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A NEW GENERALIZED REFINEMENT OF THE WEIGHTED ARITHMETIC–GEOMETRIC MEAN INEQUALITY

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Abstract. In this paper, we prove that for $i=1,2,\ldots,n$, $a_i \geqslant 0$ and $\alpha_i > 0$ satisfy $\sum_{i=1}^{n} \alpha_i = 1$, then for $m = 1,2,3,\ldots$, we have

$$\left(\prod_{i=1}^{n} a_i^{\alpha_i}\right)^m + r_0^m \left(\sum_{i=1}^{n} a_i^m - n\sqrt[n]{\prod_{i=1}^{n} a_i^m}\right) \leqslant \left(\sum_{i=1}^{n} \alpha_i a_i\right)^m$$

where $r_0 = \min\{\alpha_i : i = 1,\ldots,n\}$. This is a considerable generalization of the two refinements of the arithmetic-geometric mean inequality due to Furuichi [2], Manasrah and Kittaneh [7], which correspond to the cases $m = 1$ and $n = 2$, respectively. As application we give some generalized inequalities of determinants for positive definite matrices.

1. Introduction

The celebrated weighted arithmetic-geometric mean (AM-GM) inequality states as follows:

THEOREM 1. *Let n be a positive integer. For $i = 1,2,\ldots,n$, let $a_i \geqslant 0$, and let $\alpha_i > 0$ satisfy $\sum_{i=1}^{n} \alpha_i = 1$. Then, we have*

$$\prod_{i=1}^{n} a_i^{\alpha_i} \leqslant \sum_{i=1}^{n} \alpha_i a_i. \tag{1}$$

The inequality (1) is equivalent to

$$\left(\prod_{i=1}^{n} a_i^{\alpha_i}\right)^m \leqslant \left(\sum_{i=1}^{n} \alpha_i a_i\right)^m, \tag{2}$$

for any positive integer $m = 1,2,3,\ldots$.

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The special case of the weighted AM-GM inequality ($n = 2$) is the well-known Young's inequality; for positive real numbers a, b and $0 \leqslant \alpha \leqslant 1$, we have

$$a^{\alpha} b^{1-\alpha} \leqslant \alpha a + (1-\alpha) b. \tag{3}$$

Hirzallah and Kittaneh [4] refined Young's inequality (3) to

$$\left(a^{\alpha} b^{1-\alpha}\right)^2 + r_0^2 (a-b)^2 \leqslant \left(\alpha a + (1-\alpha) b\right)^2, \tag{4}$$

where $r_0 = \min\{\alpha, 1-\alpha\}$.

Kittaneh and Manasrah [6] refined Young's inequality so that

$$a^{\alpha} b^{1-\alpha} + r_0 \left(\sqrt{a} - \sqrt{b}\right)^2 \leqslant \alpha a + (1-\alpha) b, \tag{5}$$

where $r_0 = \min\{\alpha, 1-\alpha\}$.

Furuichi [2] refined (1) as follows:

$$\prod_{i=1}^{n} a_i^{\alpha_i} + r_0 \left(\sum_{i=1}^{n} a_i - n \sqrt[n]{\prod_{i=1}^{n} a_i} \right) \leqslant \sum_{i=1}^{n} \alpha_i a_i, \tag{6}$$

where $r_0 = \min\{\alpha_i : i = 1, \ldots, n\}$. This inequality generalizes the inequality (5).

Recently, Manasrah and Kittaneh [7] generalized the inequality (4) to

$$\left(a^{\alpha} b^{1-\alpha}\right)^m + r_0^m \left(a^{\frac{m}{2}} - b^{\frac{m}{2}}\right)^2 \leqslant \left(\alpha a + (1-\alpha) b\right)^m, \tag{7}$$

where $r_0 = \min\{\alpha, 1-\alpha\}$ and $m = 1, 2, 3, \ldots$.

2. Main result

In this section we refined the inequality (2) by adding the quantity

$$r_0^m \left(\sum_{i=1}^{n} a_i^m - n \sqrt[n]{\prod_{i=1}^{n} a_i^m} \right),$$

where $r_0 = \min\{\alpha_i : i = 1, \ldots, n\}$.

Before giving the main result, we need the following lemma.

LEMMA 1. *Let n and m be two integers and let $a_i \in \mathbb{R}^+$. Set $i_0 := m$, $i_n := 0$ and*

$$A := \{(i_1, \ldots, i_{n-1}) : 0 \leqslant i_j \leqslant i_{j-1},\ 1 \leqslant j \leqslant n-1\}.$$

Then, we have

$$\left(\sum_{i=1}^{n} a_i \right)^m = \sum_{(i_1, \ldots, i_{n-1}) \in A} \binom{i_0}{i_1} \binom{i_1}{i_2} \cdots \binom{i_{n-2}}{i_{n-1}} a_1^{i_0 - i_1} a_2^{i_1 - i_2} \ldots a_n^{i_{n-1} - i_n} \tag{8}$$

and for all $1 \leqslant j \leqslant n$, *it follows that*

$$ma_j \left(\sum_{i=1}^{n} a_i\right)^{m-1} = \sum_{(i_1,\ldots,i_{n-1})\in A} \binom{i_0}{i_1}\binom{i_1}{i_2}\cdots\binom{i_{n-2}}{i_{n-1}}(i_{j-1}-i_j)a_1^{i_0-i_1}a_2^{i_1-i_2}\ldots a_n^{i_{n-1}-i_n}. \tag{9}$$

Proof. The equality (8) is a formulation of the well-known multinomial formula (see, [1, p. 33]). And the equality (9) follows by derivation of (8) with respect to a_j.

The main result of this paper is the following theorem.

THEOREM 2. *Let n be a positive integer. For $i = 1,2,\ldots,n$, let $a_i \geqslant 0$ and let $\alpha_i > 0$ satisfy $\sum_{i=1}^{n}\alpha_i = 1$. Then for $m = 1,2,3,\ldots$, we have*

$$\left(\prod_{i=1}^{n} a_i^{\alpha_i}\right)^m + r_0^m\left(\sum_{i=1}^{n} a_i^m - n\sqrt[n]{\prod_{i=1}^{n} a_i^m}\right) \leqslant \left(\sum_{i=1}^{n}\alpha_i a_i\right)^m, \tag{10}$$

where $r_0 = \min\{\alpha_i : i = 1,\ldots,n\}$.

Proof. Let $\alpha_j = \min\{\alpha_i : i = 1,\ldots,n\}$. We claim that

$$\left(\sum_{i=1}^{n}\alpha_i a_i\right)^m - \alpha_j^m\left(\sum_{i=1}^{n} a_i^m - n\sqrt[n]{\prod_{i=1}^{n} a_i^m}\right) \geqslant \left(\prod_{i=1}^{n} a_i^{\alpha_i}\right)^m.$$

Indeed, we have the following equality

$$\left(\sum_{i=1}^{n}\alpha_i a_i\right)^m - \alpha_j^m\left(\sum_{i=1}^{n} a_i^m - n\sqrt[n]{\prod_{i=1}^{n} a_i^m}\right) = \sum_{(i_1,\ldots,i_{n-1})\in A}\binom{i_0}{i_1}\binom{i_1}{i_2}\cdots\binom{i_{n-2}}{i_{n-1}}$$
$$\alpha_1^{i_0-i_1}\alpha_2^{i_1-i_2}\ldots\alpha_n^{i_{n-1}-i_n}a_1^{i_0-i_1}a_2^{i_1-i_2}\ldots a_n^{i_{n-1}-i_n} - \alpha_j^m\left(\sum_{i=1}^{n} a_i^m - n\sqrt[n]{\prod_{i=1}^{n} a_i^m}\right).$$

Let B be a subset of A such that

$$\sum_{(i_1,\ldots,i_{n-1})\in A}\binom{i_0}{i_1}\binom{i_1}{i_2}\cdots\binom{i_{n-2}}{i_{n-1}}\alpha_1^{i_0-i_1}\alpha_2^{i_1-i_2}\ldots\alpha_n^{i_{n-1}-i_n}a_1^{i_0-i_1}a_2^{i_1-i_2}\ldots a_n^{i_{n-1}-i_n}$$
$$= \sum_{(i_1,\ldots,i_{n-1})\in B}\binom{i_0}{i_1}\binom{i_1}{i_2}\cdots\binom{i_{n-2}}{i_{n-1}}\alpha_1^{i_0-i_1}\alpha_2^{i_1-i_2}\ldots\alpha_n^{i_{n-1}-i_n}a_1^{i_0-i_1}a_2^{i_1-i_2}\ldots a_n^{i_{n-1}-i_n}$$
$$+\sum_{i=1}^{n}\alpha_i^m a_i^m.$$

Hence

$$\left(\sum_{i=1}^{n}\alpha_i a_i\right)^m-\alpha_j^m\left(\sum_{i=1}^{n}a_i^m-n\sqrt[n]{\prod_{i=1}^{n}a_i^m}\right)=\sum_{(i_1,\ldots,i_{n-1})\in B}\binom{i_0}{i_1}\binom{i_1}{i_2}\cdots\binom{i_{n-2}}{i_{n-1}}$$
$$\alpha_1^{i_0-i_1}\alpha_2^{i_1-i_2}\ldots\alpha_n^{i_{n-1}-i_n}a_1^{i_0-i_1}a_2^{i_1-i_2}\ldots a_n^{i_{n-1}-i_n}+\sum_{i=1}^{n}(\alpha_i^m-\alpha_j^m)a_i^m+n\alpha_j^m\sqrt[n]{\prod_{i=1}^{n}a_i^m}. \tag{11}$$

We have

$$\sum_{(i_1,\ldots,i_{n-1})\in B}\binom{i_0}{i_1}\binom{i_1}{i_2}\cdots\binom{i_{n-2}}{i_{n-1}}\alpha_1^{i_0-i_1}\alpha_2^{i_1-i_2}\ldots\alpha_n^{i_{n-1}-i_n}+\sum_{i=1}^{n}(\alpha_i^m-\alpha_j^m)+n\alpha_j^m=1.$$

Thus (11) is a convex combination of positive numbers. Therefore, by (1),

$$\left(\sum_{i=1}^{n}\alpha_i a_i\right)^m-\alpha_j^m\left(\sum_{i=1}^{n}a_i^m-n\sqrt[n]{\prod_{i=1}^{n}a_i^m}\right)\geqslant\prod_{i=1}^{n}a_i^{\alpha_i(m)},$$

where

$$\alpha_k(m)=m\alpha_k^m+\sum_{(i_1,\ldots,i_{n-1})\in B}\binom{i_0}{i_1}\binom{i_1}{i_2}\cdots\binom{i_{n-2}}{i_{n-1}}(i_{k-1}-i_k)\alpha_1^{i_0-i_1}\alpha_2^{i_1-i_2}\ldots\alpha_n^{i_{n-1}-i_n}$$

for all $1\leqslant k\leqslant n$. It is immediate that

$$\alpha_k(m)=\sum_{(i_1,\ldots,i_{n-1})\in A}\binom{i_0}{i_1}\binom{i_1}{i_2}\cdots\binom{i_{n-2}}{i_{n-1}}(i_{k-1}-i_k)\alpha_1^{i_0-i_1}\alpha_2^{i_1-i_2}\ldots\alpha_n^{i_{n-1}-i_n}.$$

By Lemma 1, we have $\alpha_k(m)=m\alpha_k$ for all $k=1,2,\ldots,n$. Thus

$$\left(\sum_{i=1}^{n}\alpha_i a_i\right)^m-\alpha_j^m\left(\sum_{i=1}^{n}a_i^m-n\sqrt[n]{\prod_{i=1}^{n}a_i^m}\right)\geqslant\left(\prod_{i=1}^{n}a_i^{\alpha_i}\right)^m.$$

This completes the proof.

The following corollary is a consequence of Theorem 2.

COROLLARY 1. *For $i=1,2,\ldots,n$, let $a_i\geqslant 0$, and let $\alpha_i>0$ satisfy $\sum_{i=1}^{n}\alpha_i=1$. Then for all integers $m\geqslant 1$, we have*

$$\prod_{i=1}^{n}a_i^{\alpha_i}+r_0^m\left(\sum_{i=1}^{n}a_i-n\sqrt[n]{\prod_{i=1}^{n}a_i}\right)\leqslant\left(\sum_{i=1}^{n}\alpha_i a_i^{\frac{1}{m}}\right)^m\leqslant\sum_{i=1}^{n}\alpha_i a_i,$$

where $r_0=\min\{\alpha_i : i=1,\ldots,n\}$.

Moreover, if we set $U_m:=\left(\sum_{i=1}^{n}\alpha_i a_i^{\frac{1}{m}}\right)^m$. Then $\{U_m\}$ is a decreasing sequence and we have

$$\lim_{m\to\infty}U_m=\prod_{i=1}^{n}a_i^{\alpha_i}.$$

Proof. By Theorem 2, we have

$$\prod_{i=1}^{n} a_i^{\alpha_i} + r_0^m \left(\sum_{k=1}^{n} a_i - n \sqrt[n]{\prod_{i=1}^{n} a_i} \right) \leqslant \Big(\sum_{i=1}^{n} \alpha_i a_i^{\frac{1}{m}} \Big)^m .$$

The function $f(t) = t^m$ with $m \geqslant 2$ is convex, then by Jensen's inequality we find that

$$\Big(\sum_{i=1}^{n} \alpha_i a_i^{\frac{1}{m}} \Big)^m \leqslant \sum_{i=1}^{n} \alpha_i a_i .$$

It follows that,

$$\prod_{i=1}^{n} a_i^{\alpha_i} + r_0^m \left(\sum_{k=1}^{n} a_i - n \sqrt[n]{\prod_{i=1}^{n} a_i} \right) \leqslant \Big(\sum_{i=1}^{n} \alpha_i a_i^{\frac{1}{m}} \Big)^m \leqslant \sum_{i=1}^{n} \alpha_i a_i .$$

It is well-known (see, e.g., [3, p. 13, p. 26]) that $\{U_m\}$ is a decreasing sequence and that

$$\lim_{m\to\infty} U_m = \prod_{i=1}^{n} a_i^{\alpha_i} .$$

This ends the proof.

3. Application

In this section, we give some generalized inequalities of determinants for positive definite matrices.

Let $\mathbf{M}_n(\mathbb{C})$ be the space of $n \times n$ complex matrices. A matrix A is called positive definite denoted as $A > 0$ if $\langle Ax, x \rangle > 0$ for all nonzero $x \in \mathbb{C}^n$. A determinant version of Young's inequality is also known [5, p. 467]; for positive definite matrices A, B and $0 \leqslant \alpha \leqslant 1$

$$\det \big(A^{\alpha} B^{1-\alpha} \big) \leqslant \det \big(\alpha A + (1-\alpha) B \big). \tag{12}$$

To prove the result of this section, we need the following lemma, which is a generalized Minkowski's inequality for determinants (see, e.g., [5, p. 482]).

LEMMA 2. *Let p be a positive integer and let $A_1, A_2, \ldots, A_p \in \mathbf{M}_n(\mathbb{C})$ be positive definite matrices. Then we have*

$$\det \left(\sum_{i=1}^{p} A_i \right)^{\frac{1}{n}} \geqslant \sum_{i=1}^{p} \det \big(A_i \big)^{\frac{1}{n}} . \tag{13}$$

THEOREM 3. *Let p be a positive integer. Let $A_1, A_2, \ldots, A_p \in \mathbf{M}_n(\mathbb{C})$ be positive definite matrices and let $\alpha_i > 0$ satisfy $\sum_{i=1}^{p} \alpha_i = 1$. Then, for $m = 1, 2, 3, \ldots$, we have*

$$\det\left(\prod_{i=1}^{p} A_i^{\alpha_i}\right)^m + r_0^{nm}\left(\sum_{i=1}^{p}\det(A_i)^m - p\sqrt[p]{\prod_{i=1}^{p}\det(A_i)^m}\right) \leqslant \det\left(\sum_{i=1}^{p}\alpha_i A_i\right)^m,$$

where $r_0 = \min\{\alpha_i : i = 1,\ldots,n\}$.

Proof. We have

$$\begin{aligned}
&\det\left(\sum_{i=1}^{p}\alpha_i A_i\right)^m \\
&= \left[\det\left(\sum_{i=1}^{p}\alpha_i A_i\right)^{\frac{1}{n}}\right]^{nm} \geqslant \left(\sum_{i=1}^{p}\det(\alpha_i A_i)^{\frac{1}{n}}\right)^{nm} \\
&= \left(\sum_{i=1}^{p}\alpha_i \det(A_i)^{\frac{1}{n}}\right)^{nm} \\
&\geqslant \left(\prod_{i=1}^{p}\left(\det(A_i)^{\frac{1}{n}}\right)^{\alpha_i}\right)^{nm} + r_0^{nm}\left(\sum_{i=1}^{p}\left(\det(A_i)^{\frac{1}{n}}\right)^{nm} - p\sqrt[p]{\prod_{i=1}^{p}\left(\det(A_i)^{\frac{1}{n}}\right)^{nm}}\right) \\
&= \det\left(\prod_{i=1}^{p} A_i^{\alpha_i}\right)^m + r_0^{nm}\left(\sum_{i=1}^{p}\det(A_i)^m - p\sqrt[p]{\prod_{i=1}^{p}\det(A_i)^m}\right),
\end{aligned}$$

where the first inequality is by Lemma 2 and the second is by Theorem 2. The proof is thus complete.

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