NEW NORM EQUALITIES AND INEQUALITIES
FOR CERTAIN OPERATOR MATRICES

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Abstract. We prove new norm equalities and inequalities for general $n \times n$ tridiagonal and anti-tridiagonal operator matrices, including pinching type inequalities for weakly unitarily invariant norms.

1. Introduction

Let $B(H)$ denote the C^*-algebra of all bounded linear operators on a complex separable Hilbert space H with inner product $\langle \ . \ , \ . \ \rangle$. For $T \in B(H)$, let $\omega(T) = \sup\{|\langle Tx,x \rangle| : x \in H, \|x\| = 1\}$ and $\|T\| = \sup\{|\langle Tx, y \rangle| : x, y \in H, \|x\| = \|y\| = 1\}$ be the numerical radius and the usual operator norm of T, respectively. Let C_p denote the Schatten p-class of operators in $B(H)$. For $T \in C_p$, $1 \leqslant p < \infty$, let $\|T\|_p = (tr|T|^p)^{\frac{1}{p}}$ be the Schatten p-norm of T, where $|T| = (T^*T)^{\frac{1}{2}}$ denotes the absolute value of T and tr is the usual trace functional. When we talk of $\|T\|_p$, we are assuming that $T \in C_p$. The above mentioned norms are weakly unitarily invariant. (Recall that a norm τ on $B(H)$ is called weakly unitarily invariant if $\tau(T) = \tau(UTU^*)$ for all $T \in B(H)$ and for all unitary operators $U \in B(H)$).

The problem of relating a norm of an operator matrix $T = [T_{ij}]$ to those of its entries T_{ij} has attracted the attention of several mathematicians (see, e.g., [1, 7, 11, 12], and references therein). This problem is closely related to certain problems in operator theory, mathematical physics, quantum information theory, and numerical analysis. For the general theory of unitarily invariant norms, we refer to [4, 8].

If $R_1, R_2, ..., R_n$ are operators in $B(H)$, we write the direct sum $\overset{n}{\underset{j=1}{\oplus}} R_j$ for the

$n \times n$ block-diagonal operator matrix $\begin{bmatrix} R_1 & 0 & 0 & \cdots & 0 \\ 0 & R_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & R_n \end{bmatrix}$, regarded as an operator

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on $H^{(n)}$ $\left(= \displaystyle\bigoplus_{j=1}^{n} H, \text{ the direct sum of n copies of } H \right)$. Thus,

$$\omega \left(\bigoplus_{j=1}^{n} R_j \right) = \max \{ \, \omega(R_j) : j = 1, 2, \ldots, n \},$$

$$\left\| \bigoplus_{j=1}^{n} R_j \right\| = \max \{ \, \|R_j\| : j = 1, 2, \ldots, n \},$$

and

$$\left\| \bigoplus_{j=1}^{n} R_j \right\|_p = \left(\sum_{j=1}^{n} \|R_j\|_p^p \right)^{\frac{1}{p}} \text{ for } 1 \leqslant p < \infty.$$

In particular,

$$\left\| \bigoplus_{j=1}^{n} R \right\|_p = n^{\frac{1}{p}} \|R\|_p \text{ for } 1 \leqslant p < \infty.$$

The pinching inequality for weakly unitarily invariant norms is one of the most useful inequalities for operator matrices. It asserts that if $S = [S_{ij}]$, then

$$\tau \left(\bigoplus_{i=1}^{n} S_{ii} \right) \leqslant \tau(S) \tag{1}$$

(see e.g., [6], [4, p. 107], [5, pp. 87-88], and [8, p. 82]). For the numerical radius, the usual operator norm, and the Schatten p-norms, the inequality (1) states that

$$\max_{1 \leqslant i \leqslant n} \omega(S_{ii}) \quad \leqslant \quad \omega(S), \tag{2}$$

$$\max_{1 \leqslant i \leqslant n} \| \, S_{ii} \, \| \quad \leqslant \quad \|S\|, \tag{3}$$

and

$$\left(\sum_{i=1}^{n} \|S_{ii}\|_p^p \right)^{\frac{1}{p}} \quad \leqslant \quad \|S\|_p. \tag{4}$$

For $1 < p < \infty$, equality holds in (4) if and only if S is block-diagonal, i.e., if and only if $S_{ij} = 0$ for $i \neq j$ (see, e.g., [8, p. 94]).

In [3], the authors considered norm equalities and inequalities for operator matrices. Based on the nice structures of circulant and skew-circulant operator matrices, they presented pinching type inequalities for weakly unitarily invariant norms. Many works about norm equalities and inequalities of special operator matrices can be found in [1, 6, 10, 11, 12].

Now, if T_{ij} are operators in $B(H)$ for $i, j = 1, 2, \ldots n$, then the general $n \times n$ tridiagonal operator matrix is defined as

$$T = \begin{bmatrix} T_{11} & T_{12} & 0 & \cdots & & 0 \\ T_{21} & T_{22} & T_{23} & \ddots & & \vdots \\ 0 & T_{32} & T_{33} & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & & T_{n-1,n} \\ 0 & \cdots & 0 & T_{n,n-1} & & T_{nn} \end{bmatrix}.$$

Also, the general $n \times n$ anti-tridiagonal operator matrix is defined as

$$T = \begin{bmatrix} 0 & \cdots & 0 & T_{1,n-1} & T_{1n} \\ \vdots & & \cdot^{\cdot^{\cdot}} & T_{2,n-2} & T_{2,n-1} & T_{2n} \\ 0 & & \cdot^{\cdot^{\cdot}} & T_{3,n-2} & T_{3,n-1} & 0 \\ T_{n-1,1} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & \cdot^{\cdot^{\cdot}} & \vdots \\ T_{n1} & T_{n2} & 0 & & \cdots & 0 \end{bmatrix}.$$

In [2], the authors gave numerical radius equalities and inequalities for 5×5 tridiagonal operator matrices. In Section 2, we give general norm equalities for certain $n \times n$ tridiagonal and $n \times n$ anti-tridiagonal operator matrices. In Section 3, we apply these norm equalities to obtain new pinching type inequalities, and equality conditions in these norm inequalities are also given.

2. Norm equalities for special $n \times n$ tridiagonal and $n \times n$ anti-tridiagonal operator matrices

In this section, we prove a norm equality for special $n \times n$ tridiagonal operator matrices and for $n \times n$ anti-tridiagonal operator matrices, and then we give some results concerning the numerical radius, the usual operator norm, and the Schatten p-norms.

THEOREM 1. *Let $A, B \in B(H)$ and $T = \begin{bmatrix} A & B & 0 & \cdots & 0 \\ B & A & B & \ddots & \vdots \\ 0 & B & A & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & B \\ 0 & \cdots & 0 & B & A \end{bmatrix}$ be a tridiagonal op-*

erator matrix in $B(H^{(n)})$. Then

$$\tau(T) = \tau \left(\bigoplus_{j=1}^{n} \left[A + \left(2\cos\frac{j\pi}{n+1} \right) B \right] \right)$$

Proof.

$$
\text{Let } U = \sqrt{\tfrac{2}{n+1}}
\begin{bmatrix}
\sin\dfrac{\pi}{n+1} & \sin\dfrac{2\pi}{n+1} & \sin\dfrac{3\pi}{n+1} & \cdots & \sin\dfrac{n\pi}{n+1} \\[2mm]
\sin\dfrac{2\pi}{n+1} & \sin\dfrac{4\pi}{n+1} & \sin\dfrac{6\pi}{n+1} & \cdots & \sin\dfrac{2n\pi}{n+1} \\[2mm]
\sin\dfrac{3\pi}{n+1} & \sin\dfrac{6\pi}{n+1} & \sin\dfrac{9\pi}{n+1} & \cdots & \sin\dfrac{3n\pi}{n+1} \\[2mm]
\vdots & \vdots & \vdots & \vdots & \vdots \\[2mm]
\sin\dfrac{n\pi}{n+1} & \sin\dfrac{2n\pi}{n+1} & \sin\dfrac{3n\pi}{n+1} & \cdots & \sin\dfrac{n^2\pi}{n+1}
\end{bmatrix} \otimes I,
$$

where I is the identity operator in $B(H)$. It can be seen that the column vectors of the $n \times n$ matrix given in the definition of U form an orthonormal set of vectors, (see, e.g., [9, pp. 72-73]). Thus, U is a unitary operator in $B\left(H^{(n)}\right)$ and

$$
UTU^* = diag\left(A + \left(2\cos\frac{\pi}{n+1} \right)B, \ \ A + \left(2\cos\frac{2\pi}{n+1} \right)B, \right.
$$
$$
\left. A + \left(2\cos\frac{3\pi}{n+1} \right)B, \ \ldots, \ \ A + \left(2\cos\frac{n\pi}{n+1} \right)B \right).
$$

Hence, from the invariance property of weakly unitarily invariant norms we have the desired result.

Specializing the norm equality in Theorem 1 to the numerical radius, the usual operator norm, and to the Schatten p-norms, we obtain the following equalities:

1. $\omega(T) = \max\left\{ \omega\left(A + (2\cos\frac{j\pi}{n+1})B\right) : j = 1, 2, \ldots, n \right\}$,

   in particular, (letting $n = 3$), we have $\omega(T) = \max\{ \omega(A + \sqrt{2}B), \omega(A), \omega(A - \sqrt{2}B)\}$.

2. $\|T\| = \max\left\{ \left\| A + (2\cos\frac{j\pi}{n+1})B \right\| : j = 1, 2, \ldots, n \right\}$,

   in particular, (letting $n = 3$), we have $\|T\| = \max\left\{ \left\|A + \sqrt{2}B\right\|, \|A\|, \left\|A - \sqrt{2}B\right\| \right\}$.

3. $\|T\|_p = \left(\sum_{j=1}^{n} \left\| A + (2\cos\frac{j\pi}{n+1})B \right\|_p^p \right)^{\frac{1}{p}}$ for $1 \leqslant p < \infty$,

   in particular, (letting $n = 3$), we have $\|T\|_p = \left(\left\|A + \sqrt{2}B\right\|_p^p + \|A\|_p^p + \left\|A - \sqrt{2}B\right\|_p^p \right)^{\frac{1}{p}}$.

Here, we give some special cases of Theorem 1.

1. If $A = 0$, then $\omega(T) = 2\max\left\{ \left|\cos\frac{j\pi}{n+1}\right| \omega(B) : j = 1, 2, \ldots, n \right\}$,

   $\|T\| = 2\max\left\{ \left|\cos\frac{j\pi}{n+1}\right| \|B\| : j = 1, 2, \ldots, n \right\}$, and

   $\|T\|_p = 2\left(\sum_{j=1}^{n} \left|\cos\frac{j\pi}{n+1}\right|^p \right)^{\frac{1}{p}} \|B\|_p$ for $1 \leqslant p < \infty$.

2. If $B = 0$, then $\omega(T) = \omega(A)$, $\|T\| = \|A\|$ and $\|T\|_p = n^{\frac{1}{p}}\|A\|_p$ for $1 \leqslant p < \infty$.

3. If $A = B$, then $\omega(T) = \max\left\{\left|1 + 2\cos\frac{j\pi}{n+1}\right| \omega(A) : j = 1,2,\ldots,n\right\}$,

$\|T\| = \max\left\{\left|1 + 2\cos\frac{j\pi}{n+1}\right| \|A\| : j = 1,2,\ldots,n\right\}$, and

$$\|T\|_p = \left(\sum_{j=1}^{n}\left|1 + 2\cos\frac{j\pi}{n+1}\right|^p\right)^{\frac{1}{p}}\|A\|_p \quad \text{for } 1 \leqslant p < \infty.$$

4. If $B = iA$, then $\omega(T) = \max\{\omega((1 + (2\cos\frac{j\pi}{n+1})i)A) : j = 1,2,\ldots,n\}$,

$\|T\| = \max\left\{\left|1 + (2\cos\frac{j\pi}{n+1})i\right| \|A\| : j = 1,2,\ldots,n\right\}$, and

$$\|T\|_p = \left(\sum_{j=1}^{n}\left|1 + (2\cos\frac{j\pi}{n+1})i\right|^p\right)^{\frac{1}{p}}\|A\|_p \quad \text{for } 1 \leqslant p < \infty.$$

THEOREM 2. *Let* $A, B \in B(H)$ *and* $T = \begin{bmatrix} A & k^{n-2}B & 0 & \cdots & 0 \\ B & A & k^{n-2}B & \ddots & \vdots \\ 0 & B & A & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & k^{n-2}B \\ 0 & \cdots & 0 & B & A \end{bmatrix}$ *be a tridi-*

agonal operator matrix in $B(H^{(n)})$, *and let* $k = e^{\frac{\pi i}{2n}}$ *(the nth root of the imaginary number* i). *Then* $\tau(T) = \tau\left(\bigoplus_{j=1}^{n}\left[A + (2k^{n-1}\cos\frac{j\pi}{n+1})B\right]\right)$.

Proof.

Let $V = \begin{bmatrix} I & 0 & \cdots & 0 & 0 \\ 0 & kI & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & k^{n-2}I & 0 \\ 0 & 0 & \cdots & 0 & k^{n-1}I \end{bmatrix}$. Then it is easy to prove that V is a unitary

operator in $B(H^{(n)})$ and

$$V^*TV = \begin{bmatrix} A & k^{n-1}B & 0 & \cdots & 0 \\ k^{n-1}B & A & k^{n-1}B & \ddots & \vdots \\ 0 & k^{n-1}B & A & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & k^{n-1}B \\ 0 & & \cdots & 0 & k^{n-1}B & A \end{bmatrix}.$$

Hence, from the invariance property of weakly unitarily invariant norms and Theorem 1, we have the desired result.

THEOREM 3. *Let* $A, B \in B(H)$ *and* $T = \begin{bmatrix} A & \omega^{n-1}B & 0 & \cdots & 0 \\ \omega B & A & \omega^{n-1}B & \ddots & \vdots \\ 0 & \omega B & A & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \omega^{n-1}B \\ 0 & \cdots & 0 & \omega B & A \end{bmatrix}$ *be a*

tridiagonal operator matrix in $B(H^{(n)})$, *and let* $\omega = e^{\frac{2\pi i}{n}}$ *(the* n^{th} *root of unity). Then*
$$\tau(T) = \tau\left(\bigoplus_{j=1}^{n}\left[A + (2\cos\frac{j\pi}{n+1})B\right]\right).$$

Proof.

Let $W = \begin{bmatrix} I & 0 & \cdots & 0 & 0 \\ 0 & \omega I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \omega^{n-2}I & 0 \\ 0 & 0 & \cdots & 0 & \omega^{n-1}I \end{bmatrix}$. Then it is easy to prove that W is a unitary

operator in $B(H^{(n)})$ and

$$WTW^* = \begin{bmatrix} A & B & 0 & \cdots & 0 \\ B & A & B & \ddots & \vdots \\ 0 & B & A & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & B \\ 0 & \cdots & 0 & B & A \end{bmatrix}.$$

Hence, from the invariance property of weakly unitarily invariant norms and Theorem 1, we have the desired result.

Next, we give norm equalities for $n \times n$ special anti-tridiagonal operator matrices.

THEOREM 4. *Let* $A, B \in B(H)$ *and* $T = \begin{bmatrix} 0 & \cdots & 0 & B & A \\ \vdots & \iddots & B & A & B \\ 0 & \iddots & A & B & 0 \\ B & \iddots & \iddots & \iddots & \vdots \\ A & B & 0 & \cdots & 0 \end{bmatrix}$ *be an anti-tridiagonal*

operator matrix in $B(H^{(n)})$. *Then*

$$\tau(T) = \tau\left(\bigoplus_{j=1}^{n}(-1)^{j+1}\left[A + (2\cos\frac{j\pi}{n+1})B\right]\right).$$

Proof.
By using the same U in Theorem 1, we have

$$UTU^* = \bigoplus_{j=1}^{n} (-1)^{j+1} \left[A + (2\cos\frac{j\pi}{n+1})B \right].$$

Hence, from the invariance property of weakly unitarily invariant norms, we get the desired result.

REMARK 1. We note that for the numerical radius, the usual operator norm and the Schatten p-norms, we conclude results for anti-tridiagonal operator matrices similar to those given for the tridiagonal operator matrices.

3. Pinching inequalities

The results in this section are pinching inequalities for more general $n \times n$ tridiagonal and $n \times n$ anti-tridiagonal operator matrices.

THEOREM 5. Let $A, B, C \in B(H)$ and $S = \begin{bmatrix} A & B & 0 & \cdots & 0 \\ C & A & B & \ddots & \vdots \\ 0 & C & A & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & B \\ 0 & \cdots & 0 & C & A \end{bmatrix}$ be a tridiagonal

operator matrix in $B(H^{(n)})$. Then

$$\tau(S) \geqslant \tau \left(\bigoplus_{j=1}^{n} \left[A + (\cos\frac{j\pi}{n+1})(B+C) \right] \right).$$

Proof.

It is easy to prove that $L = \begin{bmatrix} 0 & 0 & \cdots & 0 & I \\ 0 & \cdots & 0 & I & 0 \\ \vdots & \cdot^{\cdot^{\cdot}} & I & 0 & 0 \\ 0 & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ I & 0 & 0 & \cdots & 0 \end{bmatrix}$ is a unitary operator in $B(H^{(n)})$ and

$(S + LSL^*) = \begin{bmatrix} 2A & (B+C) & 0 & \cdots & & 0 \\ (B+C) & 2A & (B+C) & \ddots & & \vdots \\ 0 & (B+C) & 2A & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & & (B+C) \\ 0 & \cdots & & 0 & (B+C) & 2A \end{bmatrix}$. Hence, from Theorem 1,

we have

$$\tau(S + LSL^*) = \tau \left(\bigoplus_{j=1}^{n} \left[2A + (2\cos\frac{j\pi}{n+1})(B+C) \right] \right).$$

Now, from the invariance property of weakly unitarily invariant norms and the triangle inequality, we have $\tau(S) \geqslant \tau\left(\bigoplus\limits_{j=1}^{n}\left[A+(\cos\dfrac{j\pi}{n+1})(B+C)\right]\right)$, as required.

Specializing Theorem 5 to the numerical radius, the usual operator norm and the Schatten p-norms, we have the following corollary.

COROLLARY 1. *Let* $A, B, C \in B(H)$ *and* $S = \begin{bmatrix} A & B & 0 & \cdots & 0 \\ C & A & B & \ddots & \vdots \\ 0 & C & A & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & B \\ 0 & \cdots & 0 & C & A \end{bmatrix}$ *be a tridiagonal operator matrix in* $B(H^{(n)})$. *Then:*

(i) $\omega(S) \geqslant \max\{ \omega\left(A+(\cos\frac{j\pi}{n+1})(B+C)\right) : j=1,2,\ldots,n\}$, *with equality when* $B = C$.

(ii) $\|S\| \geqslant \max\{ \left\|A+(\cos\frac{j\pi}{n+1})(B+C)\right\| : j=1,2,\ldots,n\}$, *with equality when* $B = C$.

(iii) $\|S\|_p \geqslant \left(\sum\limits_{j=1}^{n}\left\|A+(\cos\dfrac{j\pi}{n+1})(B+C)\right\|_p^p\right)^{\frac{1}{p}}$ *for* $1 < p < \infty$, *with equality if and only if* $B = C$.

Proof.

In view of Theorem 1, it is enough to prove the "only if " part of (iii).

Assume that $\| S \|_p^p = \sum\limits_{j=1}^{n}\left\|A+(\cos\dfrac{j\pi}{n+1})(B+C)\right\|_p^p$, $1 < p < \infty$, and consider U as in Theorem 1. Then $\| S \|_p^p = \| USU^* \|_p^p$, i.e.,

$$\left\|\begin{bmatrix} A & B & 0 & \cdots & 0 \\ C & A & B & \ddots & \vdots \\ 0 & C & A & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & B \\ 0 & \cdots & 0 & C & A \end{bmatrix}\right\|_p^p =$$

$$\left\|\begin{bmatrix} A+(\cos\frac{\pi}{n+1})(B+C) & \frac{1}{2}(B-C) & 0 & \cdots & 0 \\ \frac{1}{2}(C-B) & A+(\cos\frac{2\pi}{n+1})(B+C) & \frac{1}{2}(B-C) & \ddots & \vdots \\ 0 & \frac{1}{2}(C-B) & & \ddots & 0 \\ \vdots & \ddots & & \ddots & \frac{1}{2}(B-C) \\ 0 & \cdots & 0 & \frac{1}{2}(C-B) & A+(\cos\frac{n\pi}{n+1})(B+C) \end{bmatrix}\right\|_p^p$$

$$= \sum\limits_{j=1}^{n}\left\|A+(\cos\dfrac{j\pi}{n+1})(B+C)\right\|_p^p.$$

Now employing (4), we conclude that USU^* must be block diagonal, i.e., $B-C=0$, and hence $B=C$, as required.

In view of part (iii) of Corollary 1, it should be mentioned here that the converses of parts (i) and (ii) are not true in general. To see this, consider $S_1 = \begin{bmatrix} 0&0&0&2 \\ 0&0&2&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$

and $S_2 = \begin{bmatrix} 0 & 0\,0\,1 \\ 0 & 0\,1\,0 \\ 0 & 1\,0\,0 \\ -1\,0\,0\,0 \end{bmatrix}$ for parts (i) and (ii), respectively. Also, the converse of part (iii) is not true for the trace norm (which corresponds to $p=1$) and the usual operator norm (which corresponds to $p=\infty$). To see this, consider $S_3 = \begin{bmatrix} 0\,0\,0\,1 \\ 0\,0\,1\,0 \\ 0\,1\,0\,0 \\ 0\,0\,0\,0 \end{bmatrix}$ and $S_4 = $

$\begin{bmatrix} 0 & 0\,0\,1 \\ 0 & 0\,1\,0 \\ 0 & 1\,0\,0 \\ -1\,0\,0\,0 \end{bmatrix}$ for the trace and usual operator norms, respectively.

THEOREM 6. *Let* $A,B,C \in B(H)$ *and* $S = \begin{bmatrix} 0 & \cdots & 0 & B & A \\ \vdots & \cdot^{\cdot^{\cdot}} & B & A & C \\ 0 & \cdot^{\cdot^{\cdot}} & A & C & 0 \\ B & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ A & C & 0 & \cdots & 0 \end{bmatrix}$ *be an anti-tridiagonal operator matrix in* $B(H^{(n)})$. *Then*

$$\tau(S) \geqslant \tau\left(\overset{n}{\underset{j=1}{\oplus}} (-1)^{j+1} \left[A + (\cos\frac{j\pi}{n+1})(B+C) \right] \right).$$

Proof. By using the same L in Theorem 5, it is easy to prove that

$$S+LSL^* = \begin{bmatrix} 0 & \cdots & 0 & (B+C) & 2A \\ \vdots & \cdot^{\cdot^{\cdot}} & (B+C) & 2A & (B+C) \\ 0 & \cdot^{\cdot^{\cdot}} & 2A & (B+C) & 0 \\ (B+C) & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ 2A & (B+C) & 0 & \cdots & 0 \end{bmatrix}.$$

Hence, from Theorem 4, we have

$$\tau(S+LSL^*) = \tau\left(\overset{n}{\underset{j=1}{\oplus}} (-1)^{j+1} \left[2A + \left(2\cos\frac{j\pi}{n+1}\right)(B+C) \right] \right).$$

Now, from the invariance property of weakly unitarily invariant norms, and the triangle inequality, we have

$$\tau(S) \geqslant \tau \left(\overset{n}{\underset{j=1}{\oplus}} (-1)^{j+1} \left[A + (\cos \frac{j\pi}{n+1})(B+C) \right] \right), \text{ as required.}$$

REMARK 2. We note that for the numerical radius, the usual operator norm, and the Schatten p-norms, one can conclude results for anti-tridiagonal operator matrices similar to those given in Corollary 1 for tridiagonal operator matrices.

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