C\frac{1}{|Q|}\int_Q w dx.$$

Then $w \in RH_s$ if and only if $w^s \in A_\infty$.

We now recall the definition of the weighted Hardy spaces $H^p_w(\mathbb{R}^n)$. For more details, see [23]. Let $\mathscr{S}$ denote the Schwartz class of smooth functions and $\mathscr{S}'$ its topological dual space.. Also, denote by $\mathscr{S}_\infty$ the functions $f \in \mathscr{S}$ satisfying $\int_{\mathbb{R}^n} f(x)x^\alpha dx = 0$ for all muti-indices $\alpha \in \mathbb{Z}^n_+ := (\{0,1,2,\cdots\})^n$ and $\mathscr{S}'_\infty$ its topological dual space. Let $\psi \in \mathscr{S}$, $0 < p < \infty$ and $\psi_t(x) = t^{-n}\psi(t^{-1}x)$, $x \in \mathbb{R}^n$. Denote by $\mathscr{M}$ the grand maximal operator given by $\mathscr{M}f(x) = \sup\{|\psi_t * f(x)| : t > 0, \psi \in \mathscr{F}_N\}$ for any fixed large integer N, where

$$\mathscr{F}_N = \left\{\psi \in \mathscr{S} : \int \psi(x)dx = 1, \sum_{|\alpha|\leqslant N} \sup(1+|x|)^N|\partial^\alpha \psi(x)| \leqslant 1\right\}.$$

The weighted Hardy space $H^p_w(\mathbb{R}^n)$ is the set of all $f \in \mathscr{S}'$, for which the quantity

$$\|f\|_{H^p_w} = \|\mathscr{M}f\|_{L^p_w} < \infty.$$

Denote that

$$M_\phi(f)(x) = \sup_k |\phi_k * f(x)|.$$

If $0 < p < \infty$ and $\phi \in \mathscr{F}_N$,

$$\|f\|_{H^p_w(\mathbb{R}^n)} \sim \|M_\phi(f)\|_{L^p_w(\mathbb{R}^n)}.$$

For brevity, hereafter we write $\|f\|_{X(\mathbb{R}^n)} = \|f\|_X$, where X is the function space. A function a on $\mathbb{R}^n$ is called a (N,∞)-atom, if there exists a cube Q such that $\operatorname{supp} a \subset Q$, $\|a\|_{L^\infty} \leqslant 1$ and

$$\int_{\mathbb{R}^n} a(x)x^\alpha dx = 0 \qquad \text{for} \quad |\alpha| \leqslant N-1.$$

Given $N > s_w$, define

$$\mathscr{O}_N = \left\{f \in C_0^\infty : \int_{\mathbb{R}^n} x^\beta f(x)dx = 0,\ 0 \leqslant |\beta| \leqslant N\right\}.$$

Moverover, $\mathscr{O}_N$ is dense in $H^p_w(\mathbb{R}^n)$. We here recall the following finite atomic decomposition, which was proved in [6].

PROPOSITION 1. *Given* $0 < p < \infty$ *and* $w \in A_\infty$, *fix* $N > s_w$. *Then if* $f \in \mathscr{O}_N$, *there exists a finite sequence* $\{a_i\}_{i=1}^M$ *of* (N,∞) *atoms with supports* Q_i, *and a non-negative sequence* $\{\lambda_i\}_{i=1}^M$ *such that* $f = \sum_i \lambda_i a_i$ *and*

$$\left\| \sum_{i=1}^M \lambda_i \chi_{Q_i} \right\|_{L_w^p} \leqslant C \|f\|_{H_w^p}.$$

We now recall the weighted Carleson measure space $CMO_w^p(\mathbb{R}^n)$. Note that Carleson measure spaces CMO^p have been studied by many authors. See [12, 14, 15, 17, 18].

Let $\psi \in \mathscr{S}$ satisfy

$$\begin{aligned} &\operatorname{supp}(\hat{\psi}) \subset \{\xi \in \mathbb{R}^n : 1/2 \leqslant |\xi| \leqslant 2\}, \\ &|\hat{\psi}(\xi)| \geqslant C > 0 \quad \text{if } \frac{3}{5} \leqslant \xi \leqslant \frac{5}{3} \quad \text{and} \\ &\sum_{j \in \mathbb{Z}} |\hat{\psi}(2^j \xi)|^2 = 1 \quad \text{if } \xi \neq 0. \end{aligned} \tag{2}$$

We say that a cube $Q \subset \mathbb{R}^n$ is dyadic if $Q = Q_{j\mathbf{k}} = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : 2^{-j-N} k_i \leqslant x_i < 2^{-j-N}(k_i + 1), i = 1, 2, \ldots, n\}$ for some $j \in \mathbb{Z}$, some fixed positive large integer N and $\mathbf{k} = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$. Denote by $\ell(Q) = 2^{-j}$ the side length of $Q = Q_{j\mathbf{k}}$. Denote by $\mathscr{D}$ the set of all dyadic cubes Q. Denote by $z_Q = 2^{-j}\mathbf{k}$ the left lower corner of Q and by x_Q is any point in Q when $Q = Q_{j\mathbf{k}}$. For any function ψ defined on $\mathbb{R}^n$, $j \in \mathbb{Z}$, and $Q = Q_{j\mathbf{k}}$, set

$$\psi_j(x) = 2^{jn} \psi(2^j x), \quad \psi_Q(x) = |Q|^{1/2} \psi_j(x - z_Q).$$

DEFINITION 1. Let $\psi \in \mathscr{S}$ satisfy the above conditions, $w \in A_\infty$ and $0 < p \leqslant 1$. The weighted Carleson measure space $CMO_w^p(\mathbb{R}^n)$ is the collection of all $f \in \mathscr{S}'_\infty$ fulfilling

$$\|f\|_{CMO_w^p} := \sup_{P \in \mathscr{D}} \left\{ \frac{1}{w(P)^{\frac{2}{p}-1}} \sum_{Q \subset P} \frac{|Q|}{w(Q)} |\langle f, \psi_Q \rangle|^2 \right\}^{1/2} < \infty.$$

Define a linear map S_φ by

$$S_\varphi(f) = \{\langle f, \varphi_Q \rangle\}_Q,$$

and another linear map T_ψ by

$$T_\psi(\{s_Q\}_Q) = \sum_Q s_Q \psi_Q.$$

For $g \in CMO_w^p(\mathbb{R}^n)$, define a linear functional L_g by

$$L_g(f) = \langle S_\psi(g), S_\varphi(f) \rangle = \sum_Q \langle g, \psi_Q \rangle \langle f, \varphi_Q \rangle$$

for $f \in \mathscr{S}_\infty$. Then the weighted Carleson measure space $CMO_w^p(\mathbb{R}^n)$ is the dual space of the weighted Hardy space $H_w^p(\mathbb{R}^n)$.

PROPOSITION 2. *Suppose that* $w \in A_\infty$, $0 < p \leqslant 1$. *The dual of* $H^p_w(\mathbb{R}^n)$ *is* $CMO^p_w(\mathbb{R}^n)$ *in the following sense.*
(1) For $g \in CMO^p_w(\mathbb{R}^n)$, *the linear functional* L_g, *defined initially on* $\mathscr{S}_\infty$, *extends to a continuous linear functional on* $H^p_w(\mathbb{R}^n)$ *with* $\|L_g\| \leqslant C\|g\|_{CMO^p_w}$.
(2) Conversely, every continuous linear functional L *on* H^p_w *satisfies* $L = L_g$ *for some* $g \in CMO^p_w(\mathbb{R}^n)$ *with* $\|g\|_{CMO^p_w} \leqslant C\|L\|$.

Now we state some basic results about variable exponent function spaces. For more information, see [3, 7, 9, 20, 25]. For any Lebesgue measurable function $p(\cdot) : \mathbb{R}^n \to (0,\infty]$ and for any measurable subset $E \subset \mathbb{R}^n$, we denote

$$p^-(E) = \inf_{x\in E} p(x), \qquad p^+(E) = \sup_{x\in E} p(x).$$

Especially, we denote $p^- = p^-(\mathbb{R}^n)$ and $p^+ = p^+(\mathbb{R}^n)$. Let $p(\cdot)\colon\ \mathbb{R}^n \to (0,\infty)$ be a measurable function with $0 < p^- \leqslant p^+ < \infty$ and $\mathscr{P}^0$ be the set of all these $p(\cdot)$. Let $p(\cdot) : \mathbb{R}^n \to (0,\infty]$ be a Lebesgue measurable function. The variable Lebesgue space $L^{p(\cdot)}$ consisits of all Lebesgue measurable functions f, for which the quantity $\int_{\mathbb{R}^n} |\varepsilon f(x)|^{p(x)} dx$ is finite for some $\varepsilon > 0$ and

$$\|f\|_{L^{p(\cdot)}} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \leqslant 1\right\}.$$

We also recall the following class of exponent function, which can be found in [8]. Let $\mathscr{B}$ be the set of $p(\cdot) \in \mathscr{P}$ such that the Hardy-littlewood maximal operator M is bounded on $L^{p(\cdot)}$. An important subset of $\mathscr{B}$ is the LH condition.

In the study of variable exponent function spaces it is common to assume that the exponent function $p(\cdot)$ satisfies the LH condition. We say that $p(\cdot) \in LH$, if $p(\cdot)$ satisfies

$$|p(x) - p(y)| \leqslant \frac{C}{-\log(|x-y|)}, \quad |x-y| \leqslant 1/2$$

and

$$|p(x) - p(y)| \leqslant \frac{C}{\log|x| + e}, \quad |y| \geqslant |x|.$$

It is well known that $p(\cdot) \in \mathscr{B}$ if $p(\cdot) \in \mathscr{P} \cap LH$. Let $f \in \mathscr{S}'$, $\psi \in \mathscr{S}$, $p(\cdot) \in \mathscr{P}^0$. The variable Hardy space $H^{p(\cdot)}$ is the set of all $f \in \mathscr{S}'$, for which the quantity

$$\|f\|_{H^{p(\cdot)}} = \|\mathscr{M} f\|_{L^{p(\cdot)}} < \infty.$$

If $p(\cdot) \in LH(\mathbb{R}^n) \cap \mathscr{P}^0(\mathbb{R}^n)$ and $\phi \in \mathscr{F}_N$,

$$\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} \sim \|M_\phi(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Given $N > d$, $\mathscr{O}_N$ is dense in $H^{p(\cdot)}(\mathbb{R}^n)$. We have the following finite atomic decomposition for variable Hardy space. For more results about variable Hardy spaces, we refer to [7, 20, 24, 26, 30, 31, 32].

PROPOSITION 3. *Given $p(\cdot) \in LH \cap \mathscr{P}^0$, fix $N > d$. Then if $f \in H^{p(\cdot)}(\mathbb{R}^n)$, there exists a finite sequence $\{a_i\}_{i=1}^M$ of (N,∞) atoms with supports Q_i, and a non-negative sequence $\{\lambda_i\}_{i=1}^M$ such that $f = \sum_i \lambda_i a_i$ and*

$$\left\| \sum_{i=1}^M \lambda_i \chi_{Q_i} \right\|_{L^{p(\cdot)}} \leqslant C \|f\|_{H^{p(\cdot)}}.$$

We now introduce a new space $CMO^{p(\cdot)}(\mathbb{R}^n)$ as follows.

DEFINITION 2. Let $\psi \in \mathscr{S}$ define above, and $0 < p^- \leqslant p^+ \leqslant 1$. The Carleson measure space $CMO^{p(\cdot)}(\mathbb{R}^n)$ is the collection of all $f \in \mathscr{S}'_\infty$ fulfilling

$$\|f\|_{CMO^{p(\cdot)}} := \sup_{P \in \mathscr{D}} \left\{ \frac{|P|}{\|\chi_P\|_{p(\cdot)}^2} \int_{\mathbb{R}^n} \sum_{Q \subset P} |Q|^{-1} |\langle f, \psi_Q \rangle|^2 \chi_Q(x) dx \right\}^{1/2} < \infty.$$

PROPOSITION 4. *Suppose that $p(\cdot) \in LH$, $0 < p^- \leqslant p^+ \leqslant 1$. The dual of $H^{p(\cdot)}(\mathbb{R}^n)$ is $CMO^{p(\cdot)}(\mathbb{R}^n)$ in the following sense.*
(1) For $g \in CMO^{p(\cdot)}(\mathbb{R}^n)$, the linear functional L_g, defined initially on $\mathscr{S}_\infty$, extends to a continuous linear functional on $H^{p(\cdot)}(\mathbb{R}^n)$ with $\|L_g\| \leqslant C\|g\|_{CMO^{p(\cdot)}}$.
(2) Conversely, every continuous linear functional L on $H^{p(\cdot)}(\mathbb{R}^n)$ satisfies $L = L_g$ for some $g \in CMO^{p(\cdot)}(\mathbb{R}^n)$ with $\|g\|_{CMO^{p(\cdot)}} \leqslant C\|L\|$.

3. Proofs of main results

In this section we prove the main results by applying the finite atomic decompositions in terms of L^∞ atoms and using an argument of weak density property. Note that the weak density property is very useful when we deal with the bounedness of operators on Carleson measure type spaces or Lipschitz type spaces (see [11, 13, 27, 28]). In the following, we first prove that when δ is greater than the critical index, the maximal Bochner-Riesz means are bounded on generalized weighted Hardy spaces and variable Hardy spaces. Now we state the following Fefferman-Stein vector-valued inequalities for maximal operators on weighted Lebesgue spaces.

PROPOSITION 5. [1] *Let $0 < p < \infty$ and $w \in A_p$. Then for any $q > 1$, $f = \{f_i\}_{i \in \mathbb{Z}}$, $f_i \in L_{loc}(\mathbb{R}^n)$,*

$$\| \|\mathbb{M}(f)\|_{l^q} \|_{L^p_w(\mathbb{R}^n)} \leqslant C \| \|f\|_{l^q} \|_{L^p_w(\mathbb{R}^n)},$$

where $\mathbb{M}(f) = \{M(f_i)\}_{i \in \mathbb{Z}}$.

We also need the following boundedness of the vector-valued maximal operator M on variable Lebesgue spaces, whose proof can be found in [4] via extrapolation.

PROPOSITION 6. *Let $p(\cdot) \in LH \cap \mathscr{P}$. Then for any $q>1$, $f=\{f_i\}_{i\in\mathbb{Z}}$, $f_i \in L_{loc}(\mathbb{R}^n)$,*

$$\|\|\mathbb{M}(f)\|_{l^q}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\|\|f\|_{l^q}\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where $\mathbb{M}(f)=\{M(f_i)\}_{i\in\mathbb{Z}}$.

Proof of Theorem 4. By Proposition 1, for $p_0<p<\infty$ and $w\in A_{p/p_0}$, fix $N_0>s_w$, then if $f\in\mathscr{O}_{N_0}$, there exists a finite sequence $\{a_i\}_{i=1}^M$ of (N_0,∞) atoms with supports Q_i, and a non-negative sequence $\{\lambda_i\}_{i=1}^M$ such that

$$f=\sum_{i=1}^M \lambda_i a_i$$

and

$$\left\|\sum_{i=1}^M \lambda_i \chi_{Q_i}\right\|_{L_w^p(\mathbb{R}^n)} \leqslant C\|f\|_{H_w^p(\mathbb{R}^n)}.$$

By the sublinearity of B_*^δ, we have

$$\|B_*^\delta f\|_{L_w^p(\mathbb{R}^n)} \leqslant \left\|\sum_{i=1}^M \lambda_i B_*^\delta a_i\right\|_{L_w^p(\mathbb{R}^n)}.$$

To prove our theorem, we first need the following estimate for the action of the operator on atoms, whose proof can be found in [19, page125-page127] or [29, Lemma 3.2]. Given any (N_0,∞)-atom a, we claim that for $x\in\mathbb{R}^n$,

$$|B_*^\delta a(x)| \leqslant C\frac{\ell(Q)^{n/p_0}}{(\ell(Q)+|x-c_Q|)^{n/p_0}}.$$

Hence, for all $x\in\mathbb{R}^n$ we have

$$B_*^\delta a(x) \leqslant CM(\chi_Q)(x)^{1/p_0}.$$

Therefore, by $w\in A_{p/p_0}$, Proposition 5 and Proposition 1 we get

$$\begin{aligned}
\left\|\sum_{i=1}^M \lambda_i B_*^\delta a_i\right\|_{L_w^p(\mathbb{R}^n)} &\leqslant C\left\|\left(\sum_{i=1}^M \lambda_i M(\chi_Q)^{1/p_0}\right)^{p_0}\right\|_{L_w^{p/p_0}(\mathbb{R}^n)}^{1/p_0} \\
\leqslant C\left\|\left(\sum_{i=1}^M \lambda_i \chi_Q\right)^{p_0}\right\|_{L_w^{p/p_0}(\mathbb{R}^n)}^{1/p_0} &= C\left\|\sum_{i=1}^M \lambda_i \chi_Q\right\|_{L_w^p} \leqslant C\|f\|_{H_w^p(\mathbb{R}^n)}.
\end{aligned}$$

Since that $\mathscr{O}_{N_0}$ is dense in $H_w^p(\mathbb{R}^n)$, then by the density argument B_*^δ can be extended to a bounded operator from $H_w^p(\mathbb{R}^n)$ to $L_w^p(\mathbb{R}^n)$. Therefore, we have completed the proof. $\square$

Proof of Theorem 5. The proof of Theorem 5 is very similar to the above proof and so we only need to show the differences. To prove the variable Hardy spaces estimates, we only need to show that

$$\left\|\sum_{i=1}^{M}\lambda_i B_*^\delta a_i\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant \|f\|_{H^{p(\cdot)}(\mathbb{R}^n)}.$$

Choose $N_0 > d$ and observe that $p_0 < p^- \leqslant p^+ < \infty$ and $p(\cdot) \in LH$. Therefore, by Proposition 6 and Proposition 3 we get

$$\begin{aligned}\left\|\sum_{i=1}^{M}\lambda_i B_*^\delta a_i\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leqslant C\left\|\left(\sum_{i=1}^{M}\lambda_i M(\chi_Q)^{1/p_0}\right)^{p_0}\right\|_{L^{p(\cdot)/p_0}(\mathbb{R}^n)}^{1/p_0}\\ \leqslant C\left\|\left(\sum_{i=1}^{M}\lambda_i\chi_Q\right)^{p_0}\right\|_{L^{p(\cdot)/p_0}(\mathbb{R}^n)}^{1/p_0} &= C\left\|\sum_{i=1}^{M}\lambda_i\chi_Q\right\|_{L^{p(\cdot)}} \leqslant C\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Since that $\mathscr{O}_{N_0}$ is dense in $H^{p(\cdot)}(\mathbb{R}^n)$, then by the density argument B_*^δ can be extended to a bounded operator from $H^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot)}(\mathbb{R}^n)$. Thus, we completed the proof. □

Under the same hypothesis of Theorem 4 and Theorem 5, if we replace $B_*^\delta f$ by $B_R^\delta f$, then the results can be strengthened again. Before we give the proof of Theorem 6 and Theorem 7, we recall the following propositions on the weak density property for $CMO_w^p(\mathbb{R}^n)$ and $CMO^{p(\cdot)}(\mathbb{R}^n)$.

PROPOSITION 7. [27] *Let $w \in A_\infty$ and $0 < p \leqslant 1$. If $f \in CMO_w^p(\mathbb{R}^n)$, then there exist a sequence $\{f_m\} \in CMO_w^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ such that f_m converges to f in the distribution sense. Furthermore,*

$$\|f_m\|_{CMO_w^p} \leqslant C\|f\|_{CMO_w^p}, \quad \textit{for} \quad f \in CMO_w^p(\mathbb{R}^n).$$

PROPOSITION 8. [25] *Let $p(\cdot) \in LH$ and $0 < p^- \leqslant p^+ \leqslant 1$. If $f \in CMO^{p(\cdot)}(\mathbb{R}^n)$, then there exist a sequence $\{f_m\} \in CMO^{p(\cdot)}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ such that f_m converges to f in the distribution sense. Furthermore,*

$$\|f_m\|_{CMO^{p(\cdot)}} \leqslant C\|f\|_{CMO^{p(\cdot)}}, \quad \textit{for} \quad f \in CMO^{p(\cdot)}(\mathbb{R}^n).$$

Proof of Theorem 6. By Proposition 1, for $p_0 < p < \infty$ and $w \in A_{p/p_0}$, fix $N_0 > s_w$, then if $f \in \mathscr{O}_{N_0}$, there exists a finite sequence $\{a_i\}_{i=1}^M$ of (N_0,∞) atoms with supports Q_i, and a non-negative sequence $\{\lambda_i\}_{i=1}^M$ such that

$$f = \sum_{i=1}^{M}\lambda_i a_i$$

and

$$\left\|\sum_{i=1}^{M}\lambda_i\chi_{Q_i}\right\|_{L_w^p(\mathbb{R}^n)} \leqslant C\|f\|_{H_w^p(\mathbb{R}^n)}.$$

By the linearity of B_R^δ, we have

$$\|B_R^\delta f\|_{L_w^p(\mathbb{R}^n)} \leqslant \left\| \sum_{i=1}^M \lambda_i B_R^\delta a_i \right\|_{L_w^p(\mathbb{R}^n)}.$$

To prove it, we need a maximal opertor estimate for the action of the operator B_R^δ on atoms, whose proof can be found in [29, Lemma 3.3]. Suppose that $\varphi \in C_c^{N_0}(\mathbb{R}^n)$, supp $\varphi \subset \{|x| < \frac{1}{4}\}$, $\varphi \geqslant 0$ and $\int \varphi dx = 1$. Given any (N_0,∞)-atom a, then for $0<t<\infty$ and $x \in \mathbb{R}^n$,

$$|\varphi_t * B_R^\delta a(x)| \leqslant C \frac{\ell(Q)^{n/p_0}}{(\ell(Q)+|x-c_Q|)^{n/p_0}}. \tag{3}$$

Hence, for all $x \in \mathbb{R}^n$,

$$B_R^\delta a(x) \leqslant CM(\chi_Q)(x)^{1/p_0}.$$

Therefore, repeating the same argument in Theorem 4 we can show that B_R^δ can be extended to a bounded operator from $H_w^p(\mathbb{R}^n)$ to $H_w^p(\mathbb{R}^n)$. Next we show that B_R^δ is a bounded operator on $CMO_w^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ to $CMO_w^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, when $0 < p^- \leqslant p^+ \leqslant 1$. The adjoint operator $B_R^{\delta,*}$ is defined by

$$\left\langle B_R^{\delta,*} f, g \right\rangle = \left\langle f, B_R^\delta g \right\rangle, \quad f,g \in \mathscr{S}.$$

The kernel of $B_R^{\delta,*}$ also satisfies the conditions (1) and (3). Hence, $B_R^{\delta,*}$ is also a bounded operator from $H_w^p(\mathbb{R}^n)$ to $H_w^p(\mathbb{R}^n)$. Then applying Proposition 2 yields that

$$\left|\left\langle B_R^\delta f, g \right\rangle\right| = \left|\left\langle f, B_R^{\delta,*} g \right\rangle\right| \leqslant \|f\|_{CMO_w^p(\mathbb{R}^n)} \|B_R^{\delta,*} g\|_{H_w^p(\mathbb{R}^n)} \leqslant C\|f\|_{CMO_w^p(\mathbb{R}^n)} \|g\|_{H_w^p(\mathbb{R}^n)}.$$

That is, for each $f \in CMO_w^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, $L_f(g) = \left\langle B_R^\delta f, g \right\rangle$ is a continuous linear functional on $H_w^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Since that $w \in A_\infty$, then by the fact $H_w^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $H_w^p(\mathbb{R}^n)$, L_f can be extended to a continuous linear functional on $H_w^p(\mathbb{R}^n)$ with

$$\|L_f\| \leqslant C\|f\|_{CMO_w^p}.$$

Conversely, by Propostion 2 again, there exists $h \in CMO_w^p(\mathbb{R}^n)$ such that $\left\langle B_R^\delta f, g \right\rangle = \langle h, g \rangle$ for $g \in H_w^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ with

$$\|h\|_{CMO_w^p} \leqslant C\|L_f\|.$$

Thus,

$$\|B_R^\delta f\|_{CMO_w^p} = \sup_{P \in \mathscr{D}} \left\{ \frac{1}{w(P)^{\frac{2}{p}-1}} \sum_{Q \subset P} \frac{|Q|}{w(Q)} |\left\langle B_R^\delta f, \psi_Q \right\rangle|^2 \right\}^{1/2}$$

$$= \sup_{P\in\mathscr{D}} \left\{ \frac{1}{w(P)^{\frac{2}{p}-1}} \sum_{Q\subset P} \frac{|Q|}{w(Q)} |\langle h, \psi_Q\rangle|^2 \right\}^{1/2}$$
$$= \|h\|_{CMO_w^p} \leqslant C\|L_f\| \leqslant C\|f\|_{CMO_w^p}.$$

Next we extend this result to $CMO_w^p(\mathbb{R}^n)$ via an argument of weak density property. Suppose that $f \in CMO_w^p(\mathbb{R}^n)$. By Proposition 7, we can choose a sequence $\{f_m\} \subset CMO_w^p(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ with

$$\|f_m\|_{CMO_w^p} \leqslant C\|f\|_{CMO_w^p}$$

such that f_m converges to f in the distribution sense. Therefore, for $f \in CMO_w^p(\mathbb{R}^n)$, define

$$\left\langle B_R^\delta f, g\right\rangle = \lim_{m\to\infty} \left\langle B_R^\delta f_m, g\right\rangle, \quad \text{for} \quad g \in H_w^p(\mathbb{R}^n)\cap L^2(\mathbb{R}^n).$$

In fact, we have $\langle B_R^\delta(f_i - f_j), g\rangle = \left\langle f_i - f_j, B_R^{\delta,*}(g)\right\rangle$, where $f_i - f_j$ and g belong to $L^2(\mathbb{R}^n)$. So we have $B_R^{\delta,*} g \in H_w^p(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$. Then we get that

$$\left\langle B_R^\delta(f_i - f_j), g\right\rangle = \left\langle f_i - f_j, B_R^{\delta,*}(g)\right\rangle \to 0$$

as $j,\ k \to \infty$. Therefore, $B_R^\delta f$ is well defined and

$$\left\langle B_R^\delta f, g\right\rangle = \lim_{m\to\infty} \left\langle B_R^\delta f_m, g\right\rangle$$

for any $g \in H_w^p(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ and $f_m \in CMO_w^p(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$. Then by Fatou's Lemma, for each dyadic cube P in $\mathbb{R}^n$,

$$\left\{ \frac{1}{w(P)^{\frac{2}{p}-1}} \sum_{Q\subset P} \frac{|Q|}{w(Q)} \left|\left\langle B_R^\delta f, \psi_Q\right\rangle\right|^2 \right\}^{1/2}$$
$$\leqslant \liminf_{m\to\infty} \left\{ \frac{1}{w(P)^{\frac{2}{p}-1}} \sum_{Q\subset P} \frac{|Q|}{w(Q)} \left|\left\langle B_R^\delta f_m, \psi_Q\right\rangle\right|^2 \right\}^{1/2}.$$

Hence,

$$\|B_R^\delta f\|_{CMO_w^p} \leqslant \liminf_{m\to\infty} \|B_R^\delta f_m\|_{CMO_w^p} \leqslant C\|f_m\|_{CMO_w^p} \leqslant C\|f\|_{CMO_w^p}.$$

Therefore, we have completed the proof. □

Proof of Theorem 7. The proof of Theorem 7 is nearly identical to the above proof and so we only need to show the differences. To prove the variable Hardy spaces estimates, we only need to show that

$$\left\| \sum_{i=1}^{M} \lambda_i B_R^\delta a_i \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant \|f\|_{H^{p(\cdot)}(\mathbb{R}^n)}.$$

Choose $N_0 > d$ and observe that $p_0 < p^- \leqslant p^+ < \infty$ and $p(\cdot) \in LH$. Therefore, by Proposition 6 and Proposition 3 we similarly get the desired results. The proof of variable Carleson measure spaces estimates for B_R^δ is also essentially the same as the proof of Theorem 6. By applying Proposition 4 and Proposition 8, we can obtain that

$$\|B_R^\delta f\|_{CMO^{p(\cdot)}} \leqslant \liminf_{m\to\infty} \|B_R^\delta f_m\|_{CMO^{p(\cdot)}} \leqslant C\|f_m\|_{CMO^{p(\cdot)}} \leqslant C\|f\|_{CMO^{p(\cdot)}}.$$

Thus, we complete the proof. □

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