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VOLTERRA INTEGRAL OPERATORS FROM $\mathscr{D}^p_{p-2+s}$ INTO $F(p\lambda, p\lambda+s\lambda-2, q)$

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Abstract. Let $1<p<\infty$, $0<q<\infty$, $0<s$, $\lambda \leqslant 1$ such that $q+s\lambda>1$. We characterize the boundedness and compactness of inclusion mapping from Dirichlet type spaces $\mathscr{D}^p_{p-2+s}$ into tent spaces $T_{p\lambda,q}(\mu)$. As an application, the boundedness of the Volterra operator T_g, its companion operator I_g and the multiplication operator M_g from $\mathscr{D}^p_{p-2+s}$ to $F(p\lambda, p\lambda+s\lambda-2, q)$ are given. Furthermore, we study the essential norm and compactness of T_g and I_g.

1. Introduction

Let $\mathbb{D}$ be the unit disk of the complex plane $\mathbb{C}$ and $\partial\mathbb{D}$ be its boundary, the unit circle. Let $\mathscr{H}(\mathbb{D})$ denote the space of all analytic functions in $\mathbb{D}$ endowed with the topology of uniform convergence on compact subsets. Given $0<p<\infty$ and $\alpha>-1$, the Dirichlet type space $\mathscr{D}^p_\alpha$ is the set of all functions $f\in\mathscr{H}(\mathbb{D})$ such that

$$\|f\|^p_{\mathscr{D}^p_\alpha} := |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p dA_\alpha(z) < \infty.$$

Here $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$ and $dA(z)=\frac{1}{\pi}dxdy$ is the normalized Lebesgue area measure on $\mathbb{D}$. As is known, if $p<\alpha+1$, then we have that $\mathscr{D}^p_\alpha = A^p_{\alpha-p}$, the weighted Bergman space (see, for example, Theorem 6 of [4]). If $p>\alpha+2$, then $\mathscr{D}^p_\alpha \subset \mathscr{H}^\infty$, the Banach algebra of all bounded analytic functions with the supremum norm $\|f\|_{\mathscr{H}^\infty} = \sup_{z\in\mathbb{D}}|f(z)|$ (see, for example, [31]). This means that the space $\mathscr{D}^p_\alpha$ becomes a proper Dirichlet type space when $p-2\leqslant\alpha\leqslant p-1$. The spaces $\mathscr{D}^p_{p-1}$ are closely related with Hardy spaces. In fact, $\mathscr{D}^2_1 = \mathscr{H}^2$ in the sense of equivalent norms. From [4] we have $\mathscr{D}^p_{p-1}\subseteq\mathscr{H}^p$ when $0<p\leqslant 2$. If $2\leqslant p<\infty$, then $\mathscr{H}^p\subseteq\mathscr{D}^p_{p-1}$, see [12]. The spaces $\mathscr{D}^p_{p-2}$ are the well known analytic Besov space.

The Bloch space $\mathscr{B}$ is the class of all $f\in\mathscr{H}(\mathbb{D})$ for which

$$\|f\|_{\mathscr{B}} := |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

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The little Bloch space $\mathscr{B}_0$, consists of all $f \in \mathscr{H}(\mathbb{D})$ such that

$$\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0.$$

It is known that the spaces $\mathscr{D}^p_{p-2}$ and $\mathscr{B}_0$ are subspaces of the Bloch space $\mathscr{B}$.

Let $0 < p < \infty$, $-2 < q < \infty$ and $0 < t < \infty$. The space $F(p,q,t)$ consists of those $f \in \mathscr{H}(\mathbb{D})$ such that

$$\|f\|^p_{F(p,q,t)} := \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q (1-|\varphi_a(z)|^2)^t dA(z) < \infty.$$

This space was first introduced by Zhao in [33]. When $p = 2, q = 0$, $F(p,q,t)$ is just the Q_t space. It is well known that $F(p,p-2,t) = \mathscr{B}$ for all $t > 1$ in the sense of equivalent norms, see [33].

Let μ be a nonnegative Borel measure on $\mathbb{D}$. For $0 < p < \infty$ and $0 < q < \infty$, the tent space $T_{p,q}(\mu)$ consists of all μ-measurable functions f such that

$$\|f\|^p_{T_{p,q}(\mu)} := \sup_{I\subseteq\partial\mathbb{D}} \frac{1}{|I|^q} \int_{S(I)} |f(z)|^p d\mu(z) < \infty.$$

Given $g \in \mathscr{H}(\mathbb{D})$, the Volterra integral operator T_g and its companion operator I_g with symbol g are defined by

$$T_g f(z) := \int_0^z g'(w) f(w) dw, \qquad I_g f(z) := \int_0^z g(w) f'(w) dw, \ \ f \in \mathscr{H}(\mathbb{D}), \ \ z \in \mathbb{D},$$

respectively. Recall that the multiplication operator M_g is defined by

$$M_g f(z) := g(z) f(z), \quad f \in \mathscr{H}(\mathbb{D}), \quad z \in \mathbb{D}.$$

The operators T_g and I_g are closely related to M_g. For example, note that the following relation holds

$$T_g f + I_g f = M_g f - f(0) g(0).$$

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be analytic function spaces. Denote $M(X,Y)$ the set of multipliers from X to Y, that is,

$$M(X,Y) := \{g \in \mathscr{H}(\mathbb{D}) : fg \in Y, \ \ \forall f \in X\}.$$

By the closed graph theorem, if $g \in M(X,Y)$, then we have that the operator $M_g : X \to Y$ is bounded.

Operator T_g seems studied for the first time in [15]. After that many authors have studied this, as well as some other related operators on the unit disc or the unit ball in $\mathbb{C}^n$ (see, for example, [1, 2, 5, 6, 7, 8, 9, 10, 11, 14, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 32]). Some of these papers study the operators from or to the general space $F(p,q,t)$ and the Dirichlet-type space (see [9, 14, 20, 22, 23, 26]).

Xiao in [32] studied the embedding from Q_p space $(0<p<1)$ into $T_{2,q}(\mu)$. As an application, he characterized the boundedness and compactness of the operator T_g on the Q_p space. Pau and Zhao studied the embedding from Möbius invariant Besov type space $F(p,p-2,s)$ into $T_{p,q}(\mu)$ in [14]. Liu and Lou studied the embedding from Morrey spaces to $T_{2,q}(\mu)$ in [13].

In this paper, we first characterize the boundedness and compactness of the embedding from $\mathscr{D}^p_{p-2+s}$ into tent spaces $T_{p\lambda,q}(\mu)$. Then as an application, the boundedness of T_g, I_g and M_g from $\mathscr{D}^p_{p-2+s}$ to $F(p\lambda,p\lambda+s\lambda-2,q)$ are given. Furthermore, we study the essential norm and compactness of T_g and I_g. For some previous results on essential norms of integral type and related operators see, for example, [5, 6, 7, 18, 21, 23, 27, 28].

The article is organized as follows. In the next section, we state some preliminary results. The embedding theorems from $\mathscr{D}^p_{p-2+s}$ to tent spaces and the boundedness of T_g and I_g from $\mathscr{D}^p_{p-2+s}$ to $F(p\lambda,p\lambda+s\lambda-2,q)$ are given in Sections 3 and 4, respectively. In the last section, we estimate the essential norm of T_g and I_g.

Throughout this paper, C denotes a positive constant, it is not necessary to be the same from one line to another. Let f and g be two positive functions. For convenience, we write $f\lesssim g$, if $f\leqslant Cg$ holds, where C is a positive constant independent of f and g. If $f\lesssim g$ and $g\lesssim f$, then we say $f\asymp g$.

2. Preliminary results

Let I be an arc of $\partial\mathbb{D}$ and $|I|$ be the normalized Lebesgue arc length of I. The Carleson square based on I, denoted by $S(I)$, is defined by

$$S(I):=\left\{z=re^{i\theta}\in\mathbb{D}:1-|I|\leqslant r<1,e^{i\theta}\in I\right\}.$$

Let μ be a positive Borel measure on $\mathbb{D}$. For $0<s<\infty$, μ is called an s-Carleson measure if $\sup_{I\subseteq\partial\mathbb{D}}\frac{\mu(S(I))}{|I|^s}<\infty$. If μ is an s-Carleson measure, then we set

$$\|\mu\|_s:=\sup_{I\subseteq\partial\mathbb{D}}\frac{\mu(S(I))}{|I|^s}.$$

We need the following equivalent description of s-Carleson measure (see, e.g., [14]).

LEMMA 2.1. Let μ be a nonnegative Borel measure on $\mathbb{D}$. If $0<s,t<\infty$, then μ is an s-Carleson measure if and only if

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{(1-|a|^2)^t}{|1-\bar{a}z|^{s+t}}d\mu(z)<\infty.$$

Moreover,

$$\sup_{I\subseteq\partial\mathbb{D}}\frac{\mu(S(I))}{|I|^s}\asymp\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{(1-|a|^2)^t}{|1-\bar{a}z|^{s+t}}d\mu(z).$$

The following point evaluation estimate is folklore. It is proved by using, for example, the standard methods in Lemma 3 and Lemma 4 in [20]. Hence, the proof is omitted.

LEMMA 2.2. Let $1 < p < \infty$ and $0 < s \leqslant 1$. If $f \in \mathscr{D}^p_{p-2+s}$, then

$$|f(z)| \lesssim \frac{\|f\|_{\mathscr{D}^p_{p-2+s}}}{(1-|z|^2)^{\frac{s}{p}}}, \quad z \in \mathbb{D}.$$

The following integral estimates are fundamental in function spaces and operator theory, see [16, 1.4.10. Proposition] (the case that we need can be also found in [34, Lemma 3.10]).

LEMMA 2.3. Suppose that $z \in \mathbb{D}$, c is real, $t > -1$, and

$$I_{c,t}(z) = \int_{\mathbb{D}} \frac{(1-|w|^2)^t}{|1-\bar{w}z|^{2+t+c}} dA(w).$$

(i) If $c < 0$, then as a function of z, $I_{c,t}$ is bounded on $\mathbb{D}$.
(ii) If $c = 0$, then $I_{c,t}(z) \asymp \log \frac{1}{1-|z|^2}$, as $|z| \to 1^-$.
(iii) If $c > 0$, then $I_{c,t}(z) \asymp \frac{1}{(1-|z|^2)^c}$, as $|z| \to 1^-$.

Finally, here we will also use the following lemma which has been recently proved in [7].

LEMMA 2.4. For $0 < r < 1$, let $\chi_{\{z:|z|<r\}}$ be the characteristic function of the set $\{z : |z| < r\}$. If μ is a s-Carleson measure on $\mathbb{D}$, then μ is a vanishing s-Carleson measure if and only if $\|\mu - \mu_r\|_s \to 0$ as $r \to 1^-$, where $d\mu_r = \chi_{\{z:|z|<r\}} d\mu$.

3. Embedding from $\mathscr{D}^p_{p-2+s}$ to tent spaces

In this section, we study the embedding from $\mathscr{D}^p_{p-2+s}$ to tent spaces. We give a complete characterization for the boundedness and compactness of the inclusion mapping $i : \mathscr{D}^p_{p-2+s} \to T_{p\lambda,q}(\mu)$. We say that the inclusion mapping $\mathscr{D}^p_{p-2+s} \to T_{p\lambda,q}(\mu)$ is compact if

$$\lim_{n\to\infty} \frac{1}{|I|^q} \int_{S(I)} |f_n(z)|^{p\lambda} d\mu(z) = 0$$

whenever $I \subseteq \partial\mathbb{D}$ and $\{f_n\}$ is a bounded sequence in $\mathscr{D}^p_{p-2+s}$ that converges to 0 uniformly on compact subsets of $\mathbb{D}$.

THEOREM 3.1. *Let μ be a nonnegative Borel measure on $\mathbb{D}$. Let $1 < p < \infty$, $0 < s, \lambda \leqslant 1$ and $0 < q < \infty$ such that $q + s\lambda > 1$. Then the inclusion mapping $i : \mathscr{D}^p_{p-2+s} \to T_{p\lambda,q}(\mu)$ is bounded if and only if μ is a $(q+s\lambda)$-Carleson measure. Furthermore, $\|i\|^{p\lambda} \asymp \|\mu\|_{q+s\lambda}$.*

Proof. Assume that the operator $i:\mathscr{D}^p_{p-2+s}\to T_{p\lambda,q}(\mu)$ is bounded. Given $a\in\mathbb{D}$, let

$$f_a(z):=\frac{1-|a|^2}{(1-\bar{a}z)^{1+\frac{s}{p}}},\quad z\in\mathbb{D}. \tag{1}$$

Using Lemma 2.3, we have $f_a\in\mathscr{D}^p_{p-2+s}$ with $\sup_{a\in\mathbb{D}}\|f_a\|_{\mathscr{D}^p_{p-2+s}}\lesssim 1$. Fix an arc $I\subseteq\partial\mathbb{D}$. Let $e^{i\theta}$ be the center of I and $a=(1-|I|)e^{i\theta}$. Then

$$|1-\bar{a}z|\asymp 1-|a|=|I|,$$

and

$$|f_a(z)|^{p\lambda}\asymp\frac{1}{|I|^{s\lambda}}$$

whenever $z\in S(I)$. So

$$\frac{\mu(S(I))}{|I|^{q+s\lambda}}\asymp\frac{1}{|I|^q}\int_{S(I)}|f_a(z)|^{p\lambda}d\mu(z)\leqslant\|f_a\|^{p\lambda}_{T_{p\lambda,q}(\mu)}\leqslant\|i\|^{p\lambda}\|f_a\|^{p\lambda}_{\mathscr{D}^p_{p-2+s}}\lesssim\|i\|^{p\lambda}.$$

Consequently, μ is a $(q+s\lambda)$-Carleson measure and

$$\|\mu\|_{q+s\lambda}\lesssim\|i\|^{p\lambda}.$$

Conversely, suppose that μ is a $(q+s\lambda)$-Carleson measure. Fix $f\in\mathscr{D}^p_{p-2+s}$. Let I be any arc on $\partial\mathbb{D}$ and $a=(1-|I|)e^{i\theta}$, where $e^{i\theta}$ is the midpoint of I. From Lemma 2.2,

$$|f(a)|\lesssim\frac{\|f\|_{\mathscr{D}^p_{p-2+s}}}{(1-|a|)^{\frac{s}{p}}}=\frac{\|f\|_{\mathscr{D}^p_{p-2+s}}}{|I|^{\frac{s}{p}}}.$$

Obviously,

$$\begin{aligned}\frac{1}{|I|^q}\int_{S(I)}|f(z)|^{p\lambda}d\mu(z)&\lesssim\frac{1}{|I|^q}\int_{S(I)}|f(a)|^{p\lambda}d\mu(z)+\frac{1}{|I|^q}\int_{S(I)}|f(z)-f(a)|^{p\lambda}d\mu(z)\\&=I_1(a)+I_2(a).\end{aligned}$$

It is clear that

$$I_1(a)\lesssim\frac{\mu(S(I))}{|I|^{q+s\lambda}}\|f\|^{p\lambda}_{\mathscr{D}^p_{p-2+s}}\lesssim\|\mu\|_{q+s\lambda}\|f\|^{p\lambda}_{\mathscr{D}^p_{p-2+s}}.$$

By the assumed condition and Theorem 7.4 in [34], we know that $i:A^{p\lambda}_{q-2+s\lambda}\to L^{p\lambda}(d\mu)$ is bounded. By Theorem 4.28 in [34], we have

$$\int_{\mathbb{D}}|f(z)|^p dA(z)\asymp|f(0)|^p+\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^p dA(z).$$

Based on these facts, we turn to estimate $I_2(a)$. The estimate will be divided into two cases.

Case 1: $q+s\lambda \geqslant 2$.

$$\begin{aligned}
I_2(a) &\asymp \int_{S(I)} \frac{|f(z)-f(a)|^{p\lambda}}{|1-\bar{a}z|^q} d\mu(z) \\
&\asymp (1-|a|^2)^{s\lambda} \int_{S(I)} \frac{|f(z)-f(a)|^{p\lambda}(1-|a|^2)^2}{|1-\bar{a}z|^{2+s\lambda+q}} d\mu(z) \\
&\lesssim (1-|a|^2)^{s\lambda} \int_{\mathbb{D}} \frac{|f(z)-f(a)|^{p\lambda}(1-|a|^2)^2}{|1-\bar{a}z|^{2+s\lambda+q}} d\mu(z) \\
&\lesssim (1-|a|^2)^{s\lambda} \int_{\mathbb{D}} \frac{|f(z)-f(a)|^{p\lambda}(1-|a|^2)^2}{|1-\bar{a}z|^{2+s\lambda+q}} (1-|z|^2)^{q-2+s\lambda} dA(z) \\
&\lesssim (1-|a|^2)^{s\lambda} \int_{\mathbb{D}} \frac{|f(z)-f(a)|^{p\lambda}(1-|a|^2)^2}{|1-\bar{a}z|^4} dA(z) \\
&= (1-|a|^2)^{s\lambda} \int_{\mathbb{D}} |f\circ\varphi_a(w)-f(a)|^{p\lambda} dA(w) \\
&\asymp (1-|a|^2)^{s\lambda} \int_{\mathbb{D}} |(f\circ\varphi_a)'(w)|^{p\lambda}(1-|w|^2)^{p\lambda} dA(w) \\
&= (1-|a|^2)^{s\lambda} \int_{\mathbb{D}} |f'(\varphi_a(w))|^{p\lambda}(1-|\varphi_a(w)|^2)^{p\lambda} dA(w) \\
&= \int_{\mathbb{D}} |f'(z)|^{p\lambda} \frac{(1-|z|^2)^{p\lambda}(1-|a|^2)^{2+s\lambda}}{|1-\bar{a}z|^4} dA(z).
\end{aligned}$$

If $0<\lambda<1$, then Hölder's inequality yields that

$$\begin{aligned}
I_2(a) &\lesssim \int_{\mathbb{D}} |f'(z)|^{p\lambda}(1-|z|^2)^{p\lambda-2\lambda+s\lambda} \frac{(1-|z|^2)^{2\lambda-s\lambda}(1-|a|^2)^{2+s\lambda}}{|1-\bar{a}z|^4} dA(z) \\
&\leqslant \left(\int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{p-2+s} dA(z)\right)^{\lambda} \left(\int_{\mathbb{D}} \frac{(1-|z|^2)^{\frac{2\lambda-s\lambda}{1-\lambda}}(1-|a|^2)^{\frac{2+s\lambda}{1-\lambda}}}{|1-\bar{a}z|^{\frac{4}{1-\lambda}}} dA(z)\right)^{1-\lambda}.
\end{aligned}$$

Applying Lemma 2.3, we get

$$(1-|a|^2)^{\frac{2+s\lambda}{1-\lambda}} \int_{\mathbb{D}} \frac{(1-|z|^2)^{\frac{2\lambda-s\lambda}{1-\lambda}}}{|1-\bar{a}z|^{2+\frac{2\lambda-s\lambda}{1-\lambda}+\frac{2+s\lambda}{1-\lambda}}} dA(z) \lesssim 1.$$

So

$$I_2(a) \lesssim \|f\|^{p\lambda}_{\mathscr{D}^p_{p-2+s}}.$$

If $\lambda=1$, then

$$I_2(a) \lesssim \int_{\mathbb{D}} |f'(z)|^p \frac{(1-|z|^2)^p(1-|a|^2)^{2+s}}{|1-\bar{a}z|^4} dA(z) \lesssim \int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{p-2+s} dA(z)$$

$$\lesssim \|f\|^p_{\mathscr{D}^p_{p-2+s}}.$$

Case 2: $1 < q + s\lambda < 2$.

$$\begin{aligned}
I_2(a) &\asymp (1-|a|^2)^{-q}\int_{S(I)} |f(z)-f(a)|^{p\lambda}\,d\mu(z)\\
&\asymp (1-|a|^2)^{2-q}\int_{S(I)} \frac{|f(z)-f(a)|^{p\lambda}(1-|a|^2)^2}{|1-\bar{a}z|^4}\,d\mu(z)\\
&\lesssim (1-|a|^2)^{2-q}\int_{\mathbb{D}} \frac{|f(z)-f(a)|^{p\lambda}(1-|a|^2)^2}{|1-\bar{a}z|^4}\,d\mu(z)\\
&\lesssim (1-|a|^2)^{2-q}\int_{\mathbb{D}} \frac{|f(z)-f(a)|^{p\lambda}(1-|a|^2)^2}{|1-\bar{a}z|^4}(1-|z|^2)^{q-2+s\lambda}\,dA(z)\\
&= (1-|a|^2)^{2-q}\int_{\mathbb{D}} |f\circ\varphi_a(w)-f(a)|^{p\lambda}(1-|\varphi_a(w)|^2)^{q-2+s\lambda}\,dA(w)\\
&\lesssim (1-|a|^2)^{s\lambda}\int_{\mathbb{D}} |f\circ\varphi_a(w)-f(a)|^{p\lambda}(1-|w|^2)^{q-2+s\lambda}\,dA(w)\\
&\asymp (1-|a|^2)^{s\lambda}\int_{\mathbb{D}} |(f\circ\varphi_a)'(w)|^{p\lambda}(1-|w|^2)^{p\lambda+q-2+s\lambda}\,dA(w)\\
&= \int_{\mathbb{D}} |f'(z)|^{p\lambda}\frac{(1-|a|^2)^{q+2s\lambda}(1-|z|^2)^{p\lambda+q-2+s\lambda}}{|1-\bar{a}z|^{2q+2s\lambda}}\,dA(z).
\end{aligned}$$

If $0 < \lambda < 1$, then according to Hölder's inequality, we have

$$\begin{aligned}
I_2(a) \lesssim &\left(\int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{p-2+s}\,dA(z)\right)^{\lambda}\\
&\times\left(\int_{\mathbb{D}} \frac{(1-|a|^2)^{\frac{q+2s\lambda}{1-\lambda}}(1-|z|^2)^{\frac{2\lambda+q-2}{1-\lambda}}}{|1-\bar{a}z|^{\frac{2q+2s\lambda}{1-\lambda}}}\,dA(z)\right)^{1-\lambda}.
\end{aligned}$$

It follows from Lemma 2.3 that

$$(1-|a|^2)^{\frac{q+2s\lambda}{1-\lambda}}\int_{\mathbb{D}} \frac{(1-|z|^2)^{\frac{2\lambda+q-2}{1-\lambda}}}{|1-\bar{a}z|^{2+\frac{2\lambda+q-2}{1-\lambda}+\frac{q+2s\lambda}{1-\lambda}}}\,dA(z) \lesssim 1.$$

Consequently,

$$I_2(a) \lesssim \|f\|^{p\lambda}_{\mathscr{D}^p_{p-2+s}}.$$

If $\lambda = 1$, then

$$\begin{aligned}
I_2(a) &\lesssim \int_{\mathbb{D}} |f'(z)|^p\frac{(1-|a|^2)^{q+2s}(1-|z|^2)^{p+q-2+s}}{|1-\bar{a}z|^{2q+2s}}\,dA(z)\\
&\lesssim \int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{p-2+s}\,dA(z) \leqslant \|f\|^p_{\mathscr{D}^p_{p-2+s}}.
\end{aligned}$$

Combining the estimates $I_1(a)$ and $I_2(a)$, we conclude that the inclusion mapping $i: \mathscr{D}^p_{p-2+s} \to T_{p\lambda,q}(\mu)$ is bounded and $\|i\|^{p\lambda} \lesssim \|\mu\|_{q+s\lambda}$.

THEOREM 3.2. *Let $1 < p < \infty$, $0 < s, \lambda \leqslant 1$ and $0 < q < \infty$ such that $q + s\lambda > 1$. Let μ be a positive Borel measure on $\mathbb{D}$. Then the inclusion mapping $i: \mathscr{D}^p_{p-2+s} \to T_{p\lambda,q}(\mu)$ is compact if and only if μ is a vanishing $(q+s\lambda)$-Carleson measure.*

Proof. Assume that $i: \mathscr{D}^p_{p-2+s} \to T_{p\lambda,q}(\mu)$ is compact. Given a sequence of arcs $\{I_n\}$ with $\lim_{n\to\infty} |I_n| = 0$. Denote the center of I_n by $e^{i\theta_n}$ and $a_n = (1-|I_n|)e^{i\theta_n}$. Let

$$f_n(z) := \frac{1-|a_n|^2}{(1-\bar{a}_n z)^{1+\frac{s}{p}}}, \quad z \in \mathbb{D}. \tag{2}$$

It is clear that $\{f_n\}$ is bounded in $\mathscr{D}^p_{p-2+s}$ and $\{f_n\}$ converges to zero uniformly on any compact subset of $\mathbb{D}$. Since

$$|f_n(z)| = \frac{1-|a_n|^2}{|1-\bar{a}_n z|^{1+\frac{s}{p}}} \asymp (1-|a_n|)^{-\frac{s}{p}} = |I_n|^{-\frac{s}{p}}, \quad z \in S(I_n),$$

we obtain

$$\frac{\mu(S(I_n))}{|I_n|^{q+s\lambda}} \asymp \frac{1}{|I_n|^q} \int_{S(I_n)} |f_n(z)|^{p\lambda} d\mu(z) \to 0, \quad n \to \infty.$$

By the arbitrariness of $\{I_n\}$, we deduce that μ is a vanishing $(q+s\lambda)$-Carleson measure.

Conversely, suppose that μ is a vanishing $(q+s\lambda)$-Carleson measure, then μ is also a $(q+s\lambda)$-Carleson measure and $\lim_{r\to 1^-} \|\mu - \mu_r\|_{q+s\lambda} = 0$ by Lemma 2.4. It follows from Theorem 3.1 that $i: \mathscr{D}^p_{p-2+s} \to T_{p\lambda,q}(\mu)$ is bounded. Let $\{f_n\}$ be a bounded sequence in $\mathscr{D}^p_{p-2+s}$ such that $\{f_n\}$ converges to zero uniformly on each compact subset of $\mathbb{D}$. We have

$$\begin{aligned}
\frac{1}{|I|^q} \int_{S(I)} |f_n(z)|^p d\mu(z) &\lesssim \frac{1}{|I|^q} \int_{S(I)} |f_n(z)|^{p\lambda} d\mu_r(z) + \frac{1}{|I|^q} \int_{S(I)} |f_n(z)|^{p\lambda} d(\mu-\mu_r)(z) \\
&\lesssim \frac{1}{|I|^q} \int_{S(I)} |f_n(z)|^{p\lambda} d\mu_r(z) + \|\mu-\mu_r\|_{q+s\lambda} \|f_n\|^{p\lambda}_{\mathscr{D}^p_{p-2+s}} \\
&\lesssim \frac{1}{|I|^q} \int_{S(I)} |f_n(z)|^{p\lambda} d\mu_r(z) + \|\mu-\mu_r\|_{q+s\lambda}.
\end{aligned}$$

Letting $n \to \infty$ and then $r \to 1$, we obtain $\lim_{n\to\infty} \frac{1}{|I|^q} \int_{S(I)} |f_n(z)|^p d\mu(z) = 0$. This shows that the inclusion mapping $i: \mathscr{D}^p_{p-2+s} \to T_{p\lambda,q}(\mu)$ is compact.

4. Boundedness of T_g, I_g and M_g

In the present section, via the embedding theorem (Theorem 3.1), we provide characterizations for the boundedness of Volterra integral operator T_g and its companion operator I_g from $\mathscr{D}^p_{p-2+s}$ to $F(p\lambda, p\lambda+s\lambda-2, q)$. Moreover, we study the multiplication operator M_g from $\mathscr{D}^p_{p-2+s}$ to $F(p\lambda, p\lambda+s\lambda-2, q)$.

THEOREM 4.1. *Let* $1<p<\infty$, $0<s,\lambda \leqslant 1$ *and* $0<q<\infty$ *such that* $p\lambda \geqslant 1$ *and* $q+s\lambda>1$. *If* $g \in \mathscr{H}(\mathbb{D})$, *then* $T_g : \mathscr{D}^p_{p-2+s} \to F(p\lambda, p\lambda+s\lambda-2, q)$ *is bounded if and only if* $g \in \mathscr{B}$. *Furthermore,* $\|T_g\| \asymp \|g\|_{\mathscr{B}}$.

Proof. Assume that $T_g : \mathscr{D}^p_{p-2+s} \to F(p\lambda, p\lambda+s\lambda-2, q)$ is bounded. For a fixed $a \in \mathbb{D}$, define f_a as in (1). Then $f_a \in \mathscr{D}^p_{p-2+s}$ with $\sup_{a\in\mathbb{D}} \|f_a\|_{\mathscr{D}^p_{p-2+s}} \lesssim 1$. So

$$\|T_g f_a\|_{F(p\lambda,p\lambda+s\lambda-2,q)} \leqslant \|T_g\|\ \|f_a\|_{\mathscr{D}^p_{p-2+s}} \lesssim \|T_g\|.$$

In addition, Lemma 4.12 of [34] gives

$$\begin{aligned}
\|T_g f_a\|^{p\lambda}_{F(p\lambda,p\lambda+s\lambda-2,q)} &\geqslant \int_{\mathbb{D}} |g'(z)|^{p\lambda} \frac{(1-|a|^2)^{p\lambda}}{|1-\bar{a}z|^{p\lambda+s\lambda}} (1-|z|^2)^{p\lambda-2+s\lambda}(1-|\varphi_a(z)|^2)^q dA(z) \\
&= \int_{\mathbb{D}} |g'(z)|^{p\lambda} \frac{(1-|a|^2)^{p\lambda+q}(1-|z|^2)^{p\lambda-2+s\lambda+q}}{|1-\bar{a}z|^{p\lambda+s\lambda+2q}} dA(z) \\
&= \int_{\mathbb{D}} |g'(\varphi_a(w))|^{p\lambda} \frac{(1-|a|^2)^{p\lambda}(1-|w|^2)^{p\lambda-2+s\lambda+q}}{|1-\bar{a}w|^{p\lambda+s\lambda}} dA(w) \\
&\gtrsim |g'(a)|^{p\lambda}(1-|a|^2)^{p\lambda}.
\end{aligned}$$

It follows that

$$|g'(a)|(1-|a|^2) \lesssim \|T_g f_a\|_{F(p\lambda,p\lambda+s\lambda-2,q)} \lesssim \|T_g\|.$$

Thus, $g \in \mathscr{B}$ and $\|g\|_{\mathscr{B}} \lesssim \|T_g\|$.

Now suppose $g \in \mathscr{B}$. Using the equivalent norm of Bloch function [33], we have

$$\begin{aligned}
\|g\|^{p\lambda}_{\mathscr{B}} &\asymp \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |g'(z)|^{p\lambda}(1-|z|^2)^{p\lambda-2}(1-|\varphi_a(z)|^2)^{q+s\lambda} dA(z) \\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |g'(z)|^{p\lambda}(1-|z|^2)^{p\lambda-2+s\lambda+q} \left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^{q+s\lambda} dA(z) \\
&\asymp \sup_{I\subset\partial\mathbb{D}} \frac{1}{|I|^{q+s\lambda}} \int_{S(I)} |g'(z)|^{p\lambda}(1-|z|^2)^{p\lambda-2+s\lambda+q} dA(z).
\end{aligned}$$

This means that $d\mu_g(z) = |g'(z)|^{p\lambda}(1-|z|^2)^{p\lambda-2+s\lambda+q}dA(z)$ is a $(q+s\lambda)$-Carleson measure and $\|\mu_g\|_{q+s\lambda} \asymp \|g\|^{p\lambda}_{\mathscr{B}}$. From Theorem 3.1, the inclusion mapping $i : \mathscr{D}^p_{p-2+s} \to T_{p\lambda,q}(\mu_g)$ is bounded. If $f \in \mathscr{D}^p_{p-2+s}$, then

$$\|T_g f\|^{p\lambda}_{F(p\lambda,p\lambda+s\lambda-2,q)} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^{p\lambda}|g'(z)|^{p\lambda}(1-|z|^2)^{p\lambda-2+s\lambda}(1-|\varphi_a(z)|^2)^q dA(z)$$

$$
\begin{aligned}
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^{p\lambda} |g'(z)|^{p\lambda} (1-|z|^2)^{p\lambda-2+s\lambda+q} \left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^q dA(z) \\
&\asymp \sup_{I\subset\partial\mathbb{D}} \frac{1}{|I|^q} \int_{S(I)} |f(z)|^{p\lambda} d\mu_g(z) = \|f\|^{p\lambda}_{T_{p\lambda,q}(\mu_g)} \\
&\lesssim \|\mu_g\|_{q+s\lambda} \|f\|^{p\lambda}_{\mathscr{D}^p_{p-2+s}} \asymp \|g\|^{p\lambda}_{\mathscr{B}} \|f\|^{p\lambda}_{\mathscr{D}^p_{p-2+s}}.
\end{aligned}
$$

As a consequence, $T_g : \mathscr{D}^p_{p-2+s} \to F(p\lambda, p\lambda+s\lambda-2, q)$ is bounded and $\|T_g\| \lesssim \|g\|_{\mathscr{B}}$.

THEOREM 4.2. *Let $1<p<\infty$, $0<s,\lambda \leqslant 1$ and $0<q<\infty$ such that $p\lambda \geqslant 1$ and $q+s\lambda>1$. If $g\in\mathscr{H}(\mathbb{D})$, then $I_g : \mathscr{D}^p_{p-2+s} \to F(p\lambda, p\lambda+s\lambda-2, q)$ is bounded if and only if $g\in\mathscr{H}^\infty$. Furthermore, $\|I_g\| \asymp \|g\|_{\mathscr{H}^\infty}$.*

Proof. Let $g\in\mathscr{H}^\infty$. Given $f\in\mathscr{D}^p_{p-2+s}$, for any $a\in\mathbb{D}$, let

$$
I(a) = \int_{\mathbb{D}} |g(z)|^{p\lambda} |f'(z)|^{p\lambda} (1-|z|^2)^{p\lambda-2+s\lambda} (1-|\varphi_a(z)|^2)^q dA(z).
$$

If $0<\lambda<1$, then Hölder's inequality gives

$$
\begin{aligned}
I(a) &\leqslant \|g\|^{p\lambda}_{\mathscr{H}^\infty} \int_{\mathbb{D}} |f'(z)|^{p\lambda} (1-|z|^2)^{p\lambda-2+s\lambda} (1-|\varphi_a(z)|^2)^q dA(z) \\
&\leqslant \|g\|^{p\lambda}_{\mathscr{H}^\infty} \left(\int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2+s} dA(z)\right)^\lambda \\
&\quad \times \left(\int_{\mathbb{D}} \frac{(1-|a|^2)^{\frac{q}{1-\lambda}} (1-|z|^2)^{\frac{2\lambda+q-2}{1-\lambda}}}{|1-\bar{a}z|^{\frac{2q}{1-\lambda}}} dA(z)\right)^{1-\lambda}.
\end{aligned}
$$

Set

$$
J(a) = \int_{\mathbb{D}} \frac{(1-|a|^2)^{\frac{q}{1-\lambda}} (1-|z|^2)^{\frac{2\lambda+q-2}{1-\lambda}}}{|1-\bar{a}z|^{\frac{2q}{1-\lambda}}} dA(z).
$$

It follows from Lemma 2.3 that

$$
J(a) = (1-|a|^2)^{\frac{q}{1-\lambda}} \int_{\mathbb{D}} \frac{(1-|z|^2)^{\frac{2\lambda+q-2}{1-\lambda}}}{|1-\bar{a}z|^{2+\frac{2\lambda+q-2}{1-\lambda}+\frac{q}{1-\lambda}}} dA(z) \lesssim 1. \tag{3}
$$

Consequently, $I(a) \lesssim \|g\|^{p\lambda}_{\mathscr{H}^\infty} \|f\|^{p\lambda}_{\mathscr{D}^p_{p-2+s}}$ and hence

$$
\|I_g f\|_{F(p\lambda, p\lambda+s\lambda-2, q)} \lesssim \|g\|_{\mathscr{H}^\infty} \|f\|_{\mathscr{D}^p_{p-2+s}}. \tag{4}
$$

If $\lambda=1$, then

$$
\|I_g f\|^p_{F(p\lambda, p\lambda+s\lambda-2, q)} \leqslant \|g\|^p_{\mathscr{H}^\infty} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2+s} (1-|\varphi_a(z)|^2)^q dA(z)
$$

$$\leqslant \|g\|_{\mathscr{H}^\infty}^p \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2+s} dA(z)$$
$$\leqslant \|g\|_{\mathscr{H}^\infty}^p \|f\|_{\mathscr{D}^p_{p-2+s}}^p.$$

So (4) is also true. We conclude that $I_g : \mathscr{D}^p_{p-2+s} \to F(p\lambda, p\lambda + s\lambda - 2, q)$ is bounded and $\|I_g\| \leqslant \|g\|_{\mathscr{H}^\infty}$.

Conversely, for a fixed $a \in \mathbb{D}$ with $|a| \geqslant \frac{1}{2}$, we define f_a as in (1). We know that $\sup_{a\in\mathbb{D}} \|f_a\|_{\mathscr{D}^p_{p-2+s}} \lesssim 1$ and hence

$$\|I_g f_a\|_{F(p\lambda,p\lambda+s\lambda-2,q)} \leqslant \|I_g\| \; \|f_a\|_{\mathscr{D}^p_{p-2+s}} \lesssim \|I_g\|.$$

Furthermore, Lemma 4.12 of [34] gives

$$\begin{aligned} \|I_g f_a\|^{p\lambda}_{F(p\lambda,p\lambda+s\lambda-2,q)} &\gtrsim \int_{\mathbb{D}} |g(z)|^{p\lambda} \frac{(1-|a|^2)^{p\lambda}}{|1-\bar{a}z|^{2p\lambda+s\lambda}} (1-|z|^2)^{p\lambda-2+s\lambda} (1-|\varphi_a(z)|^2)^q dA(z) \\ &= \int_{\mathbb{D}} |g\circ\varphi_a(w)|^{p\lambda} \frac{(1-|w|^2)^{p\lambda-2+s\lambda+q}}{|1-\bar{a}w|^{s\lambda}} dA(w) \\ &\gtrsim |g(a)|^{p\lambda}. \end{aligned}$$

Therefore, $|g(a)| \lesssim \|I_g\|$. By the choice of a, we deduce that $g \in \mathscr{H}^\infty$ and $\|g\|_{\mathscr{H}^\infty} \lesssim \|I_g\|$.

In the following, by using Theorems 4.1 and 4.2, we characterize the multipliers from $\mathscr{D}^p_{p-2+s}$ to $F(p\lambda, p\lambda+s\lambda-2,q)$.

THEOREM 4.3. *Let* $1 < p < \infty$, $0 < s, \lambda \leqslant 1$ *and* $0 < q < \infty$ *such that* $p\lambda \geqslant 1$ *and* $q + s\lambda > 1$. *Then* $M(\mathscr{D}^p_{p-2+s}, F(p\lambda, p\lambda+s\lambda-2,q)) = \mathscr{H}^\infty$.

Proof. Given $g \in \mathscr{H}^\infty$. It follows from Theorems 4.1 and 4.2 that both integral operators

$$T_g : \mathscr{D}^p_{p-2+s} \to F(p\lambda, p\lambda+s\lambda-2,q) \text{ and } I_g : \mathscr{D}^p_{p-2+s} \to F(p\lambda, p\lambda+s\lambda-2,q)$$

are bounded. So $M_g : \mathscr{D}^p_{p-2+s} \to F(p\lambda, p\lambda+s\lambda-2,q)$ is bounded.

Conversely, let $f \in F(p\lambda, p\lambda+s\lambda-2,q)$ and $a \in \mathbb{D}$. From Lemma 4.12 of [34], we have

$$\begin{aligned} \|f\|^{p\lambda}_{F(p\lambda,p\lambda+s\lambda-2,q)} &\geqslant \int_{\mathbb{D}} |f'(z)|^{p\lambda} (1-|z|^2)^{p\lambda-2+s\lambda} (1-|\varphi_a(z)|^2)^q dA(z) \\ &= \int_{\mathbb{D}} |f'(\varphi_a(w))|^{p\lambda} \frac{(1-|a|^2)^{p\lambda+s\lambda}(1-|w|^2)^{p\lambda-2+s\lambda+q}}{|1-\bar{a}w|^{2p\lambda+2s\lambda}} dA(w) \\ &\gtrsim (1-|a|^2)^{p\lambda+s\lambda} |f'(a)|^{p\lambda}. \end{aligned}$$

Namely,

$$|f'(a)| \lesssim \frac{\|f\|_{F(p\lambda,p\lambda+s\lambda-2,q)}}{(1-|a|^2)^{1+\frac{s}{p}}}.$$

Since a is arbitrary, we get

$$|f(a)| \lesssim \frac{\|f\|_{F(p\lambda,p\lambda+s\lambda-2,q)}}{(1-|a|^2)^{\frac{s}{p}}}.$$

For any $a\in\mathbb{D}$, let f_a be defined as in (1). Then $\{f_a\}$ is bounded in $\mathscr{D}^p_{p-2+s}$. It follows that $M_g f_a\in F(p\lambda,p\lambda+s\lambda-2,q)$ and then

$$|M_g f_a(z)| \lesssim \frac{\|M_g f_a\|_{F(p\lambda,p\lambda+s\lambda-2,q)}}{(1-|z|^2)^{\frac{s}{p}}} \lesssim \frac{\|M_g\|\;\|f_a\|_{\mathscr{D}^p_{p-2+s}}}{(1-|z|^2)^{\frac{s}{p}}} \lesssim \frac{\|M_g\|}{(1-|z|^2)^{\frac{s}{p}}}.$$

As a consequence,

$$\left|\frac{1-|a|^2}{(1-\bar{a}z)^{1+\frac{s}{p}}}g(z)\right| \lesssim \frac{\|M_g\|}{(1-|z|^2)^{\frac{s}{p}}}.$$

Taking $z=a$, we obtain $|g(a)|\lesssim\|M_g\|$. By the arbitrariness of $a\in\mathbb{D}$, we conclude that $g\in\mathscr{H}^\infty$ and $\|g\|_{\mathscr{H}^\infty}\lesssim\|M_g\|$.

5. Essential norm of T_g and I_g

In this section, we discuss the essential norm of T_g and I_g from $\mathscr{D}^p_{p-2+s}$ to $F(p\lambda,p\lambda+s\lambda-2,q)$. We start by recalling some related definitions and notations. Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be Banach spaces and $T:X\to Y$ be a bounded linear operator. The essential norm of $T:X\to Y$, denoted by $\|T\|_e$, is defined by

$$\|T\|_e := \inf_K\{\|T-K\|_{X\to Y} : K \text{ is compact from } X \text{ to } Y\}.$$

It is not difficult to check that $T:X\to Y$ is compact if and only if $\|T\|_e=0$. So the estimation of $\|T\|_e$ gives the requirement for T to be compact. Let Z be a closed subspace of X. Given $f\in X$, the distance from f to Z, denoted by $\text{dist}_X(f,Z)$, is defined by

$$\text{dist}_X(f,Z) := \inf_{g\in Z}\|f-g\|_X.$$

The following lemma gives the distance from the Bloch function to the little Bloch space, see [3, 30].

LEMMA 5.1. If $f\in\mathscr{B}$, then

$$\limsup_{|z|\to1^-}(1-|z|^2)|f'(z)| \asymp \text{dist}_{\mathscr{B}}(f,\mathscr{B}_0) \asymp \limsup_{r\to1^-}\|f-f_r\|_{\mathscr{B}}.$$

Here $f_r(z)=f(rz),\ 0<r<1, z\in\mathbb{D}$.

To give the essential norm of T_g from $\mathscr{D}^p_{p-2+s}$ to $F(p\lambda,p\lambda+s\lambda-2,q)$, we need the following lemma.

LEMMA 5.2. Let $1 < p < \infty$, $0 < s, \lambda \leqslant 1$ and $0 < q < \infty$ such that $p\lambda \geqslant 1$ and $q + s\lambda > 1$. If $0 < r < 1$ and $g \in \mathscr{B}$, then $T_{g_r} : \mathscr{D}^p_{p-2+s} \to F(p\lambda, p\lambda + s\lambda - 2, q)$ is compact.

Proof. Given $\{f_n\} \subset \mathscr{D}^p_{p-2+s}$ such that $\{f_n\}$ converges to zero uniformly on any compact subset of $\mathbb{D}$ and $\sup_n \|f_n\|_{\mathscr{D}^p_{p-2+s}} \leqslant 1$. For each $a \in \mathbb{D}$, let

$$I(a) = \int_{\mathbb{D}} |f_n(z)|^{p\lambda}(1-|z|^2)^{p\lambda-2+s\lambda}(1-|\varphi_a(z)|^2)^q dA(z).$$

If $0 < \lambda < 1$, then by Hölder's inequality and (3),

$$\begin{aligned} I(a) &\leqslant \left(\int_{\mathbb{D}} |f_n(z)|^p (1-|z|^2)^{p-2+s} dA(z) \right)^{\lambda} \\ &\quad \times \left((1-|a|^2)^{\frac{q}{1-\lambda}} \int_{\mathbb{D}} \frac{(1-|z|^2)^{\frac{2\lambda+q-2}{1-\lambda}}}{|1-\bar{a}z|^{\frac{2q}{1-\lambda}}} dA(z) \right)^{1-\lambda} \\ &\lesssim \left(\int_{\mathbb{D}} |f_n(z)|^p (1-|z|^2)^{p-2+s} dA(z) \right)^{\lambda}. \end{aligned}$$

Since $g \in \mathscr{B}$, we get $|g_r'(z)| \lesssim \frac{\|g\|_{\mathscr{B}}}{1-r^2}$, $z \in \mathbb{D}$. It follows that

$$\begin{aligned} \|T_{g_r} f_n\|^{p\lambda}_{F(p\lambda, p\lambda+s\lambda-2, q)} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_n(z)|^{p\lambda} |g_r'(z)|^{p\lambda} (1-|z|^2)^{p\lambda-2+s\lambda} (1-|\varphi_a(z)|^2)^q dA(z) \\ &\lesssim \frac{\|g\|^{p\lambda}_{\mathscr{B}}}{(1-r^2)^{p\lambda}} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_n(z)|^{p\lambda} (1-|z|^2)^{p\lambda-2+s\lambda} (1-|\varphi_a(z)|^2)^q dA(z) \\ &\lesssim \frac{\|g\|^{p\lambda}_{\mathscr{B}}}{(1-r^2)^{p\lambda}} \left(\int_{\mathbb{D}} |f_n(z)|^p (1-|z|^2)^{p-2+s} dA(z) \right)^{\lambda}. \end{aligned}$$

If $\lambda = 1$, similarly we have

$$\begin{aligned} \|T_{g_r} f_n\|^{p}_{F(p\lambda, p\lambda+s\lambda-2, q)} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_n(z)|^{p} |g_r'(z)|^{p} (1-|z|^2)^{p-2+s} (1-|\varphi_a(z)|^2)^q dA(z) \\ &\lesssim \frac{\|g\|^{p}_{\mathscr{B}}}{(1-r^2)^{p}} \int_{\mathbb{D}} |f_n(z)|^p (1-|z|^2)^{p-2+s} dA(z). \end{aligned}$$

Since

$$|f_n(z)|^p (1-|z|^2)^{p-2+s} \lesssim \|f_n\|^p_{\mathscr{D}^p_{p-2+s}} (1-|z|^2)^{p-2} \lesssim (1-|z|^2)^{p-2}$$

and $\int_{\mathbb{D}} (1-|z|^2)^{p-2} dA(z) < \infty$, applying the Dominated Convergence Theorem we get

$$\lim_{n\to\infty} \int_{\mathbb{D}} |f_n(z)|^p (1-|z|^2)^{p-2+s} dA(z) = \int_{\mathbb{D}} \lim_{n\to\infty} |f_n(z)|^p (1-|z|^2)^{p-2+s} dA(z) = 0,$$

which implies that $\lim_{n\to\infty}\|T_{g_r}f_n\|^p_{F(p\lambda,p\lambda+s\lambda-2,q)}=0$. Hence $T_{g_r}:\mathscr{D}^p_{p-2+s}\to F(p\lambda,p\lambda+s\lambda-2,q)$ is compact, as desired.

The following result is an important tool to study the essential norm of operators on some analytic function spaces, see [29].

LEMMA 5.3. Let X,Y be two Banach spaces of analytic functions on $\mathbb{D}$. Suppose that:

(1) The point evaluation functionals on Y are continuous.

(2) The closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets.

(3) $T:X\to Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets.

Then, T is a compact operator if and only if for any bounded sequence $\{f_n\}$ in X such that $\{f_n\}$ converges to zero uniformly on every compact set of $\mathbb{D}$, then the sequence $\{Tf_n\}$ converges to zero in the norm of Y.

The following result provide the estimation of the essential norm of T_g from $\mathscr{D}^p_{p-2+s}$ to $F(p\lambda,p\lambda+s\lambda-2,q)$.

THEOREM 5.1. *Let $1<p<\infty$, $0<s,\lambda\leqslant 1$ and $0<q<\infty$ such that $p\lambda\geqslant 1$ and $q+s\lambda>1$. If $g\in\mathscr{H}(\mathbb{D})$ and $T_g:\mathscr{D}^p_{p-2+s}\to F(p\lambda,p\lambda+s\lambda-2,q)$ is bounded, then*

$$\|T_g\|_e\asymp\limsup_{|z|\to 1^-}(1-|z|^2)|g'(z)|\asymp \mathrm{dist}_{\mathscr{B}}(g,\mathscr{B}_0).$$

Proof. Let $\{a_n\}$ be a sequence in $\mathbb{D}$ such that $\lim_{n\to\infty}|a_n|=1$. For each n, let f_n be defined as in (2). Then $\{f_n\}$ is bounded in $\mathscr{D}^p_{p-2+s}$. Furthermore, $\{f_n\}$ converges to zero uniformly on every compact subset of $\mathbb{D}$. Given a compact operator $K:\mathscr{D}^p_{p-2+s}\to F(p\lambda,p\lambda+s\lambda-2,q)$, by Lemma 5.3 we have $\lim_{n\to\infty}\|Kf_n\|_{F(p\lambda,p\lambda+s\lambda-2,q)}=0$. So

$$\begin{aligned}\|T_g-K\|&\gtrsim\limsup_{n\to\infty}\|(T_g-K)f_n\|_{F(p\lambda,p\lambda+s\lambda-2,q)}\\&\gtrsim\limsup_{n\to\infty}\Big(\|T_gf_n\|_{F(p\lambda,p\lambda+s\lambda-2,q)}-\|Kf_n\|_{F(p\lambda,p\lambda+s\lambda-2,q)}\Big)\\&=\limsup_{n\to\infty}\|T_gf_n\|_{F(p\lambda,p\lambda+s\lambda-2,q)}\\&\geqslant\limsup_{n\to\infty}\left(\int_{\mathbb{D}}|f_n(z)|^{p\lambda}|g'(z)|^{p\lambda}(1-|z|^2)^{p\lambda-2+s\lambda}(1-|\varphi_{a_n}(z)|^2)^q dA(z)\right)^{\frac{1}{p\lambda}}\\&\gtrsim\limsup_{n\to\infty}(1-|a_n|^2)|g'(a_n)|,\end{aligned}$$

and hence

$$\|T_g\|_e \gtrsim \limsup_{n\to\infty}(1-|a_n|^2)|g'(a_n)|.$$

It follows from Lemma 5.1 and the arbitrariness of $\{a_n\}$ that

$$\|T_g\|_e \gtrsim \limsup_{|z|\to 1^-}(1-|z|^2)|g'(z)| \asymp \mathrm{dist}_{\mathscr{B}}(g,\mathscr{B}_0).$$

On the other hand, by Lemma 5.2, $T_{g_r}:\mathscr{D}^p_{p-2+s}\to F(p\lambda,p\lambda+s\lambda-2,q)$ is compact. Then

$$\|T_g\|_e \leqslant \|T_g - T_{g_r}\| = \|T_{g-g_r}\| \asymp \|g-g_r\|_{\mathscr{B}}.$$

Using Lemma 5.1 again, we have

$$\|T_g\|_e \lesssim \limsup_{r\to 1^-}\|g-g_r\|_{\mathscr{B}} \asymp \mathrm{dist}_{\mathscr{B}}(g,\mathscr{B}_0).$$

The proof is complete.

By Theorem 5.1 we easily get the following corollary.

COROLLARY 5.1. *Let $1<p<\infty$, $0<s,\lambda\leqslant 1$ and $0<q<\infty$ such that $p\lambda\geqslant 1$ and $q+s\lambda>1$. If $g\in\mathscr{H}(\mathbb{D})$, then $T_g:\mathscr{D}^p_{p-2+s}\to F(p\lambda,p\lambda+s\lambda-2,q)$ is compact if and only if $g\in\mathscr{B}_0$.*

We next estimate the essential norm of I_g from $\mathscr{D}^p_{p-2+s}$ to $F(p\lambda,p\lambda+s\lambda-2,q)$.

THEOREM 5.2. *Let $1<p<\infty$, $0<s,\lambda\leqslant 1$ and $0<q<\infty$ such that $p\lambda\geqslant 1$ and $q+s\lambda>1$. If $g\in\mathscr{H}(\mathbb{D})$ and $I_g:\mathscr{D}^p_{p-2+s}\to F(p\lambda,p\lambda+s\lambda-2,q)$ is bounded, then*

$$\|I_g\|_e \asymp \|g\|_{\mathscr{H}^\infty}.$$

Proof. Let $\{a_n\}$, $\{f_n\}$ and K be defined as in the proof of Theorem 5.1. Since $K:\mathscr{D}^p_{p-2+s}\to F(p\lambda,p\lambda+s\lambda-2,q)$ is compact, we get $\lim_{n\to\infty}\|Kf_n\|_{F(p\lambda,p\lambda+s\lambda-2,q)}=0$ by Lemma 5.3. Hence,

$$\begin{aligned}\|I_g-K\| &\gtrsim \limsup_{n\to\infty}\|(I_g-K)f_n\|_{F(p\lambda,p\lambda+s\lambda-2,q)}\\ &\gtrsim \limsup_{n\to\infty}\Big(\|I_gf_n\|_{F(p\lambda,p\lambda+s\lambda-2,q)}-\|Kf_n\|_{F(p\lambda,p\lambda+s\lambda-2,q)}\Big)\\ &=\limsup_{n\to\infty}\|I_gf_n\|_{F(p\lambda,p\lambda+s\lambda-2,q)}.\end{aligned}$$

Therefore,

$$\|I_g\|_e \gtrsim \limsup_{n\to\infty}\|I_gf_n\|_{F(p\lambda,p\lambda+s\lambda-2,q)}.$$

Similar argument as in the proof of Theorem 4.2 shows that

$$\begin{aligned}
&\|I_g f_n\|^{p\lambda}_{F(p\lambda,p\lambda+s\lambda-2,q)} \\
&\gtrsim \sup_{b\in\mathbb{D}} \int_{\mathbb{D}} |g(z)|^{p\lambda} \frac{(1-|a_n|^2)^{p\lambda}}{|1-\bar{a_n}z|^{2p\lambda+s\lambda}} (1-|z|^2)^{p\lambda-2+s\lambda}(1-|\varphi_b(z)|^2)^q dA(z) \\
&\gtrsim \int_{\mathbb{D}} |g(z)|^{p\lambda} \frac{(1-|a_n|^2)^{p\lambda}}{|1-\bar{a_n}z|^{2p\lambda+s\lambda}} (1-|z|^2)^{p\lambda-2+s\lambda}(1-|\varphi_{a_n}(z)|^2)^q dA(z) \\
&= \int_{\mathbb{D}} |g\circ\varphi_{a_n}(w)|^{p\lambda} \frac{(1-|w|^2)^{p\lambda-2+s\lambda+q}}{|1-\bar{a_n}w|^{s\lambda}} dA(w) \gtrsim |g(a_n)|^{p\lambda},
\end{aligned}$$

which implies that $\|I_g\|_e \gtrsim \|g\|_{\mathscr{H}^\infty}$.

On the other hand, Theorem 4.2 gives

$$\|I_g\|_e = \inf_K \|I_g - K\| \leqslant \|I_g\| \lesssim \|g\|_{\mathscr{H}^\infty}.$$

The proof is complete.

COROLLARY 5.2. *Let $1<p<\infty$, $0<s,\lambda\leqslant 1$ and $0<q<\infty$ such that $p\lambda\geqslant 1$ and $q+s\lambda>1$. If $g\in\mathscr{H}(\mathbb{D})$, then $I_g : \mathscr{D}^p_{p-2+s} \to F(p\lambda, p\lambda+s\lambda-2, q)$ is compact if and only if $g=0$.*

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