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ON STEVIĆ–SHARMA OPERATORS FROM WEIGHTED BERGMAN SPACES TO WEIGHTED–TYPE SPACES

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Abstract. Let $\mathscr{H}(\mathbb{D})$ be the space of analytic functions on the unit disc $\mathbb{D}$. Let φ be an analytic self-map of $\mathbb{D}$ and $\psi_1, \psi_2 \in \mathscr{H}(\mathbb{D})$. Let C_φ, M_ψ and $\mathscr{D}$ denote the composition, multiplication and differentiation operators, respectively. In order to treat the products of these operators in a unified manner, Stević et al. introduced the following operator

$$T_{\psi_1,\psi_2,\varphi} f = \psi_1 \cdot f \circ \varphi + \psi_2 \cdot f' \circ \varphi, \qquad f \in \mathscr{H}(\mathbb{D}).$$

We characterize the boundedness and compactness of the operators $T_{\psi_1,\psi_2,\varphi}$ from weighted Bergman spaces to weighted-type and little weighted-type spaces of analytic functions. Also, we give examples of bounded, unbounded, compact and non compact operators $T_{\psi_1,\psi_2,\varphi}$.

1. Introduction

Let $\mathscr{H}(\mathbb{D})$ be the space of analytic functions on the unit disc $\mathbb{D}$ in the complex plane $\mathbb{C}$. Let φ be an analytic self-map of $\mathbb{D}$ and $\psi \in \mathscr{H}(\mathbb{D})$. The weighted composition operator $W_{\psi,\varphi} : \mathscr{H}(\mathbb{D}) \to \mathscr{H}(\mathbb{D})$ is defined as

$$W_{\psi,\varphi} f = \psi \cdot f \circ \varphi,$$

for $f \in \mathscr{H}(\mathbb{D})$. If $\psi = 1$, then $W_{\psi,\varphi}$ reduces to the composition operator and it is denoted by C_φ. Also, if φ is the identity map, then $W_{\psi,\varphi}$ reduces to the multiplication operator and it is denoted by M_ψ. Thus the class of weighted composition operators $W_{\psi,\varphi} = M_\psi C_\varphi$ is the product of multiplication operators and composition operators. Weighted composition operators have been appearing in a natural way on different spaces of functions. For example: the isometries of Hardy spaces, Bergman spaces and many other spaces of analytic functions are weighted composition operators. For details on this, we refer to the monographs of Fleming and Jamison [8, 9]. Also, we refer to the monographs of Cowen and MacCluer [6], Shapiro [36] and Singh and Manhas [38] for more information on composition operators and weighted composition operators.

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Let $\mathscr{D} = \mathscr{D}^1$ be the differentiation operator on $\mathscr{H}(\mathbb{D})$ defined by $\mathscr{D}f = f'$. If $n \in \mathbb{N}_0$, then the operator $\mathscr{D}^n$ is defined by $\mathscr{D}^n f = f^{(n)}$, $f \in \mathscr{H}(\mathbb{D})$. Since the differentiation operator $\mathscr{D}$ is typically unbounded on many analytic function spaces, recently many mathematicians have started exploring different properties of the following products of multiplication, composition and differentiation operators on different spaces of analytic functions.

$$\begin{aligned} \mathscr{D}M_\psi C_\varphi f &= \psi' \cdot f\circ\varphi + \psi\varphi' \cdot f'\circ\varphi; & M_\psi C_\varphi \mathscr{D} f &= \psi \cdot f'\circ\varphi; \\ C_\varphi \mathscr{D} M_\psi f &= \psi'\circ\varphi \cdot f\circ\varphi + \psi\circ\varphi \cdot f'\circ\varphi; & M_\psi \mathscr{D} C_\varphi f &= \psi\varphi' \cdot f'\circ\varphi; \\ \mathscr{D} C_\varphi M_\psi f &= \psi'\circ\varphi\cdot\varphi' \cdot f\circ\varphi + \psi\circ\varphi\cdot\varphi' \cdot f'\circ\varphi; & C_\varphi M_\psi \mathscr{D} f &= \psi\circ\varphi \cdot f'\circ\varphi. \end{aligned} \tag{1}$$

If $\psi(z) = 1$, for all $z \in \mathbb{D}$, then we get the product of composition operators and differentiation operators $\mathscr{D}C_\varphi M_\psi = \mathscr{D}C_\varphi$ and $C_\varphi \mathscr{D} M_\psi = C_\varphi \mathscr{D}$. The boundedness and compactness of the products $\mathscr{D}C_\varphi$ and $C_\varphi\mathscr{D}$ of composition operators and differentiation operators between Bergman spaces and Hardy spaces were first studied by Hibschweiler and Portnoy in [12] and then on Hardy spaces by Ohno [35]. Furthermore, Li and Stević in [17, 18, 19, 20, 21] studied the boundedness and compactness of the operator $\mathscr{D}C_\varphi$ between Bloch-type spaces, weighted Bergman spaces A^p_α and Bloch-type spaces $\mathscr{B}^\beta$, the space of bounded analytic functions H^∞ and α-Bloch spaces, mixed-norm spaces and α-Bloch spaces as well as Zygmund spaces and Bloch-type (Bers spaces). Also, Stević [39, 41] studied these product operators between Bergman spaces as well as from H^∞ and Bloch spaces to nth weighted-type spaces. The property of boundedness from below of the operator $\mathscr{D}C_\varphi$ on Bloch-type spaces has been studied by Liu and Li in [23]. If φ is the identity map, then we get the product of multiplication and differentiation operator $\mathscr{D}M_\psi C_\varphi = \mathscr{D}M_\psi$ and $M_\psi \mathscr{D} C_\varphi = M_\psi \mathscr{D}$. Yu and Liu in [49] studied the products $\mathscr{D}M_\psi$ and $M_\psi\mathscr{D}$ from mixed norm spaces to the Bloch-type spaces and Zhu [52] investigated these products from Bergman-type spaces to Bers-type spaces. Also, Liu and Li in [22] studied the operator $\mathscr{D}M_\psi$ from H^∞ to Zygmund spaces. Further, the product of weighted composition operators and differentiation operators $\mathscr{D}W_{\psi,\varphi} = \mathscr{D}M_\psi C_\varphi$ and $W_{\psi,\varphi}\mathscr{D} = M_\psi C_\varphi \mathscr{D}$ were studied by Sharma [37] between weighted Bergman-Nevanlinna and Bloch-type spaces. Li, Wang and Zhang [16] investigated $\mathscr{D}W_{\psi,\varphi}$ between weighted Bergman space and H^∞. Also, Jiang in [13] explored $\mathscr{D}W_{\psi,\varphi}$ and $W_{\psi,\varphi}\mathscr{D}$ from weighted Bergman spaces to Zygmund-type and Bloch-type spaces. In [40, 42], Stević studied weighted differentiation composition operators $W_{\psi,\varphi}\mathscr{D}^n$ from mixed-norm spaces to weighted-type spaces whereas Zhu [53] studied generalized weighted composition operators $W_{\psi,\varphi}\mathscr{D}^n$ from Bloch spaces into Bers-type spaces. Manhas and Zhao [30, 31, 32] studied the operators $\mathscr{D}W_{\psi,\varphi}$ and $W_{\psi,\varphi}\mathscr{D}$ between weighted Banach spaces of analytic functions and weighted Zygmund (Bloch) type spaces.

Let $\psi_1, \psi_2 \in \mathscr{H}(\mathbb{D})$ and $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic map. Then in [44], Stević et al. introduced the following operator which unifies all the products of multiplication, composition and differentiation operators given in (1).

$$T_{\psi_1,\psi_2,\varphi} f = \psi_1 \cdot f\circ\varphi + \psi_2 \cdot f'\circ\varphi, \qquad f \in \mathscr{H}(\mathbb{D}). \tag{2}$$

By choosing appropriate ψ_1, ψ_2 and φ in (2), we get all product type operators given in (1). More specifically, we have $T_{\psi',\psi\varphi',\varphi} = \mathscr{D}W_{\psi,\varphi} = \mathscr{D}M_\psi C_\varphi$, $T_{0,\psi,\varphi} = W_{\psi,\varphi}\mathscr{D} = M_\psi C_\varphi \mathscr{D}$, $T_{0,\psi\varphi',\varphi} = M_\psi \mathscr{D} C_\varphi$, $T_{0,\psi\circ\varphi,\varphi} = C_\varphi M_\psi \mathscr{D}$, $T_{(\psi'\circ\varphi)\varphi',(\psi\circ\varphi)\varphi',\varphi} = \mathscr{D}C_\varphi M_\psi$ and $T_{\psi'\circ\varphi,\psi\circ\varphi,\varphi} = C_\varphi \mathscr{D} M_\psi$.

In [43, 44, 45], Stević and co-workers studied these operators $T_{\psi_1,\psi_2,\varphi}$ on weighted Bergman spaces and between Hardy and α-Bloch spaces. Also, Jiang [14] investigated these operators from Zygmund spaces to the Bloch-Orlicz spaces, whereas Zhang and Liu [50] studied these operators from mixed-norm spaces to weighted-type spaces. In [2], Bai studied the Stević-Sharma operators from area Nevanlinna spaces to Bloch-Orlicz type spaces and in [26], Liu and Yu studied these operators from Besov spaces into weighted-type spaces H_μ^∞. In [10], Guo and Shu investigated the boundedness and compactness of Stević-Sharma operators from Hardy spaces to Stević weighted spaces. Also, in [24], Liu et al. studied an extension of Stević-Sharma operator from mixed-norm spaces to weighted-type spaces on the unit ball and then further Liu and Yu in [25] studied these operators from the general space $F(p,q,s)$ to weighted-type spaces on the unit ball.

In this paper our aim is to characterize the boundedness and compactness of $T_{\psi_1,\psi_2,\varphi}$ from weighted Bergman spaces to weighted-type and little weighted-type spaces of analytic functions generalizing the results of Li, Wang and Zhang [16], Wolf [48] and Jiang [13]. Also, we give examples of bounded, unbounded, compact and non compact operators $T_{\psi_1,\psi_2,\varphi}$ illustrating the role of inducing maps and weights.

2. Preliminaries

Let v be a strictly positive, continuous and bounded function on $\mathbb{D}$. We will call such a function v as a weight function or simply a weight. We define the weighted-type and little weighted-type spaces of analytic functions as follows:

$$H_v^\infty = \{f \in \mathscr{H}(\mathbb{D}) : ||f||_v := \sup_{z\in\mathbb{D}} v(z)\,|f(z)| < \infty\}.$$

and

$$H_{v,0}^\infty = \{f \in \mathscr{H}(\mathbb{D}) : \lim_{|z|\to 1} v(z)|f(z)| = 0\}.$$

Clearly H_v^∞ is a Banach space under the norm $||f||_v := \sup_{z\in\mathbb{D}} v(z)\,|f(z)|$ and it is a natural space in the sense that the norm convergence in H_v^∞ implies uniform convergence on compact subsets of $\mathbb{D}$. Also, $H_{v,0}^\infty$ is a closed subspace of H_v^∞. In case $v(z) = 1$, then $H_v^\infty = H^\infty$. The weighted Bergman space of analytic functions is defined as follows:

$$A_{v,p} = \left\{ f \in \mathscr{H}(\mathbb{D}) : \|f\|_{v,p} := \left(\int_{\mathbb{D}} |f(z)|^p v(z) dA(z) \right)^{\frac{1}{p}} < \infty \right\}, 1 \leqslant p < \infty$$

where $dA(z)$ denotes the normalized area measure. If $v(z) = 1$, then $A_{v,p} = A_p$, the classical Bergman space. If $v(z) = (1-|z|^2)^\alpha$, $\alpha > -1$, then $A_{v,p} = A_{\alpha,p}$ is the standard weighted Bergman space. For more information on Bergman spaces, we refer to Hedenmalm, Korenblum and Zhu [11], Duren and Schuster [7] and Zhu [51].

The associated weight $\widetilde{v}$ for a given weight v is defined as follows:

$$\widetilde{v}(z) = \left(\sup_{z\in\mathbb{D}} |f(z)| : f \in H_v^\infty,\ \|f\|_v \leqslant 1 \right)^{-1} = \frac{1}{\|\delta_z\|_v},$$

where $\delta_z : H_v^\infty \to \mathbb{C}$ is the point evaluation linear functional. In the setting of general weighted spaces of analytic functions, the associated weight plays an important role. It has been seen in [4] that the following relations between v and $\widetilde{v}$ hold:

$$0 < v \leqslant \widetilde{v}, \text{ and } \widetilde{v} \text{ is bounded and continuous;} \tag{3}$$

$$\|f\|_v \leqslant 1 \text{ if and only if } \|f\|_{\widetilde{v}} \leqslant 1; \tag{4}$$

for each $z \in \mathbb{D}$ there exists f_z in the closed unit ball B_v^∞ of H_v^∞ such that

$$|f_z(z)| = \frac{1}{\widetilde{v}(z)}. \tag{5}$$

A weight v is said to be radial if $v(z) = v(|z|)$ for every $z \in \mathbb{D}$. Also, a weight v is called essential if there is a constant $k > 0$ such that

$$v(z) \leqslant \widetilde{v}(z) \leqslant kv(z) \tag{6}$$

for every $z \in \mathbb{D}$.

In [28], Lusky introduced the following condition $(L1)$ which plays an important role in this paper:

$$\inf_{n\in\mathbb{N}} \frac{v(1-2^{-n-1})}{v(1-2^{-n})} > 0. \tag{L1}$$

Radial weights which satisfy condition $(L1)$ are always essential (see [5]).

The standard weights $v_\alpha(z) = (1-|z|^2)^\alpha$, where $\alpha > 0$, and the logarithmic weights $v_\beta(z) = (1-\log(1-|z|^2))^\beta, \beta < 0$ satisfy condition $(L1)$. For more details on the weighted-type and little weighted-type spaces of analytic functions which have important applications in functional analysis, complex analysis, partial differential equations, convolution equations and distribution theory, we refer to [3, 4, 27, 28].

Next, in our paper we also consider the following weights. We define the weight v as

$$\upsilon(z) := \nu(|z|^2) \text{ for every } z \in \mathbb{D}, \tag{7}$$

where ν is an analytic function on $\mathbb{D}$, non-vanishing, strictly positive on $[0,1)$ and satisfying $\lim_{r\to 1} \nu(r) = 0$.

The following are some of the examples (see [46]) illustrating these type of weights:

(w_1) If we consider $\nu_\alpha(z) = (1-z)^\alpha$, where $\alpha \geqslant 1$, then we have the standard weights $\upsilon_\alpha(z) = (1-|z|^2)^\alpha$.

(w_2) If we consider $\nu_\alpha(z) = \exp^{-\frac{1}{(1-z)^\alpha}}$, where $\alpha \geqslant 1$, then we have the exponential weights $\upsilon_\alpha(z) = \exp^{-\frac{1}{(1-|z|^2)^\alpha}}$.

(w_3) If we define $\nu(z) = \sin(1-z)$, then we have the weight $\upsilon(z) = \sin(1-|z|^2)$.

These examples also satisfy condition (L1) (see [28]).

For $a \in \mathbb{D}$, we define the functions $\upsilon_a(z) := \nu(\overline{a}z)$, $\varphi_a(z) = \frac{a-z}{1-\overline{a}z}$ and $\rho(z,a) = |\varphi_a(z)|$, for every $z \in \mathbb{D}$. The function υ_a is holomorphic since ν is holomorphic. Also, φ_a is called Mobius transformation that interchanges a and 0 and ρ is known as the pseudohyperbolic metric on $\mathbb{D}$. Note that $\varphi_a(\varphi_a(z)) = z$ and

$$\varphi_a'(z) = -\frac{1-|a|^2}{(1-\overline{a}z)^2}, z \in \mathbb{D}.$$

The notation $A \preceq B$ means that there is a positive constant C such that $A \leqslant CB$. In this paper we use the notation $A \asymp B$, which means that both $A \preceq B$ and $B \preceq A$.

3. Boundedness and compactness of the operators $T_{\psi_1,\psi_2,\varphi} : A_{v,p} \to H_w^\infty$

In order to obtain our main results of boundedness and compactness of the operators $T_{\psi_1,\psi_2,\varphi}$, we need to state and prove the following lemmas.

We begin with stating the first lemma which is proved in ([46], Lemma 3).

LEMMA 1. *Let v be a radial weight as defined in (7) such that*

$$\sup_{a\in\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{v(z)|v_a(\varphi_a(z))|}{v(\varphi_a(z))} \leqslant C < \infty,$$

and v satisfies condition (L1). Then there exists $C_v > 0$ such that for every $f \in A_{v,p}$

$$|f(z) - f(a)| \leqslant C_v \|f\|_{v,p} \max\left\{ \frac{1}{(1-|z|^2)^{\frac{2}{p}} v(z)^{\frac{1}{p}}}, \frac{1}{(1-|a|^2)^{\frac{2}{p}} v(a)^{\frac{1}{p}}} \right\} \rho(z,a)$$

for every $z, a \in \mathbb{D}$.

LEMMA 2. *Let v be a weight as defined in Lemma 1. Then there exists $C_v > 0$ such that for every $f \in A_{v,p}$*

$$|f'(z)| \leqslant \frac{C_v \|f\|_{v,p}}{(1-|z|^2)^{1+\frac{2}{p}} v(z)^{\frac{1}{p}}}$$

for every $z \in \mathbb{D}$.

Proof. By Lemma 1, the proof is completely analogous to the proof of Lemma 3 given in [48] and hence we omit the proof here.

Next we state the following lemma for the space $A_{v,p}$ whose proof can be obtained by using Lemma 2 and the techniques of Lemma 3 of [47] which is proved for H_v^∞.

LEMMA 3. *Let v be a weight as defined in Lemma 1. Then there exists $r \in (0,1)$ and a constant $M > 0$ such that for $f \in A_{v,p}$*

$$|f'(z) - f'(a)| \leqslant M \frac{\|f\|_{v,p}\rho(z,a)}{rv(a)^{\frac{1}{p}}(1-|a|^2)^{1+\frac{2}{p}}}$$

for every $z, a \in \mathbb{D}$ with $\rho(z,a) \leqslant \frac{r}{2}$.

LEMMA 4. *Let v be a weight as defined in Lemma 1. Then there is a constant K such that for every $f \in A_{v,p}$,*

$$|f'(z) - f'(a)| \leqslant K\|f\|_{v,p} \max\left\{\frac{1}{(1-|z|^2)^{1+\frac{2}{p}}v(z)^{\frac{1}{p}}}, \frac{1}{(1-|a|^2)^{1+\frac{2}{p}}v(a)^{\frac{1}{p}}}\right\}\rho(z,a)$$

for every $z, a \in \mathbb{D}$.

Proof. By Lemma 3, we can find $r \in (0,1)$ and a constant $M > 0$ such that

$$|f'(z) - f'(a)| \leqslant M \frac{\|f\|_{v,p}\rho(z,a)}{rv(a)^{\frac{1}{p}}(1-|a|^2)^{1+\frac{2}{p}}}$$

for every $z, a \in \mathbb{D}$ with $\rho(z,a) \leqslant \frac{r}{2}$. Now in case that $\rho(z,a) > \frac{r}{2}$ then by Lemma 2,

$$\begin{aligned}
|f'(z) - f'(a)| &\leqslant |f'(z)| + |f'(a)| \\
&\leqslant \frac{C_v\|f\|_{v,p}}{(1-|z|^2)^{1+\frac{2}{p}}v(z)^{\frac{1}{p}}} + \frac{C_v\|f\|_{v,p}}{(1-|a|^2)^{1+\frac{2}{p}}v(a)^{\frac{1}{p}}} \\
&\leqslant 2C_v\|f\|_{v,p} \max\left\{\frac{1}{(1-|z|^2)^{1+\frac{2}{p}}v(z)^{\frac{1}{p}}}, \frac{1}{(1-|a|^2)^{1+\frac{2}{p}}v(a)^{\frac{1}{p}}}\right\} \\
&\leqslant \frac{4}{r}C_v\|f\|_{v,p} \max\left\{\frac{1}{(1-|z|^2)^{1+\frac{2}{p}}v(z)^{\frac{1}{p}}}, \frac{1}{(1-|a|^2)^{1+\frac{2}{p}}v(a)^{\frac{1}{p}}}\right\}\rho(z,a).
\end{aligned}$$

Now if we put

$$K = \max\left\{\frac{4C_v}{r}, M\right\},$$

then we get

$$|f'(z) - f'(a)| \leqslant K\|f\|_{v,p} \max\left\{\frac{1}{(1-|z|^2)^{1+\frac{2}{p}}v(z)^{\frac{1}{p}}}, \frac{1}{(1-|a|^2)^{1+\frac{2}{p}}v(a)^{\frac{1}{p}}}\right\}\rho(z,a).$$

The corresponding estimate to Lemma 2 can be obtained for the second derivative by using Lemma 4. Inductively we can have the following two lemmas.

LEMMA 5. *Let v be a weight as defined in Lemma 1. Then there is $C_v > 0$ such that for every $f \in A_{v,p}$*

$$|f^{(n)}(z)| \leqslant \frac{C_v \|f\|_{v,p}}{(1-|z|^2)^{n+\frac{2}{p}} v(z)^{\frac{1}{p}}}$$

for every $z \in \mathbb{D}$ and every $n \in \mathbb{N}_0$.

LEMMA 6. *Let v be a weight as defined in Lemma 1. Then there is $C_v > 0$ such that for every $f \in A_{v,p}$*

$$|f^{(n)}(z) - f^{(n)}(w)| \leqslant C_v \|f\|_{v,p} \max \left\{ \frac{1}{(1-|z|^2)^{n+\frac{2}{p}} v(z)^{\frac{1}{p}}}, \frac{1}{(1-|w|^2)^{n+\frac{2}{p}} v(w)^{\frac{1}{p}}} \right\} \rho(z,w)$$

for every $z, w \in \mathbb{D}$ and every $n \in \mathbb{N}_0$.

In order to prove the compactness of the operator $T_{\psi_1,\psi_2,\varphi}$, we need the following result and the proof can be deduced from Proposition 3.11 in [6].

LEMMA 7. *Let $\psi_1, \psi_2 \in \mathscr{H}(\mathbb{D})$ and $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic map. Then the operator $T_{\psi_1,\psi_2,\varphi} : A_{v,p} \longrightarrow H_w^\infty$ is compact if and only if $T_{\psi_1,\psi_2,\varphi} : A_{v,p} \longrightarrow H_w^\infty$ is bounded and for any bounded sequence $\{f_n\}$ in $A_{v,p}$ such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$, $||T_{\psi_1,\psi_2,\varphi} f_n||_w \to 0$ as $n \to \infty$.*

In the following theorem, we characterize the self map $\varphi : \mathbb{D} \to \mathbb{D}$ and $\psi_1, \psi_2 \in \mathscr{H}(\mathbb{D})$ which induce bounded operator $T_{\psi_1,\psi_2,\varphi}$

THEOREM 1. *Let v be a weight as defined in Lemma 1 and let w be an arbitrary weight. Let $\psi_1, \psi_2 \in \mathscr{H}(\mathbb{D})$ and $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic map. Then the operator $T_{\psi_1,\psi_2,\varphi} : A_{v,p} \longrightarrow H_w^\infty$ is bounded if and only if*

(i) $M_1 = \sup_{z \in \mathbb{D}} \frac{w(z)|\psi_1(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}} v(\varphi(z))^{\frac{1}{p}}} < \infty;$

(ii) $M_2 = \sup_{z \in \mathbb{D}} \frac{w(z)|\psi_2(z)|}{(1-|\varphi(z)|^2)^{1+\frac{2}{p}} v(\varphi(z))^{\frac{1}{p}}} < \infty.$

Moreover, if the operator $T_{\psi_1,\psi_2,\varphi} : A_{v,p} \longrightarrow H_w^\infty$ is bounded, then

$$||T_{\psi_1,\psi_2,\varphi}||_{A_{v,p} \longrightarrow H_w^\infty} \asymp M_1 + M_2. \tag{8}$$

Proof. First assume that the conditions (i) and (ii) hold. Let $f \in A_{v,p}$. Then by using Lemma 5, we have

$$\begin{aligned} \|T_{\psi_1,\psi_2,\varphi} f\|_w &= \sup_{z \in \mathbb{D}} w(z)|\psi_1(z) f(\varphi(z)) + \psi_2(z) f'(\varphi(z))| \\ &\leqslant \sup_{z \in \mathbb{D}} w(z)|\psi_1(z)||f(\varphi(z))| + \sup_{z \in \mathbb{D}} w(z)|\psi_2(z)||f'(\varphi(z))| \end{aligned}$$

$$\leqslant C_v\|f\|_{v,p}\left(\sup_{z\in\mathbb{D}}\frac{w(z)|\psi_1(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}}+\sup_{z\in\mathbb{D}}\frac{w(z)|\psi_2(z)|}{(1-|\varphi(z)|^2)^{1+\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}}\right)$$
$$\leqslant C_v\|f\|_{v,p}(M_1+M_2).$$

Thus

$$\|T_{\psi_1,\psi_2,\varphi}\|_{A_{v,p}\to H_w^\infty}\leqslant C_v(M_1+M_2). \tag{9}$$

This proves that the operator $T_{\psi_1,\psi_2,\varphi}: A_{v,p}\longrightarrow H_w^\infty$ is bounded.
Conversely, assume that the operator $T_{\psi_1,\psi_2,\varphi}: A_{v,p}\longrightarrow H_w^\infty$ is bounded. Fix $a\in\mathbb{D}$. By (5), there exist $f_{\varphi(a)}^p\in B_v^\infty$ such that

$$|f_{\varphi(a)}(\varphi(a))|^p=\frac{1}{\widetilde{v}(\varphi(a))}.$$

Since v satisfies the condition (L1), it is essential and hence we can replace $\widetilde{v}$ by v. Now define

$$g_{\varphi(a)}(z)=\varphi_{\varphi(a)}(z)f_{\varphi(a)}(z)\varphi'_{\varphi(a)}(z)^{\frac{2}{p}}$$

for all $z\in\mathbb{D}$. Then

$$\begin{aligned}||g_{\varphi(a)}||_{v,p}^p&=\int_{\mathbb{D}}|g_{\varphi(a)}(z)|^p v(z)dA(z)\\&=\int_{\mathbb{D}}v(z)|\varphi_{\varphi(a)}(z)|^p|f_{\varphi(a)}(z)|^p|\varphi'_{\varphi(a)}(z)|^2dA(z)\\&\leqslant\sup_{z\in\mathbb{D}}v(z)|f_{\varphi(a)}(z)|^p\int_{\mathbb{D}}|\varphi_{\varphi(a)}(z)|^p|\varphi'_{\varphi(a)}(z)|^2dA(z)\leqslant 1.\end{aligned}$$

Therefore $g_{\varphi(a)}\in A_{v,p}$, $g_{\varphi(a)}(\varphi(a))=0$ and

$$|g'_{\varphi(a)}(\varphi(a))|=\frac{1}{(1-|\varphi(a)|^2)^{1+\frac{2}{p}}v(\varphi(a))^{\frac{1}{p}}}.$$

Thus

$$\begin{aligned}||T_{\psi_1,\psi_2,\varphi}||_{A_{v,p}\to H_w^\infty}&\geqslant||T_{\psi_1,\psi_2,\varphi}g_{\varphi(a)}||_w\\&\geqslant w(a)|(T_{\psi_1,\psi_2,\varphi}g_{\varphi(a)})(a)|\\&\geqslant w(a)|\psi_1(a)g_{\varphi(a)}(\varphi(a))+\psi_2(a)g'_{\varphi(a)}(\varphi(a))|\\&=\frac{w(a)|\psi_2(a)|}{(1-|\varphi(a)|^2)^{1+\frac{2}{p}}v(\varphi(a))^{\frac{1}{p}}}.\end{aligned}$$

Thus

$$M_2=\sup_{a\in\mathbb{D}}\frac{w(a)|\psi_2(a)|}{(1-|\varphi(a)|^2)^{1+\frac{2}{p}}v(\varphi(a))^{\frac{1}{p}}}\leqslant||T_{\psi_1,\psi_2,\varphi}||_{A_{v,p}\to H_w^\infty}<\infty. \tag{10}$$

This proves the condition (ii). Now we establish the condition (i). Let $a \in \mathbb{D}$. Then again according to (5), there exist $f^{p}_{\varphi(a)}$ in B^{∞}_{v} such that

$$|f_{\varphi(a)}(\varphi(a))|^{p} = \frac{1}{\widetilde{v}(\varphi(a))}.$$

Define

$$h_{\varphi_{(a)}}(z) = f_{\varphi(a)}(z)\varphi'_{\varphi(a)}(z)^{\frac{2}{p}},\ z \in \mathbb{D}.$$

Clearly $h_{\varphi(a)} \in A_{v,p}$ and $||h_{\varphi(a)}||_{v,p} \leqslant 1$. Thus by using Lemma 5, we have

$$\begin{aligned}
||T_{\psi_1,\psi_2,\varphi}||_{A_{v,p}\to H^{\infty}_{w}} &\geqslant ||T_{\psi_1,\psi_2,\varphi}h_{\varphi(a)}||_{w} \\
&\geqslant w(a)|(T_{\psi_1,\psi_2,\varphi}h_{\varphi(a)})(a)| \\
&\geqslant w(a)|\psi_1(a)h_{\varphi(a)}(\varphi(a)) + \psi_2(a)h'_{\varphi(a)}(\varphi(a))| \\
&\geqslant w(a)|\psi_1(a)h_{\varphi(a)}(\varphi(a))| - w(a)|\psi_2(a)h'_{\varphi(a)}(\varphi(a))| \\
&= w(a)|\psi_1(a)||f_{\varphi(a)}(\varphi(a))||\varphi'_{\varphi(a)}(\varphi(a))^{\frac{2}{p}}| - w(a)|\psi_2(a)||h'_{\varphi(a)}(\varphi(a))| \\
&\geqslant \frac{w(a)|\psi_1(a)|}{(1-|\varphi(a)|^2)^{\frac{2}{p}}v(\varphi(a))^{\frac{1}{p}}} - \frac{w(a)|\psi_2(a)|C_v\|h_{\varphi(a)}\|_{v,p}}{(1-|\varphi(a)|^2)^{1+\frac{2}{p}}v(\varphi(a))^{\frac{1}{p}}}. \qquad (11)
\end{aligned}$$

Now from condition (10) and (11), we have

$$\begin{aligned}
M_1 &= \sup_{a\in\mathbb{D}} \frac{w(a)|\psi_1(a)|}{(1-|\varphi(a)|^2)^{\frac{2}{p}}v(\varphi(a))^{\frac{1}{p}}} \\
&\leqslant ||T_{\psi_1,\psi_2,\varphi}||_{A_{v,p}\to H^{\infty}_{w}} + C_v\|h_{\varphi(a)}\|_{v,p} \sup_{a\in\mathbb{D}} \frac{w(z)|\psi_2(a)|}{(1-|\varphi(a)|^2)^{1+\frac{2}{p}}v(\varphi(a))^{\frac{1}{p}}} \\
&\leqslant (1+C_v)||T_{\psi_1,\psi_2,\varphi}||_{A_{v,p}\to H^{\infty}_{w}} < \infty. \qquad (12)
\end{aligned}$$

This proves the condition (i). Also, the asymptotic relation (8) follows from (9), (10) and (12).

In the next theorem we characterize the compactness of the operator $T_{\psi_1,\psi_2,\varphi}$.

THEOREM 2. *Let v be a weight as defined in Lemma 1 and let w be an arbitrary weight. Let $\psi_1,\psi_2 \in \mathscr{H}(\mathbb{D})$ and $\varphi:\mathbb{D}\to\mathbb{D}$ be an analytic map. Then the operator $T_{\psi_1,\psi_2,\varphi}: A_{v,p} \longrightarrow H^{\infty}_{w}$ is compact if and only if*

(i) $K_1 = \sup_{z\in\mathbb{D}} w(z)\big|\psi_1(z)\big| < \infty,$

(ii) $K_2 = \sup_{z\in\mathbb{D}} w(z)|\psi_2(z)| < \infty,$

(iii) $\lim_{r\to 1} \sup_{|\varphi(z)|>r} \frac{w(z)|\psi_1(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}} = 0,$

(iv) $\lim_{r\to 1}\sup_{|\varphi(z)|>r}\frac{w(z)|\psi_2(z)|}{(1-|\varphi(z)|^2)^{1+\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}}=0.$

Proof. First, we assume that the conditions (i)-(iv) hold. Let $\{f_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $A_{v,p}$ such that it converges to zero uniformly on compact subsets of $\mathbb{D}$. To show that $T_{\psi_1,\psi_2,\varphi}$ is compact, in view of Lemma 7, it is enough to show that the operator $T_{\psi_1,\psi_2,\varphi}$ is bounded and $||T_{\psi_1,\psi_2,\varphi}f_n||_w\to 0$ as $n\to\infty$. We begin with showing that $T_{\psi_1,\psi_2,\varphi}$ is bounded. In view of (iii) and (iv), for $\varepsilon=1$, there is $r\in(0,1)$ such that whenever $r<|\varphi(z)|<1$, we have

$$\frac{w(z)|\psi_1(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}}<1 \tag{13}$$

and

$$\frac{w(z)|\psi_2(z)|}{(1-|\varphi(z)|^2)^{1+\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}}<1. \tag{14}$$

Since the weight v is a strictly positive and continuous function on $\mathbb{D}$, there exists a constant $\delta>0$ such that $v(\varphi(z))>\delta$ for all $z\in\mathbb{D}$ satisfying $|\varphi(z)|\leqslant r$. Hence for $|\varphi(z)|\leqslant r$, we use (i) and (ii) to get the following inequalities:

$$\frac{w(z)|\psi_1(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}}\leqslant\frac{K_1}{(1-r^2)^{\frac{2}{p}}\delta^{\frac{1}{p}}} \tag{15}$$

and

$$\frac{w(z)|\psi_2(z)|}{(1-|\varphi(z)|^2)^{1+\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}}\leqslant\frac{K_2}{(1-r^2)^{1+\frac{2}{p}}\delta^{\frac{1}{p}}}. \tag{16}$$

From (13), (14), (15) and (16), we have

$$\begin{aligned}\sup_{z\in\mathbb{D}}\frac{w(z)|\psi_1(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}}&=\max\Bigg\{\sup_{|\varphi(z)|\leqslant r}\frac{w(z)|\psi_1(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}},\\&\qquad\sup_{|\varphi(z)|>r}\frac{w(z)|\psi_1(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}}\Bigg\}\\&\leqslant\sup_{|\varphi(z)|\leqslant r}\frac{w(z)|\psi_1(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}}\\&\quad+\sup_{|\varphi(z)|>r}\frac{w(z)|\psi_1(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}}\\&<\frac{K_1}{(1-r^2)^{\frac{2}{p}}\delta^{\frac{1}{p}}}+1\end{aligned} \tag{17}$$

and

$$\sup_{z\in\mathbb{D}}\frac{w(z)|\psi_2(z)|}{(1-|\varphi(z)|^2)^{1+\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}}=\max\Bigg\{\sup_{|\varphi(z)|\leqslant r}\frac{w(z)|\psi_2(z)|}{(1-|\varphi(z)|^2)^{1+\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}},$$

$$\left.\sup_{|\varphi(z)|>r} \frac{w(z)|\psi_2(z)|}{(1-|\varphi(z)|^2)^{1+\frac{2}{p}} v(\varphi(z))^{\frac{1}{p}}}\right\}$$

$$\leqslant \sup_{|\varphi(z)|\leqslant r} \frac{w(z)|\psi_2(z)|}{(1-|\varphi(z)|^2)^{1+\frac{2}{p}} v(\varphi(z))^{\frac{1}{p}}} + \sup_{|\varphi(z)|>r} \frac{w(z)|\psi_2(z)|}{(1-|\varphi(z)|^2)^{1+\frac{2}{p}} v(\varphi(z))^{\frac{1}{p}}}$$

$$< \frac{K_2}{(1-r^2)^{1+\frac{2}{p}} \delta^{\frac{1}{p}}} + 1. \tag{18}$$

Thus (17) and (18) implies that the conditions (i) and (ii) of Theorem 1 are satisfied. Hence $T_{\psi_1,\psi_2,\varphi} : A_{v,p} \longrightarrow H_w^\infty$ is bounded.

Next, we show that $||T_{\psi_1,\psi_2,\varphi} f_n||_w \to 0$ as $n \to \infty$. From conditions (iii) and (iv), it follows that for every $\varepsilon > 0$, there is $r \in (0,1)$ such that whenever $r < |\varphi(z)| < 1$, we have

$$\frac{w(z)|\psi_1(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}} v(\varphi(z))^{\frac{1}{p}}} < \varepsilon \tag{19}$$

and

$$\frac{w(z)|\psi_2(z)|}{(1-|\varphi(z)|^2)^{1+\frac{2}{p}} v(\varphi(z))^{\frac{1}{p}}} < \varepsilon. \tag{20}$$

Now since f_n converges to zero uniformly on compact subsets of $\mathbb{D}$, Cauchy's estimates implies that f_n' converges to zero uniformly on compact subsets of $\mathbb{D}$. Hence there is an $n_0 \in N$ such that, if $|\varphi(z)| \leqslant r$ and $n \geqslant n_0$, then $|f_n(\varphi(z))| < \varepsilon$ and $|f_n'(\varphi(z))| < \varepsilon$. By using the conditions (i) and (ii), we have

$$\begin{aligned}
&\sup_{|\varphi(z)|\leqslant r} w(z)|\psi_1(z)f_n(\varphi(z)) + \psi_2(z)f_n'(\varphi(z))| \\
\leqslant&\varepsilon\left(\sup_{|\varphi(z)|\leqslant r} w(z)|\psi_1(z)| + \sup_{|\varphi(z)|\leqslant r} w(z)|\psi_2(z)|\right) \\
\leqslant&\varepsilon\left(\sup_{z\in\mathbb{D}} w(z)|\psi_1(z)| + \sup_{z\in\mathbb{D}} w(z)|\psi_2(z)|\right) \\
\leqslant&\varepsilon(K_1+K_2).
\end{aligned} \tag{21}$$

Since $\{f_n\}_{n\in\mathbb{N}}$ is bounded in $A_{v,p}$, $\sup_n ||f_n||_{v,p} \leqslant M$. Thus from (19), (20) and (21), it follows that

$$\begin{aligned}
||T_{\psi_1,\psi_2,\varphi} f_n||_w &= \sup_{z\in\mathbb{D}} w(z)|\psi_1(z)f_n(\varphi(z)) + \psi_2(z)f_n'(\varphi(z))| \\
&= \max\left\{\sup_{r<|\varphi(z)|<1} w(z)|\psi_1(z)f_n(\varphi(z)) + \psi_2(z)f_n'(\varphi(z))|,\right. \\
&\qquad \left.\sup_{|\varphi(z)|\leqslant r} w(z)|\psi_1(z)f_n(\varphi(z)) + \psi_2(z)f_n'(\varphi(z))|\right\}
\end{aligned}$$

$$\leqslant \sup_{r<|\varphi(z)|<1} w(z)|\psi_1(z)f_n(\varphi(z))+\psi_2(z)f_n'(\varphi(z))|$$
$$+\sup_{|\varphi(z)|\leqslant r} w(z)|\psi_1(z)f_n(\varphi(z))+\psi_2(z)f_n'(\varphi(z))|$$
$$\leqslant \sup_{r<|\varphi(z)|<1} \frac{w(z)|\psi_1(z)|C_v||f_n||_{v,p}}{(1-|\varphi(z)|^2)^{\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}}$$
$$+\sup_{r<|\varphi(z)|<1} \frac{w(z)|\psi_2(z)|C_v||f_n||_{v,p}}{(1-|\varphi(z)|^2)^{1+\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}}+\varepsilon(K_1+K_2)$$
$$\leqslant (2MC_v+K_1+K_2)\varepsilon.$$

This proves that the operator $T_{\psi_1,\psi_2,\varphi}: A_{v,p} \longrightarrow H_w^\infty$ is compact.

Conversely, assume that the operator $T_{\psi_1,\psi_2,\varphi}: A_{v,p} \longrightarrow H_w^\infty$ is compact. Since $T_{\psi_1,\psi_2,\varphi}$ is bounded, by taking $f(z)=1$ and $g(z)=z$, we have

$$||T_{\psi_1,\psi_2,\varphi}f||_w = \sup_{z\in\mathbb{D}} w(z)|\psi_1(z)|,$$

and

$$||T_{\psi_1,\psi_2,\varphi}g||_w = \sup_{z\in\mathbb{D}} w(z)|\psi_1(z)\varphi(z)+\psi_2(z)|.$$

Thus it follows that

$$\sup_{z\in\mathbb{D}} w(z)\big|\psi_1(z))\big| = ||T_{\psi_1,\psi_2,\varphi}f||_w \leqslant ||T_{\psi_1,\psi_2,\varphi}||_{A_{v,p}\to H_w^\infty} < \infty, \tag{22}$$

and

$$\sup_{z\in\mathbb{D}} w(z)|\psi_1(z)\varphi(z)+\psi_2(z)| = ||T_{\psi_1,\psi_2,\varphi}g||_w \leqslant ||T_{\psi_1,\psi_2,\varphi}||_{A_{v,p}\to H_w^\infty} < \infty. \tag{23}$$

Also, we have

$$\sup_{z\in\mathbb{D}} w(z)|\psi_2(z)| \leqslant \sup_{z\in\mathbb{D}} w(z)|\psi_1(z)\varphi(z)+\psi_2(z)| + \sup_{z\in\mathbb{D}} w(z)|\psi_1(z)| \sup_{z\in\mathbb{D}} |\varphi(z)|. \tag{24}$$

From (22), (23) and (24) and $\|\varphi\|<1$, one can conclude

$$\sup_{z\in\mathbb{D}} w(z)|\psi_2(z)| < \infty.$$

Thus conditions (i) and (ii) are proved.

Now to prove condition (iv), let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ with $|\varphi(z_n)|\to 1$ such that

$$\lim_{r\to 1}\sup_{|\varphi(z)|>r} \frac{w(z)|\psi_2(z)|}{(1-|\varphi(z)|^2)^{1+\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}} = \lim_{n\to\infty} \frac{w(z_n)|\psi_2(z_n)|}{(1-|\varphi(z_n)|^2)^{1+\frac{2}{p}}v(\varphi(z_n))^{\frac{1}{p}}}.$$

By choosing a subsequence we may assume that there exist $n_0 \in \mathbb{N}$ such that $|\varphi(z_n)|^n \geqslant \frac{1}{2}$, for every $n \geqslant n_0$. According to (5), there exist $f^p_{\varphi(z_n)} \in B^\infty_v$, such that

$$|f_{\varphi(z_n)}(\varphi(z_n))|^p = \frac{1}{\widetilde{v}(\varphi(z_n))}.$$

Since v satisfies the condition (L1), it is essential and hence we can replace $\widetilde{v}$ by v. For each n, consider the following function

$$g_n(z) = \varphi_{\varphi(z_n)}(z)\varphi'_{\varphi(z_n)}(z)^{\frac{2}{p}} f_{\varphi(z_n)}(z)z^n.$$

Clearly $g_n \in A_{v,p}$ and $\sup\limits_n ||g_n||_{v,p} \leqslant C$. Also, $g_n(\varphi(z_n)) = 0$ and

$$|g'_n(\varphi(z_n))| = \frac{(\varphi(z_n))^n}{(1-|\varphi(z_n)|^2)^{1+\frac{2}{p}} v(\varphi(z_n))^{\frac{1}{p}}}.$$

Since $g_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, by Lemma 7, $||T_{\psi_1,\psi_2,\varphi} g_n||_w \to 0$ as $n \to \infty$. Thus

$$\begin{aligned} ||T_{\psi_1,\psi_2,\varphi} g_n||_w &\geqslant w(z_n)|\psi_1(z_n)g_n(\varphi(z_n)) + \psi_2(z_n)g'_n(\varphi(z_n))| \\ &= \frac{w(z_n)|\psi_2(z_n)||\varphi(z_n)|^n}{(1-|\varphi(z_n)|^2)^{1+\frac{2}{p}} v(\varphi(z_n))^{\frac{1}{p}}} \\ &\geqslant \frac{1}{2}\frac{w(z_n)|\psi_2(z_n)|}{(1-|\varphi(z_n)|^2)^{1+\frac{2}{p}} v(\varphi(z_n))^{\frac{1}{p}}}. \end{aligned} \tag{25}$$

From (25), it follows that

$$\lim_{n\to\infty} \frac{w(z_n)|\psi_2(z_n)|}{(1-|\varphi(z_n)|^2)^{1+\frac{2}{p}} v(\varphi(z_n))^{\frac{1}{p}}} = 0. \tag{26}$$

This proves the condition (iv).

Next to prove condition (iii), again let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$, with $|\varphi(z_n)| \to 1$ such that

$$\lim_{r\to 1} \sup_{|\varphi(z)|>r} \frac{w(z)|\psi_1(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}} v(\varphi(z))^{\frac{1}{p}}} = \lim_{n\to\infty} \frac{w(z_n)|\psi_1(z_n)|}{(1-|\varphi(z_n)|^2)^{\frac{2}{p}} v(\varphi(z_n))^{\frac{1}{p}}}. \tag{27}$$

Again, using function $f_{\varphi(z_n)}$ as obtained earlier, we define

$$h_n(z) = \varphi'_{\varphi(z_n)}(z)^{\frac{2}{p}} f_{\varphi(z_n)}(z)z^n.$$

Clearly, $h_n \in A_{v,p}$ with $\sup\limits_n ||h_n||_{v,p} \leqslant C$ and

$$h_n(\varphi(z_n)) = \frac{(\varphi(z_n))^n}{(1-|\varphi(z_n)|^2)^{\frac{2}{p}} v(\varphi(z_n))^{\frac{1}{p}}}.$$

Since $h_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, again by Lemma (7), $||T_{\psi_1,\psi_2,\varphi}h_n||_w \to 0$ as $n \to \infty$. Now

$$\begin{aligned}||T_{\psi_1,\psi_2,\varphi}h_n||_w &\geqslant w(z_n)|\psi_1(z_n)h_n(\varphi(z_n)) + \psi_2(z_n)h_n'(\varphi(z_n))| \\ &\geqslant w(z_n)|\psi_1(z_n)h_n(\varphi(z_n))| - w(z_n)|\psi_2(z_n)h_n'(\varphi(z_n))| \\ &= \frac{w(z_n)|\psi_1(z_n)||\varphi(z_n)|^n}{(1-|\varphi(z_n)|^2)^{\frac{2}{p}}v(\varphi(z_n))^{\frac{1}{p}}} - w(z_n)|\psi_2(z_n)||h_n'(\varphi(z_n))| \\ &\geqslant \frac{1}{2}\frac{w(z_n)|\psi_1(z_n)|}{(1-|\varphi(z_n)|^2)^{\frac{2}{p}}v(\varphi(z_n))^{\frac{1}{p}}} - \frac{C_v w(z_n)|\psi_2(z_n)|||h_n||_{v,p}}{(1-|\varphi(z_n)|^2)^{1+\frac{2}{p}}v(\varphi(z_n))^{\frac{1}{p}}}. \quad (28)\end{aligned}$$

Further, (28) implies that

$$\frac{w(z_n)|\psi_1(z_n)|}{(1-|\varphi(z_n)|^2)^{\frac{2}{p}}v(\varphi(z_n))^{\frac{1}{p}}} \leqslant \frac{2C_vCw(z_n)|\psi_2(z_n)|}{(1-|\varphi(z_n)|^2)^{1+\frac{2}{p}}v(\varphi(z_n))^{\frac{1}{p}}} + 2||T_{\psi_1,\psi_2,\varphi}h_n||_w. \quad (29)$$

From (26) and (29),it follows that

$$\lim_{n\to\infty}\frac{w(z_n)|\psi_1(z_n)|}{(1-|\varphi(z_n)|^2)^{\frac{2}{p}}v(\varphi(z_n))^{\frac{1}{p}}} = 0,$$

which proves the condition (iii). This completes the proof of the theorem.

If $\psi \in \mathscr{H}(\mathbb{D})$ and $\varphi : \mathbb{D} \to \mathbb{D}$ is an analytic map, then the boundedness and compactness of the product of weighted composition operators and differentiation operators $\mathscr{D}W_{\psi,\varphi} = T_{\psi',\psi\varphi',\varphi}$ and $W_{\psi,\varphi}\mathscr{D} = T_{0,\psi,\varphi}$ follows from Theorem 1 and Theorem 2 which we state in the following two corollaries.

COROLLARY 1. *Let v be a weight as defined in Lemma 1 and let w be an arbitrary weight. Let $\psi \in \mathscr{H}(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then:*

(a) the operator $\mathscr{D}W_{\psi,\varphi} : A_{v,p} \longrightarrow H_w^\infty$ is bounded if and only if

(i) $M_1 = \sup_{z\in\mathbb{D}} \frac{w(z)|\psi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}} < \infty;$

(ii) $M_2 = \sup_{z\in\mathbb{D}} \frac{w(z)|\psi(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{1+\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}} < \infty.$

Moreover, if the operator $\mathscr{D}W_{\psi,\varphi} : A_{v,p} \longrightarrow H_w^\infty$ is bounded, then

$$||\mathscr{D}W_{\psi,\varphi}||_{A_{v,p}\longrightarrow H_w^\infty} \asymp M_1 + M_2.$$

(b) the operator $\mathscr{D}W_{\psi,\varphi} : A_{v,p} \longrightarrow H_w^\infty$ is compact if and only if

(i) $K_1 = \sup_{z\in\mathbb{D}} w(z)\left|\psi'(z)\right| < \infty;$

(ii) $K_2 = \sup_{z\in\mathbb{D}} w(z)\left|\psi(z)\varphi'(z)\right| < \infty;$

(iii) $\lim_{r\to 1}\sup_{|\varphi(z)|>r}\frac{w(z)|\psi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}}=0$;

(iv) $\lim_{r\to 1}\sup_{|\varphi(z)|>r}\frac{w(z)|\psi(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{1+\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}}=0$.

COROLLARY 2. *Let v be a weight as defined in Lemma 1 and let w be an arbitrary weight. Let $\psi\in\mathscr{H}(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then*

(a) the operator $W_{\psi,\varphi}\mathscr{D}:A_{v,p}\longrightarrow H_w^\infty$ is bounded if and only if

$$L=\sup_{z\in\mathbb{D}}\frac{w(z)|\psi(z)|}{(1-|\varphi(z)|^2)^{1+\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}}<\infty.$$

Moreover, if the operator $W_{\psi,\varphi}\mathscr{D}:A_{v,p}\longrightarrow H_w^\infty$ is bounded, then

$$\|W_{\psi,\varphi}\mathscr{D}\|_{A_{v,p}\longrightarrow H_w^\infty}\asymp L.$$

(b) the operator $W_{\psi,\varphi}\mathscr{D}:A_{v,p}\longrightarrow H_w^\infty$ is compact if and only if $\psi\in H_w^\infty$ and

$$\lim_{r\to 1}\sup_{|\varphi(z)|>r}\frac{w(z)|\psi(z)|}{(1-|\varphi(z)|^2)^{1+\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}}=0.$$

REMARK 1. If we consider the weight $v(z)=(1-|z|^2)^\alpha$ where $\alpha\geqslant 1$, then clearly v satisfies the condition of Lemma 1. Now if we take $w(z)=(1-|z|^2)^\beta$, $\beta\geqslant 0$ for every $z\in\mathbb{D}$, then Corollary 1 will reduce to Theorem 4.3 and Theorem 4.4 obtained by Jiang [13]. Also, if $\psi(z)=1$ for every $z\in\mathbb{D}$, Corollary 1 will reduce to Proposition 1 and Proposition 2 obtained by Wolf [48].

Let $\psi\in\mathscr{H}(\mathbb{D})$. In Theorem 1 and Theorem 2, if we take $\varphi:\mathbb{D}\to\mathbb{D}$ as the identity map, then we get the boundedness and compactness of the product of multiplication operators and differentiation operators $\mathscr{D}M_\psi=T_{\psi',\psi,\varphi}$ and $M_\psi\mathscr{D}=T_{0,\psi,\varphi}$ which we state in the following two corollaries.

COROLLARY 3. *Let v be a weight as defined in Lemma 1 and let w be an arbitrary weight. Let $\psi\in\mathscr{H}(\mathbb{D})$. Then:*

(a) the operator $\mathscr{D}M_\psi:A_{v,p}\longrightarrow H_w^\infty$ is bounded if and only if

(i) $S_1=\sup_{z\in\mathbb{D}}\frac{w(z)|\psi'(z)|}{(1-|z|^2)^{\frac{2}{p}}v(z)^{\frac{1}{p}}}<\infty$;

(ii) $S_2=\sup_{z\in\mathbb{D}}\frac{w(z)|\psi(z)|}{(1-|z|^2)^{1+\frac{2}{p}}v(z)^{\frac{1}{p}}}<\infty$.

Moreover, if the operator $\mathscr{D}M_\psi:A_{v,p}\longrightarrow H_w^\infty$ is bounded, then

$$\|\mathscr{D}M_\psi\|_{A_{v,p}\longrightarrow H_w^\infty}\asymp S_1+S_2.$$

(b) the operator $\mathscr{D}M_\psi : A_{v,p} \longrightarrow H_w^\infty$ *is compact if and only if*

(i) $\lim_{|z|\to 1} \frac{w(z)|\psi'(z)|}{(1-|z|^2)^{\frac{2}{p}} v(z)^{\frac{1}{p}}} = 0;$

(ii) $\lim_{|z|\to 1} \frac{w(z)|\psi(z)|}{(1-|z|^2)^{1+\frac{2}{p}} v(z)^{\frac{1}{p}}} = 0.$

COROLLARY 4. *Let* v *be a weight as defined in Lemma 1 and let* w *be an arbitrary weight. Let* $\psi \in \mathscr{H}(\mathbb{D})$. *Then:*

(a) the operator $M_\psi\mathscr{D} : A_{v,p} \longrightarrow H_w^\infty$ *is bounded if and only if*

$$S = \sup_{z\in\mathbb{D}} \frac{w(z)|\psi(z)|}{(1-|z|^2)^{1+\frac{2}{p}} v(z)^{\frac{1}{p}}} < \infty.$$

Moreover, if the operator $M_\psi\mathscr{D} : A_{v,p} \longrightarrow H_w^\infty$ *is bounded, then*

$$\|M_\psi\mathscr{D}\|_{A_{v,p}\longrightarrow H_w^\infty} \asymp S.$$

(b) the operator $M_\psi\mathscr{D} : A_{v,p} \longrightarrow H_w^\infty$ *is compact if and only if*

$$\lim_{|z|\to 1} \frac{w(z)|\psi(z)|}{(1-|z|^2)^{1+\frac{2}{p}} v(z)^{\frac{1}{p}}} = 0.$$

COROLLARY 5. *Let* v *be a weight as defined in Lemma 1 and let* w *be a weight such that* $\frac{v^{\frac{1}{p}}}{w}$ *is bounded. Then the operator* $\mathscr{D}M_\psi(M_\psi\mathscr{D}) : A_{v,p} \longrightarrow H_w^\infty$ *is bounded if and only if* $\mathscr{D}M_\psi(M_\psi\mathscr{D})$ *is compact if and only if* $\psi \equiv 0$.

Proof. If the operator $\mathscr{D}M_\psi$ or $M_\psi\mathscr{D}$ is bounded, then according to Corollary 3 (a)(ii) or Corollary 4(a), it follow that there exist a constant $C > 0$ such that

$$\frac{w(z)|\psi(z)|}{(1-|z|^2)^{1+\frac{2}{p}} v(z)^{\frac{1}{p}}} \leqslant C, \tag{30}$$

for all $z \in \mathbb{D}$. Since $\frac{v^{\frac{1}{p}}}{w}$ is bounded, there exist $\lambda > 0$ such that $\frac{v^{\frac{1}{p}}(z)}{w(z)} \leqslant \lambda$, for all $z \in \mathbb{D}$. Thus from (30), we get

$$\frac{|\psi(z)|}{(1-|z|^2)^{1+\frac{2}{p}}} \leqslant \lambda C.$$

Further,

$$|\psi(z)| \leqslant \lambda C(1-|z|^2)^{1+\frac{2}{p}}.$$

By the maximum modulus theorem, $\psi \equiv 0$.

REMARK 2. From Corollary 3 and Corollary 4, it is clear that if $\mathscr{D}M_\psi$ is bounded (compact), then $M_\psi\mathscr{D}$ is bounded (compact). But the converse need not be true. This we explain in the following Example 1.

EXAMPLE 1. Let $p=1$, $w(z)=(1-|z|^2)^4e^{-\frac{1}{1-|z|^2}}$, $v(z)=(1-|z|^2)$ and $\psi(z)=e^{\frac{1}{1-z^2}}$. Then we have

$$\frac{w(z)|\psi(z)|}{(1-|z|^2)^{1+\frac{2}{p}}v(z)^{\frac{1}{p}}}=\frac{(1-|z|^2)^4e^{-\frac{1}{1-|z|^2}}|e^{\frac{1}{1-z^2}}|}{(1-|z|^2)^3(1-|z|^2)}\leqslant e^{-\frac{1}{1-|z|^2}}e^{\frac{1}{1-|z|^2}}=1.$$

Thus by Corollary 4(a), $M_\psi\mathscr{D}:A_{v,p}\longrightarrow H_w^\infty$ is bounded. But for $z=r$, we have

$$\frac{w(r)|\psi'(r)|}{(1-|r|^2)^{\frac{2}{p}}v(r)^{\frac{1}{p}}}=\frac{(1-|r|^2)^4e^{-\frac{1}{1-|r|^2}}|\frac{2r}{(1-r^2)^2}e^{\frac{1}{1-r^2}}|}{(1-|r|^2)^3}\leqslant\frac{2r}{1-r^2}\to\infty,\ \text{as } r\longrightarrow 1.$$

Hence according to Corollary 3(a), $\mathscr{D}M_\psi$ is not bounded.

On the other hand, if we take $w(z)=(1-|z|^2)^5e^{-\frac{1}{1-|z|^2}}$ and v,ψ same as above, then we have

$$\frac{w(z)|\psi(z)|}{(1-|z|^2)^{1+\frac{2}{p}}v(z)^{\frac{1}{p}}}\leqslant(1-|z|^2). \tag{31}$$

Further, (31) implies that

$$\lim_{|z|\to1}\frac{w(z)|\psi(z)|}{(1-|z|^2)^{1+\frac{2}{p}}v(z)^{\frac{1}{p}}}=0$$

and hence by Corollary 4(b), $M_\psi\mathscr{D}$ is compact. But for $z=r$ we have

$$\frac{w(r)|\psi'(r)|}{(1-|r|^2)^{\frac{2}{p}}v(r)^{\frac{1}{p}}}=2r\to2\neq0,\quad \text{as } r\longrightarrow1.$$

Thus by Corollary 3(b), $\mathscr{D}M_\psi$ is not compact.

In the following two examples we construct bounded, unbounded, compact, non-compact operators $T_{\psi_1,\psi_2,\varphi}$.

EXAMPLE 2. In this example, we give bounded and unbounded operators $T_{\psi_1,\psi_2,\varphi}$. Consider $p=1$. let $v(z)=(1-|z|^2)$ and $w(z)=(1-|z|)^5$. Define $\varphi(z)=\frac{z+1}{2}$, $\psi_1(z)=2z$ and $\psi_2(z)=\frac{z^2}{2}$. Then we obtain

$$\sup_{z\in\mathbb{D}}\frac{w(z)|\psi_1(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}}=\sup_{z\in\mathbb{D}}\frac{(1-|z|)^5 2|z|}{(1-|\frac{z+1}{2}|^2)^3}$$

$$\leqslant \sup_{z\in\mathbb{D}} \frac{2(1-|z|)^5}{\left(\frac{1-|z|}{2}\right)^3\left(\frac{|z|+1}{2}\right)^3}$$
$$\leqslant \sup_{z\in\mathbb{D}} 128(1-|z|)^2 < \infty.$$

This satisfies the first condition of Theorem 1. Now we verify the condition (ii).

$$\sup_{z\in\mathbb{D}} \frac{w(z)|\psi_2(z)|}{(1-|\varphi(z)|^2)^{1+\frac{2}{p}} v(\varphi(z))^{\frac{1}{p}}} = \sup_{z\in\mathbb{D}} \frac{(1-|z|)^5|z|^2\frac{1}{2}}{\left(1-|\frac{z+1}{2}|^2\right)^4}$$
$$\leqslant \sup_{z\in\mathbb{D}} \frac{\frac{1}{2}(1-|z|)^5|z|^2}{\left(\frac{1-|z|}{2}\right)^4\left(\frac{|z|+1}{2}\right)^4}$$
$$\leqslant \sup_{z\in\mathbb{D}} 128(1-|z|)|z|^2 < \infty.$$

Hence by Theorem 1, $T_{\psi_1,\psi_2,\varphi}$ is bounded. Next, let the weight v and the map φ be the same as before. Now if we define $w(z) = (1-|z|)^4$ and $\psi_2(z) = \frac{1}{(1-z)^4}$, then we have

$$\frac{w(z)|\psi_2(z)|}{(1-|\varphi(z)|^2)^{1+\frac{2}{p}} v(\varphi(z))^{\frac{1}{p}}} = \frac{(1-|z|)^4\frac{1}{|1-z|^4}}{\left(1-|\frac{z+1}{2}|^2\right)^4}.$$

For $z = r$,

$$\frac{w(r)|\psi_2(r)|}{(1-|\varphi(r)|^2)^3 v(\varphi(r))} = \frac{1}{\left(1-\left(\frac{r+1}{2}\right)^2\right)^4} \to \infty, \qquad \text{as } r \to 1.$$

This shows that the condition (ii) of Theorem 1 is not satisfied. Hence $T_{\psi_1,\psi_2,\varphi}$ is unbounded.

EXAMPLE 3. In this example, we give compact and non compact operators $T_{\psi_1,\psi_2,\varphi}$. First, we give an example of compact operator $T_{\psi_1,\psi_2,\varphi}$. Consider $p = 1$. Let $v(z) = (1-|z|^2)$ and $w(z) = (1-|z|)^6$. Define $\varphi(z) = \frac{z+1}{2}$, $\psi_1(z) = \frac{1}{(1-z)^2}$ and $\psi_2(z) = \frac{1}{1-z}$. Clearly, $|\varphi(z)| \to 1$ implies $|z| \to 1$. It is easy to verify conditions (i) and (ii) of Theorem 2. Next, we establish conditions (iii) and (iv) of Theorem 2.

$$\frac{w(z)|\psi_1(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}} v(\varphi(z))^{\frac{1}{p}}} = \frac{(1-|z|)^6}{\left(1-|\frac{z+1}{2}|^2\right)^3|1-z|^2}$$
$$\leqslant \frac{(1-|z|)^6}{\left(\frac{1-|z|}{2}\right)^3\left(\frac{|z|+1}{2}\right)^3(1-|z|)^2}$$
$$\leqslant 64(1-|z|) \to 0, \qquad \textit{as } |z| \to 1. \tag{32}$$

Also,

$$\frac{w(z)|\psi_2(z)|}{(1-|\varphi(z)|^2)^{1+\frac{2}{p}} v(\varphi(z))^{\frac{1}{p}}} = \frac{(1-|z|)^6}{\left(1-|\frac{z+1}{2}|^2\right)^4|1-z|}$$

$$
\begin{aligned}
&\leqslant \frac{(1-|z|)^6}{\left(\frac{1-|z|}{2}\right)^4\left(\frac{|z|+1}{2}\right)^4(1-|z|)} \\
&\leqslant 256(1-|z|) \to 0, \qquad as\ |z| \to 1. \qquad (33)
\end{aligned}
$$

It follows from (32) and (33) that the conditions (iii) and (iv) of Theorem 2 are satisfied. Hence the operator $T_{\psi_1,\psi_2,\varphi}$ is compact.

Next, we give an example of a bounded operator $T_{\psi_1,\psi_2,\varphi}$ which is not compact. For this, if we define $w(z) = (1-|z|)^5$ and take v, φ, ψ_1 and ψ_2 as before, then from (32) and (33) we have

$$
\sup_{z\in\mathbb{D}} \frac{w(z)|\psi_1(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}} v(\varphi(z))^{\frac{1}{p}}} < \infty,
$$

and

$$
\sup_{z\in\mathbb{D}} \frac{w(z)|\psi_2(z)|}{(1-|\varphi(z)|^2)^{1+\frac{2}{p}} v(\varphi(z))^{\frac{1}{p}}} < \infty.
$$

Thus by Theorem 1, $T_{\psi_1,\psi_2,\varphi}$ is bounded. Next, we show $T_{\psi_1,\psi_2,\varphi}$ is not compact. For, let $z = r$, then we have

$$
\begin{aligned}
\frac{w(r)|\psi_2(r)|}{(1-|\varphi(r)|^2)^{1+\frac{2}{p}} v(\varphi(r))^{\frac{1}{p}}} &= \frac{(1-|r|)^5}{\left(1-|\frac{r+1}{2}|^2\right)^4 |1-r|} \\
&= \frac{(1-r)^5}{\left(\frac{1-r}{2}\right)^4\left(\frac{r+3}{2}\right)^4(1-r)} \\
&= \frac{256}{(r+3)^4} \to 1 \neq 0, \text{ as } r \to 1.
\end{aligned}
$$

This shows that condition (iv) of Theorem 2 is not satisfied. Hence the operator $T_{\psi_1,\psi_2,\varphi}$ is not compact.

4. Boundedness and compactness of the operators $T_{\psi_1,\psi_2,\varphi} : A_{v,p} \to H^{\infty}_{w,0}$

We begin with stating the following lemma (see Lemma 2.1 in [34] or Lemma 1 in [29]) which we shall use to characterize the compactness of the operator $T_{\psi_1,\psi_2,\varphi}$.

LEMMA 8. *Let w be an arbitrary weight and K be closed set in $H^{\infty}_{w,0}$. Then K is compact if and only if it is bounded and*

$$
\lim_{|z|\to 1} \sup_{f\in K} w(z)|f(z)| = 0.
$$

REMARK 3. In Lemma 8, if the set K is not closed, then we can replace the word compact by the word relatively compact.

THEOREM 3. *Let v be a weight as defined in Lemma 1 and let w be an arbitrary weight. Let $\psi_1, \psi_2 \in \mathscr{H}(\mathbb{D})$ and $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic map. Then the operator $T_{\psi_1,\psi_2,\varphi} : A_{v,p} \longrightarrow H^\infty_{w,0}$ is bounded if and only if $T_{\psi_1,\psi_2,\varphi} : A_{v,p} \longrightarrow H^\infty_w$ is bounded, and $\psi_1, \psi_2 \in H^\infty_{w,0}$.*

Proof. First, assume that the operator $T_{\psi_1,\psi_2,\varphi} : A_{v,p} \longrightarrow H^\infty_{w,0}$ is bounded. Then it is clear that the operator $T_{\psi_1,\psi_2,\varphi} : A_{v,p} \longrightarrow H^\infty_w$ is bounded. Now if $f_1(z) = 1$ and $f_2(z) = z$, then clearly $f_1, f_2 \in A_{v,p}$, and hence $T_{\psi_1,\psi_2,\varphi} f_1 = \psi_1 \in H^\infty_{w,0}$ and $T_{\psi_1,\psi_2,\varphi} f_2 \in H^\infty_{w,0}$. Thus

$$\lim_{|z|\to 1} w(z)|\psi_1(z)| = 0, \tag{34}$$

and

$$\lim_{|z|\to 1} w(z)|\psi_1(z)\varphi(z) + \psi_2(z)| = 0. \tag{35}$$

Since

$$w(z)|\psi_2(z)| \leqslant w(z)|\psi_1(z)\varphi(z) + \psi_2(z)| + w(z)|\psi_1(z)|,$$

from (34) and (35), it follows that

$$\lim_{|z|\to 1} w(z)|\psi_2(z)| = 0. \tag{36}$$

Hence $\psi_2 \in H^\infty_{w,0}$.

Conversely, suppose that the operator $T_{\psi_1,\psi_2,\varphi} : A_{v,p} \longrightarrow H^\infty_w$ is bounded and $\psi_1, \psi_2 \in H^\infty_{w,0}$. Let $f \in A_{v,p}$. Then for each polynomial $p(z)$, we have

$$\begin{aligned} \lim_{|z|\to 1} w(z)|T_{\psi_1,\psi_2,\varphi} p(z)| &= \lim_{|z|\to 1} w(z)|\psi_1(z)p(\varphi(z)) + \psi_2(z)p'(\varphi(z))| \\ &\leqslant \lim_{|z|\to 1} w(z)|\psi_1(z)|\|p\|_\infty + \lim_{|z|\to 1} w(z)|\psi_2(z)|\|p'\|_\infty = 0 \end{aligned}$$

Thus $T_{\psi_1,\psi_2,\varphi} p \in H^\infty_{w,0}$. Since it is well known that the set of polynomials is dense in $A_{v,p}$ for the radial weight v (see [1, p. 10] or [15, p. 343] or [33, p. 134]), there exist a sequence of polynomials $\{p_n\}$ such that $\|f - p_n\|_{v,p} \to 0$ as $n \to \infty$. Hence

$$\|T_{\psi_1,\psi_2,\varphi} f - T_{\psi_1,\psi_2,\varphi} p_n\|_w \leqslant \|T_{\psi_1,\psi_2,\varphi}\|_{A_{v,p}\to H^\infty_w} \|f - p_n\|_{v,p} \to 0, \ \textit{as}\ n \to \infty.$$

Since $H^\infty_{w,0}$ is closed subspace of H^∞_w, we get $T_{\psi_1,\psi_2,\varphi}(A_{v,p}) \subseteq H^\infty_{w,0}$. Hence the operator $T_{\psi_1,\psi_2,\varphi} : A_{v,p} \longrightarrow H^\infty_{w,0}$ is bounded.

REMARK 4. From Theorem 3, it follows that if the operator $T_{\psi_1,\psi_2,\varphi} : A_{v,p} \longrightarrow H^\infty_{w,0}$ is bounded, then $T_{\psi_1,\psi_2,\varphi} : A_{v,p} \longrightarrow H^\infty_w$ is also bounded. But the converse may not be true. This we shall explain in the following example.

EXAMPLE 4. Let $p=1$, $v(z)=1-|z|^2$, $w(z)=\left(1-\frac{|z|^2}{4}\right)^4$, $\varphi(z)=\frac{z}{2}$, $\psi_1(z)=e^z$ and $\psi_2(z)=e^{z^2}$. Then we have

$$\frac{w(z)|\psi_1(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}}=\frac{\left(1-\frac{|z|^2}{4}\right)^4|e^z|}{\left(1-\frac{|z|^2}{4}\right)^2\left(1-\frac{|z|^2}{4}\right)}\leqslant\left(1-\frac{|z|^2}{4}\right)e^{|z|}<e,\qquad\text{for all } z\in\mathbb{D},$$

and

$$\frac{w(z)|\psi_2(z)|}{(1-|\varphi(z)|^2)^{1+\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}}=\frac{\left(1-\frac{|z|^2}{4}\right)^4|e^{z^2}|}{\left(1-\frac{|z|^2}{4}\right)^3\left(1-\frac{|z|^2}{4}\right)}<e^{|z|^2}<e,\qquad\text{for all } z\in\mathbb{D}.$$

Thus both the conditions of Theorem 1 are satisfied and hence $T_{\psi_1,\psi_2,\varphi}:A_{v,p}\longrightarrow H_w^\infty$ is bounded. But for $z=r$, we have

$$\lim_{|z|\to1}w(z)|\psi_1(z)|=\lim_{r\to1}\left(1-\frac{r^2}{4}\right)^4e^r=\left(\frac{3}{4}\right)^4e\neq0.$$

That is, $\psi_1\notin H_{w,0}^\infty$. Thus according to Theorem 3, $T_{\psi_1,\psi_2,\varphi}:A_{v,p}\longrightarrow H_{w,0}^\infty$ is not bounded.

THEOREM 4. *Let v be a weight as defined in Lemma 1 and let w be an arbitrary weight. Let $\psi_1,\psi_2\in\mathscr{H}(\mathbb{D})$ and $\varphi:\mathbb{D}\to\mathbb{D}$ be an analytic map. Then the operator $T_{\psi_1,\psi_2,\varphi}:A_{v,p}\longrightarrow H_{w,0}^\infty$ is compact if and only if*

(i) $\lim_{|z|\to1}\frac{w(z)|\psi_2(z)|}{(1-|\varphi(z)|^2)^{1+\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}}=0$;

(ii) $\lim_{|z|\to1}\frac{w(z)|\psi_1(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}}=0$.

Proof. If the conditions (i) and (ii) hold, then clearly conditions (i) and (ii) of Theorem 1 are satisfied. Thus the operator $T_{\psi_1,\psi_2,\varphi}:A_{v,p}\longrightarrow H_w^\infty$ is bounded. Also, since

$$(1-|\varphi(z)|^2)^{1+\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}\leqslant C_1$$

and

$$(1-|\varphi(z)|^2)^{\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}\leqslant C_2,$$

from (i) and (ii), we have

$$\lim_{|z|\to1}w(z)|\psi_2(z)|=\lim_{|z|\to1}\frac{w(z)|\psi_2(z)|(1-|\varphi(z)|^2)^{1+\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}}{(1-|\varphi(z)|^2)^{1+\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}}$$

$$\leqslant \lim_{|z|\to 1} \frac{C_1 w(z)|\psi_2(z)|}{(1-|\varphi(z)|^2)^{1+\frac{2}{p}} v(\varphi(z))^{\frac{1}{p}}} = 0,$$

and

$$\begin{aligned}\lim_{|z|\to 1} w(z)|\psi_1(z)| &= \lim_{|z|\to 1} \frac{w(z)|\psi_1(z)|(1-|\varphi(z)|^2)^{\frac{2}{p}} v(\varphi(z))^{\frac{1}{p}}}{(1-|\varphi(z)|^2)^{\frac{2}{p}} v(\varphi(z))^{\frac{1}{p}}} \\ &\leqslant \lim_{|z|\to 1} \frac{C_2 w(z)|\psi_1(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}} v(\varphi(z))^{\frac{1}{p}}} = 0.\end{aligned}$$

Thus $\psi_1, \psi_2 \in H^\infty_{w,0}$ and hence Theorem 3 implies that the operator $T_{\psi_1,\psi_2,\varphi} : A_{v,p} \longrightarrow H^\infty_{w,0}$ is bounded. Let $f \in A_{v,p}$. Now by using Lemma 5, we have

$$\begin{aligned} w(z)|(T_{\psi_1,\psi_2,\varphi} f)(z)| &= w(z)|\psi_1(z) f(\varphi(z)) + \psi_2(z) f'(\varphi(z))| \\ &\leqslant \frac{w(z)|\psi_1(z)|C_v\|f\|_{v,p}}{(1-|\varphi(z)|^2)^{\frac{2}{p}} v(\varphi(z))^{\frac{1}{p}}} + \frac{w(z)|\psi_2(z)|C_v\|f\|_{v,p}}{(1-|\varphi(z)|^2)^{1+\frac{2}{p}} v(\varphi(z))^{\frac{1}{p}}}. \end{aligned} \tag{37}$$

Let

$$K = T_{\psi_1,\psi_2,\varphi}\{f \in A_{v,p} : \|f\|_{v,p} \leqslant 1\}.$$

Then clearly the set K is bounded in $H^\infty_{w,0}$ and hence using conditions (i), (ii) in (37), it follows that

$$\lim_{|z|\to 1} \sup_{\|f\|_{v,p}\leqslant 1} w(z)|\left(T_{\psi_1,\psi_2,\varphi} f\right)(z)| = 0. \tag{38}$$

Thus in view of Lemma 8, we get the compactness of the operator $T_{\psi_1,\psi_2,\varphi} : A_{v,p} \longrightarrow H^\infty_{w,0}$.

Conversely, suppose that the operator $T_{\psi_1,\psi_2,\varphi} : A_{v,p} \longrightarrow H^\infty_{w,0}$ is compact. Now again by using the same argument of Theorem 3, we can get

$$\lim_{|z|\to 1} w(z)|\psi_1(z)| = 0, \tag{39}$$

and

$$\lim_{|z|\to 1} w(z)|\psi_2(z)| = 0. \tag{40}$$

That is, $\psi_1, \psi_2 \in H^\infty_{w,0}$. Since the operator $T_{\psi_1,\psi_2,\varphi} : A_{v,p} \longrightarrow H^\infty_w$ is compact, Theorem 2 implies that

$$\lim_{r\to 1} \sup_{|\varphi(z)|>r} \frac{w(z)|\psi_1(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}} v(\varphi(z))^{\frac{1}{p}}} = 0, \tag{41}$$

and

$$\lim_{r\to 1} \sup_{|\varphi(z)|>r} \frac{w(z)|\psi_2(z)|}{(1-|\varphi(z)|^2)^{1+\frac{2}{p}} v(\varphi(z))^{\frac{1}{p}}} = 0. \tag{42}$$

Let $\varepsilon > 0$. Then from (41) and (42), it follows that there exist $r_1, r_2 \in (0,1)$ such that whenever $r_1 < |\varphi(z)| < 1$ and $r_2 < |\varphi(z)| < 1$, we have

$$\frac{w(z)|\psi_1(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}} v(\varphi(z))^{\frac{1}{p}}} < \varepsilon, \tag{43}$$

and

$$\frac{w(z)|\psi_2(z)|}{(1-|\varphi(z)|^2)^{1+\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}} < \varepsilon. \tag{44}$$

Let

$$M_1 = \inf_{|t|\leqslant r}(1-|t|^2)^{\frac{2}{p}}v(t)^{\frac{1}{p}}, \tag{45}$$

and

$$M_2 = \inf_{|t|\leqslant r}(1-|t|^2)^{1+\frac{2}{p}}v(t)^{\frac{1}{p}}. \tag{46}$$

Thus for $|\varphi(z)| \leqslant r_1$ and $|\varphi(z)| \leqslant r_2$, (45) and (46) implies that

$$M_1 \leqslant (1-|\varphi(z)|^2)^{\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}, \tag{47}$$

and

$$M_2 \leqslant (1-|\varphi(z)|^2)^{1+\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}. \tag{48}$$

Let $\varepsilon_1 = \varepsilon M_1$ and $\varepsilon_2 = \varepsilon M_2$. Then according to (39) and (40), there exist $\delta_1, \delta_2 \in (0,1)$ such that whenever $\delta_1 < |z| < 1$ and $\delta_2 < |z| < 1$, we have

$$w(z)|\psi_1(z)| < \varepsilon_1, \tag{49}$$

$$w(z)|\psi_2(z)| < \varepsilon_2. \tag{50}$$

Further, using (47) and (48), (49) and (50) implies that whenever $|z| > \delta_1$, $|\varphi(z)| \leqslant r_1$ and $|z| > \delta_2$, $|\varphi(z)| \leqslant r_2$, we have

$$\frac{w(z)|\psi_1(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}} < \varepsilon, \tag{51}$$

and

$$\frac{w(z)|\psi_2(z)|}{(1-|\varphi(z)|^2)^{1+\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}} < \varepsilon. \tag{52}$$

Thus on combining (43),(44), (51) and (52), we have

$$\frac{w(z)|\psi_1(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}} < \varepsilon \tag{53}$$

and

$$\frac{w(z)|\psi_2(z)|}{(1-|\varphi(z)|^2)^{1+\frac{2}{p}}v(\varphi(z))^{\frac{1}{p}}} < \varepsilon, \tag{54}$$

whenever, $|z| > \delta_1$ and $|z| > \delta_2$. This proves the conditions (i) and (ii). This completes the proof of the theorem.

Let $\psi \in \mathscr{H}(\mathbb{D})$. In Theorem 2, Theorem 3 and Theorem 4 if $\varphi : \mathbb{D} \to \mathbb{D}$ is the identity map, then we get the boundedness and compactness of the product of multiplication operators and differentiation operators $\mathscr{D}M_\psi = T_{\psi',\psi,\varphi}$ and $M_\psi\mathscr{D} = T_{0,\psi,\varphi}$ which we state in the following three corollaries.

COROLLARY 6. *Let v be a weight as defined in Lemma 1 and let w be an arbitrary weight. Then:*

(a) $\mathscr{D}M_\psi : A_{v,p} \to H^\infty_{w,0}$ *is bounded if and only if* $\mathscr{D}M_\psi : A_{v,p} \to H^\infty_w$ *is bounded.*

(b) $M_\psi\mathscr{D} : A_{v,p} \to H^\infty_{w,0}$ *is bounded if and only if* $M_\psi\mathscr{D} : A_{v,p} \to H^\infty_w$ *is bounded.*

Proof. (a) If $\mathscr{D}M_\psi : A_{v,p} \to H^\infty_{w,0}$ is bounded, then clearly $\mathscr{D}M_\psi : A_{v,p} \to H^\infty_w$ is bounded. Conversely, suppose that the operator $\mathscr{D}M_\psi : A_{v,p} \to H^\infty_w$ is bounded. Then according to Corollary 3(a), we have

$$\frac{w(z)|\psi'(z)|}{(1-|z|^2)^{\frac{2}{p}}v(z)^{\frac{1}{p}}} \leqslant S_1, \qquad \text{for all } z \in \mathbb{D} \tag{55}$$

and

$$\frac{w(z)|\psi(z)|}{(1-|z|^2)^{1+\frac{2}{p}}v(z)^{\frac{1}{p}}} \leqslant S_2, \qquad \text{for all } z \in \mathbb{D}. \tag{56}$$

Further, (55) and (56), implies that

$$w(z)|\psi'(z)| \leqslant S_1(1-|z|^2)^{\frac{2}{p}}v(z)^{\frac{1}{p}},\ z \in \mathbb{D} \tag{57}$$

and

$$w(z)|\psi(z)| \leqslant S_2(1-|z|^2)^{1+\frac{2}{p}}v(z)^{\frac{1}{p}},\ z \in \mathbb{D}. \tag{58}$$

Since the weight v is bounded, (57) and (58) implies that

$$\lim_{|z|\to 1} w(z)|\psi'(z)| = 0 \tag{59}$$

and

$$\lim_{|z|\to 1} w(z)|\psi(z)| = 0. \tag{60}$$

That is, $\psi, \psi' \in H^\infty_{w,0}$. Thus from Theorem 3, it follows that $\mathscr{D}M_\psi : A_{v,p} \to H^\infty_{w,0}$ is bounded.
(b) The proof is analogous to the proof of part (a).

COROLLARY 7. *Let v be a weight as defined in Lemma 1 and let w be an arbitrary weight. Then the following statements are equivalent:*

(a) The operator $\mathscr{D}M_\psi : A_{v,p} \to H^\infty_w$ *is compact.*

(b) The operator $\mathscr{D}M_\psi : A_{v,p} \to H^\infty_{w,0}$ *is compact.*

(c) The following conditions hold:

(i) $\lim_{|z|\to 1} \frac{w(z)|\psi'(z)|}{(1-|z|^2)^{\frac{2}{p}}v(z)^{\frac{1}{p}}} = 0.$

(ii) $\lim_{|z|\to 1} \frac{w(z)|\psi(z)|}{(1-|z|^2)^{1+\frac{2}{p}}v(z)^{\frac{1}{p}}} = 0.$

COROLLARY 8. *Let v be a weight as defined in Lemma 1 and let w be an arbitrary weight. Then the following statements are equivalent:*

(a) The operator $M_\psi\mathscr{D} : A_{v,p} \to H_w^\infty$ *is compact.*

(b) The operator $M_\psi\mathscr{D} : A_{v,p} \to H_{w,0}^\infty$ *is compact.*

(c) $\lim_{|z|\to 1} \frac{w(z)|\psi(z)|}{(1-|z|^2)^{1+\frac{2}{p}} v(z)^{\frac{1}{p}}} = 0.$

COROLLARY 9. *Let v be a weight as defined in Lemma 1 and let w be a weight such that* $\frac{v^{\frac{1}{p}}}{w}$ *is bounded. Then the following statements are equivalent:*

(a) The operator $\mathscr{D}M_\psi(M_\psi\mathscr{D}) : A_{v,p} \longrightarrow H_w^\infty$ *is bounded.*

(b) The operator $\mathscr{D}M_\psi(M_\psi\mathscr{D}) : A_{v,p} \longrightarrow H_w^\infty$ *is compact.*

(c) The operator $\mathscr{D}M_\psi(M_\psi\mathscr{D}) : A_{v,p} \longrightarrow H_{w,0}^\infty$ *is compact.*

(d) $\psi \equiv 0.$

Proof. The proof follows from Corollary 5, Corollary 7 and Corollary 8.

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