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PERTURBATION BOUNDS FOR MATRIX FUNCTIONS

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Abstract. In this article, we present some bounds for $|||f(A)-f(B)|||$, where f is a real function and is continuously differentiable on an open interval J, $|||\cdot|||$ is a unitarily invariant norm, and A,B are Hermitian matrices such that the eigenvalues of A and B are in $[\alpha,\beta]\subset J$. Also, we illustrate upper bounds for $\|f(A)-f(B)\|$ for special functions f and norms $\|\cdot\|$.

1. Introduction

Suppose that $\mathscr{M}_{m,n}$ denote the set of all $m\times n$ complex matrices. We indicate $\mathscr{M}_{n,n}$ by $\mathscr{M}_n$. We use notations $\mathscr{H}_n$ and $\mathscr{U}_n$ for the set of all Hermitian matrices and unitary matrices in $\mathscr{M}_n$, respectively. For $A\in\mathscr{M}_n$, matrix A^* denote the conjugate transpose of the matrix A. The symbol I denotes the identity matrix in $\mathscr{M}_n$. For $A,B\in\mathscr{M}_n$, we use notation $A\circ B$ for Schur product of matrices A and B. Let $A\in\mathscr{H}_n$ and $\sigma(A)\subseteq[\alpha,\beta]$. If $A=U^*DU$ is the spectral decomposition of the Hermitian matrix A and f is a complex function on $[\alpha,\beta]$, then we define $f(A):=U^*f(D)U$ (for more details see [8]). For a matrix $A\in\mathscr{H}_n$, we write $A\geqslant 0$ ($A>0$), if A is a positive semi-definite (definite) matrix. For two Hermitian matrices A and B, the notation $A\geqslant B$ ($A>B$) means that $A-B\geqslant 0$ ($A-B>0$). We define a matrix interval by $[A,B]=\{X\in\mathscr{H}_n|A\leqslant X\leqslant B\}$ and $(A,B)=\{X\in\mathscr{H}_n|A<X<B\}$. Suppose that f is a real function. We say that f is an operator monotone, if $f(A)\geqslant f(B)$, whenever $A\geqslant B$. Let J be an open interval in $\mathbb{R}$. We write $f\in\mathscr{C}^1(J)$, if real function f is continuously differentiable on J. Let $A\in\mathscr{M}_{m,n}$ and $m\leqslant n$. We denote i^{th} singular value of the matrix A by $s_i(A)$, $1\leqslant i\leqslant m$. We say that a norm $\|\cdot\|$ is matrix norm on $\mathscr{M}_n$, if $\|AB\|\leqslant\|A\|\|B\|$. A norm $|||\cdot|||$ on $\mathscr{M}_{m,n}$ is called unitarily invariant norm if $|||A|||=|||UAV^*|||$, for all $A\in\mathscr{M}_{m,n}$ and unitary matrices $U\in\mathscr{U}_m$ and $V\in\mathscr{U}_n$. Also a norm $\|\cdot\|$ on $\mathscr{M}_n$ is said to be unitary similarity invariant if $\|A\|=\|UAU^*\|$, for all $A\in\mathscr{M}_n$ and all unitary matrices $U\in\mathscr{U}_n$. Let $A=(a_{ij})\in\mathscr{M}_{m,n}$. Define $\|A\|_2:=s_1(A)$, $\|A\|_F^2:=\sum_{i,j}|a_{ij}|^2$ and indicate

$$\|A\|_1:=\max_j\sum_i|a_{ij}|,\quad \|A\|_\infty:=\max_i\sum_j|a_{ij}|.$$

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The Schur product or the Hadamard product of two matrices A and B is defined to be the matrix $A \circ B$ whose (i,j)-entry is $a_{ij}b_{ij}$ [9].

One of the problems in perturbation theory is to find a bound for $\|f(A)-f(B)\|$ in terms of $\|A-B\|$. For example, consider the differential equation

$$\frac{d^2y}{dt^2} + Ay = 0 \ (t>0), \ \ y(0) = y_0, \ \ y'(0) = y_0', \tag{1.1}$$

where A is a Hermitian positive definite matrix and $y_0, y_0' \in \mathbb{C}^n$ [8, p. 36]. We know that for all $t>0$, the matrix function

$$y(t) = \cos(\sqrt{A}t)y_0 + (\sqrt{A})^{-1}\sin(\sqrt{A}t)y_0', \tag{1.2}$$

is the solution of equation (1.1). Suppose that nonsingular matrix $\tilde{A}$ is an approximation to the matrix A and let $\tilde{y}(t)$ be the solution of equation (1.1) with the matrix $\tilde{A}$. If $A, \tilde{A} \in [\alpha I, \beta I]$ $(\alpha > 0)$, we want to obtain a bound for $\|y(t)-\tilde{y}(t)\|$ in terms of $\|A-\tilde{A}\|$.

The perturbation bounds for several matrices, special matrix norms, and operator functions have been obtained by many authors. For example Bhatia et al. [3] considered the exponential function and the power functions $f(x) = x^p$, $-\infty < p < \infty$, on the Hilbert space $\mathbf{H}$. Loan in [12] studied $f(x) = e^{xt}$ on $\mathscr{M}_n$. Hemmen and Ando [7] proved that if A, B are positive definite and $A+B \geqslant cI$ for some $c > 0$ and f is a matrix monotone increasing function on $[0,\infty)$, then $|||f(A)-f(B)||| \leqslant \left(\frac{f(c/2)-f(0)}{c/2}\right) |||A-B|||$. Gil in [5, Theorem 1.1] for diagonalizable matrices $A, B \in \mathscr{M}_n$ obtained a bound for $\|f(A)-f(B)\|_F$. A bound for $\|f(A)-f(B)\|_F$ with arbitrary matrices and functions regular on the closed convex hull of the spectra has been derived in [6, Lemma 2.1] and has been generalized to infinite dimensional operators in [4, Chapter 13].

In this paper, we will find some bounds for $|||A \circ B|||$, where $A, B \in \mathscr{M}_{m,n}$ and $|||\cdot|||$ is a unitarily invariant norm and then we will present some bounds for $|||f(A)-f(B)|||$, where $A, B \in \mathscr{H}_n$ and f is a real function. Also, some special cases are considered.

2. Bounds for $|||\mathbf{A} \circ \mathbf{B}|||$

Let $A, B \in \mathscr{M}_n$ and $|||\cdot|||$ be a unitarily invariant norm on $\mathscr{M}_n$. At the first, we obtain some bounds for $|||A \circ B|||$. Then, for some special norms, we present a better bound for $\|A \circ B\|$.

DEFINITION 1. Let $A = (a_{ij}) \in \mathscr{H}_n$ be a Hermitian matrix. We indicate $d_i(A)$ for i^{th} entry of decreasingly ordered diagonal entries of the matrix A. So $d_1(A) \geqslant d_2(A) \geqslant \cdots \geqslant d_n(A)$. Let $A \in \mathscr{M}_{m,n}$ and $|A| := (A^*A)^{\frac{1}{2}}$ be absolute value of the matrix A. Then

$$r_i(A) := \left(d_i(|A^*|^2)\right)^{\frac{1}{2}}, \ p_i(A) := d_i(|A^*|); \ 1 \leqslant i \leqslant m,$$

$$c_j(A) := \left(d_j(|A|^2)\right)^{\frac{1}{2}}, \ q_j(A) := d_j(|A|); \ 1 \leqslant j \leqslant n.$$

Suppose that $A = USV^* \in \mathscr{M}_{m,n}$, where $m \leqslant n$, is the singular value decomposition of A and let $\alpha = [\alpha_1, \alpha_2, \cdots, \alpha_m]^T \in \mathbb{R}^m$. Define

$$P(A)^{\alpha} := US^{\alpha}U^* := U diag\left(s_1(A)^{\alpha_1}, s_2(A)^{\alpha_2}, \cdots, s_m(A)^{\alpha_m}\right)U^*,$$
$$Q(A)^{\alpha} := VS^{\alpha}V^* := V diag\left(s_1(A)^{\alpha_1}, s_2(A)^{\alpha_2}, \cdots, s_m(A)^{\alpha_m}, 0, \cdots, 0\right)V^*.$$

To remove the ambiguities, we define $0^{\beta} := 0$, for all $\beta \in \mathbb{R}^n$. Suppose that

$$p_i(A, \alpha) := d_i\left(P(A)^{2\alpha}\right), \quad i = 1, 2, \cdots, m,$$
$$q_j(A, \alpha) := d_j\left(Q(A)^{2\alpha}\right), \quad j = 1, 2, \cdots, n.$$

Also, $t_i(A, \alpha) := (p_i(A, \alpha) q_i(A, e - \alpha))^{\frac{1}{2}}$, $i = 1, 2, \cdots, m$, where $e = [1, 1, \cdots, 1]^T \in \mathbb{R}^m$.

The class of Ky Fan k-norms defined as

$$\|A\|_{(K)} = \sum_{j=1}^{k} s_j(A), \quad 1 \leqslant k \leqslant n. \tag{2.1}$$

LEMMA 2.1. [2, Theorem IV.2.2] *(Fan Dominance Theorem) Let A, B be two $n \times n$ matrices. If*

$$\|A\|_{(K)} \leqslant \|B\|_{(K)} \ \textit{for}\ 1 \leqslant k \leqslant n,$$

then

$$|||A||| \leqslant |||B||| \ \textit{ for all unitarily invariant norms.}$$

In the next, we obtain some bounds for $|||A \circ B|||$, where $|||\cdot|||$ is a unitarily invariant norm on $\mathscr{M}_{m,n}$.

THEOREM 2.2. *For every unitarily invariant norm $|||\cdot|||$ and $A, B \in \mathscr{M}_{m,n}$, with $m \leqslant n$, we have*

$$|||A \circ B||| \leqslant \inf_{X,Y,\ X^*Y=A} (c_1(X)c_1(Y))\, |||B||| \leqslant \inf_{\alpha \in \mathbb{R}^m} t_1(A, \alpha)\, |||B|||.$$

Proof. Let $m \leqslant n$ and $A = X^*Y$. By [9, Theorem 5.6.2], for $k = 1, 2, \cdots, m$, we have

$$\sum_{i=1}^{k} s_i(A \circ B) \leqslant \sum_{i=1}^{k} c_i(X)c_i(Y)s_i(B) \leqslant (c_1(X)c_1(Y)) \sum_{i=1}^{k} s_i(B).$$

Therefore, by the Fan dominance theorem, for all unitarily invariant norm $|||\cdot|||$, we have $|||A \circ B||| \leqslant c_1(X)c_1(Y)\, |||B|||$ and hence

$$|||A \circ B||| \leqslant \inf_{X,Y,\ X^*Y=A} (c_1(X)c_1(Y))\, |||B|||.$$

For the second inequality, let $\alpha \in \mathbb{R}^m$ and $A = USV^*$ be the singular value decomposition of the matrix A. Choose $X = S^{\alpha}U^*$ and $Y = S^{e-\alpha}V^*$. Hence,

$$c_1(X)c_1(Y) = (p_1(A,\alpha)\,q_1(A,e-\alpha))^{\frac{1}{2}} = t_1(A,\alpha)$$

and so

$$\inf_{X,Y,\; X^*Y=A} c_1(X)c_1(Y) \leqslant \inf_{\alpha\in\mathbb{R}^m} t_1(A,\alpha). \tag{2.2}$$

Therefore proof is completed. □

COROLLARY 2.3. *For all $A,B \in \mathcal{M}_{m,n}$ and unitarily invariant norm $|||\cdot|||$, we have*

$$|||A\circ B||| \leqslant (p_1(A)\,q_1(A))^{\frac{1}{2}}\,|||B||| \leqslant (r_1(A)\,c_1(A))^{\frac{1}{2}}\,|||B||| \leqslant s_1(A)\,|||B|||, \tag{2.3}$$

$$|||A\circ B||| \leqslant \min\{p_1(A),q_1(A)\}\,|||B||| \leqslant \min\{c_1(A),r_1(A)\}\,|||B|||. \tag{2.4}$$

Proof. Since $t_1(A,\frac{1}{2}e) = (p_1(A)\,q_1(A))^{\frac{1}{2}}$, by Theorem 2.2, we have

$$|||A\circ B||| \leqslant (p_1(A)\,q_1(A))^{\frac{1}{2}}\,|||B|||.$$

We know that $p_1(A) \leqslant r_1(A)$ and $q_1(A) \leqslant c_1(A)$ (See [9, p. 342]. Also, we have $\max\{c_1(A),r_1(A)\} \leqslant s_1(A)$. Therefore (2.3) is proved.

By choosing $\alpha = e$ and $\alpha = 0$ in Theorem 2.2, we obtain the inequality (2.4). □

If A is a Hermitian matrix, then $p_1(A) = q_1(A)$ and $r_1(A) = c_1(A)$. Moreover, if $A \geqslant 0$, then $p_1(A) = q_1(A) = d_1(A)$ and so we have the following corollary:

COROLLARY 2.4. [1, page 59] *If $A,B \in \mathcal{M}_n$ and $A \geqslant 0$, then*

$$|||A\circ B||| \leqslant d_1(A)\,|||B||| = \max_i\{a_{ii}\}\,|||B|||. \tag{2.5}$$

REMARK 1. Let $\|\cdot\|$ be one of the norms $\|\cdot\|_1$, $\|\cdot\|_\infty$, or $\|\cdot\|_F$. Then, for all $A,B \in \mathcal{M}_{m,n}$, we have

$$\|A\circ B\| = \left\|(a_{ij}b_{ij})\right\| \leqslant \max_{i,j}|a_{ij}|\,\|B\|. \tag{2.6}$$

The inequality (2.6) is not true for every unitarily invariant norm $|||\cdot|||$. For example, let $A = \begin{bmatrix} 28 & 100 \\ 100 & 102 \end{bmatrix}$ and $B = \begin{bmatrix} 17 & 33 \\ -30 & 116 \end{bmatrix}$. Therefore,

$$\max_{i,j}|a_{ij}|\,\|B\|_2 = 12557 < \|A\circ B\|_2 = 12593.$$

3. Bounds for $\|\mathbf{f}(\mathbf{A})-\mathbf{f}(\mathbf{B})\|$

Let $f \in \mathscr{C}^1(J)$ and $D = diag\,(d_1, d_2, \cdots, d_n)$, $d_i \in J$. We denote the first divided differences of f at D by $f^{[1]}(D)$ as $\left(f^{[1]}(D)\right)_{i,j} := f^{[1]}(d_i, d_j)$, where:

$$f^{[1]}(d_i, d_j) := \begin{cases} f'(d_i), & d_i = d_j \\ \frac{f(d_i) - f(d_j)}{d_i - d_j}, & d_i \neq d_j. \end{cases}$$

Let $A \in \mathscr{H}_n$ and $A = U^*DU$, where $U \in \mathscr{U}_n$ and D is a diagonal matrix. We define $f^{[1]}(A) = U^* f^{[1]}(D) U$. The map f is called (Frechet) differentiable at A if there exists a linear transformation $\mathscr{D}f(A)$ on $\mathscr{H}_n$ such that for all $H \in \mathscr{H}_n$

$$\|f(A+H) - f(A) - \mathscr{D}f(A)(H)\| = o(\|H\|).$$

The linear operator $\mathscr{D}f(A)$ is called the derivative of f at A. Now, in the following, we state the relationship between the derivative $\mathscr{D}f(A)$ and the matrix $f^{[1]}(A)$.

LEMMA 3.1. [2, Theorem V.3.3] *Let $f \in \mathscr{C}^1(J)$ and let A be a Hermitian matrix with all its eigenvalues in J. Then*

$$\mathscr{D}f(A)(H) = U\left(f^{[1]}(D) \circ U^*HU\right)U^*,$$

where $A = UDU^*$ is the spectral decomposition of A and $\circ$ denotes the Schur-product.

LEMMA 3.2. [2, Theorem X.4.5] *Let f be a differentiable map from a convex subset U of a Banach space X into a Banach space Y. Let $a, b \in U$ and let L be the line segment joining them. Then*

$$\|f(b) - f(a)\| \leqslant \sup_{u \in L} \|\mathscr{D}f(u)\| \|a - b\|.$$

Suppose that $A, B \in [\alpha I, \beta I]$ and $L_t := tA + (1-t)B$, for all $0 \leqslant t \leqslant 1$. Then $L_t \in [\alpha I, \beta I]$. Let

$$L_t = U_t D_t U_t^*, \text{ for all } 0 \leqslant t \leqslant 1, \tag{3.1}$$

where D_t and U_t are diagonal and unitary matrices, respectively. For a given $A \in M_n$, assume that S_A be the linear map on $\mathscr{M}_n$, where is defined by $S_A(Z) := A \circ Z$.

THEOREM 3.3. *Let $f \in \mathscr{C}^1(J)$ and $A, B \in [\alpha I, \beta I]$, where $[\alpha, \beta] \subset J$. Suppose that $\|\cdot\|$ is a unitary similarity invariant norm and $M := \sup_{0 \leqslant t \leqslant 1} \left\|S_{f^{[1]}(D_t)}\right\|$, where D_t is defined in (3.1). Then*

$$\|f(A) - f(B)\| \leqslant M\|A - B\|. \tag{3.2}$$

Moreover, if $|||\cdot|||$ is a unitarily invariant norm, then

$$\begin{aligned} |||f(A) - f(B)||| &\leqslant \sup_{0 \leqslant t \leqslant 1} \inf_{X_t, Y_t,\, X_t^* Y_t = f^{[1]}(D_t)} (c_1(X_t) c_1(Y_t))\, |||A - B||| \\ &\leqslant \sup_{0 \leqslant t \leqslant 1} \inf_{\alpha \in \mathbb{R}^m} t_1\left(f^{[1]}(D_t), \alpha\right) |||A - B|||. \end{aligned} \tag{3.3}$$

Proof. By Lemma 3.2,

$$\|f(A)-f(B)\| \leqslant \sup_{0\leqslant t\leqslant 1} \|\mathscr{D}f(L_t)\|\|A-B\|.$$

Using Lemma 3.1, for all $0\leqslant t\leqslant 1$, we have

$$\begin{aligned}\|\mathscr{D}f(L_t)\| &= \sup_{\|Z\|=1}\|\mathscr{D}f(L_t)(Z)\| = \sup_{\|Z\|=1}\|\mathscr{D}f(U_tD_tU_t^*)(Z)\| \\ &= \sup_{\|Z\|=1}\left\|U_t\left(f^{[1]}(D_t)\circ U_t^*ZU_t\right)U_t^*\right\| \\ &= \sup_{\|Z\|=1}\left\|f^{[1]}(D_t)\circ U_t^*ZU_t\right\| = \left\|S_{f^{[1]}(D_t)}\right\|.\end{aligned}$$

Therefore,

$$\|f(A)-f(B)\| \leqslant \sup_{0\leqslant t\leqslant 1}\left\|S_{f^{[1]}(D_t)}\right\|\|A-B\| = M\|A-B\|.$$

Now, let $|||\cdot|||$ be a unitarily invariant norm and $0\leqslant t\leqslant 1$. By using Theorem 2.2, for all $Z\in M_n$, we have

$$\left|\left|\left|f^{[1]}(D_t)\circ Z\right|\right|\right| \leqslant \inf_{X_t,Y_t,\ X_t^*Y_t=f^{[1]}(D_t)}(c_1(X_t)c_1(Y_t))\,|||Z|||.$$

Hence,

$$M=\sup_{0\leqslant t\leqslant 1}\left\|S_{f^{[1]}(D_t)}\right\| = \sup_{0\leqslant t\leqslant 1}\left|\left|\left|f^{[1]}(D_t)\circ Z\right|\right|\right| \leqslant \sup_{0\leqslant t\leqslant 1}\ \inf_{X_t,Y_t,\ X_t^*Y_t=f^{[1]}(D_t)}(c_1(X_t)c_1(Y_t)).$$

Therefore, the first inequality of (3.3) obtain by (3.2) and the second inequality of (3.3), obtain by relation (2.2) in Theorem 2.2. □

Let $0\leqslant t\leqslant 1$. Since the matrix $f^{[1]}(D_t)$ is a symmetric matrix, we obtain that $p_1(f^{[1]}(D_t)) = q_1(f^{[1]}(D_t))$ and $r_1(f^{[1]}(D_t)) = c_1(f^{[1]}(D_t))$. Since for all $A\in M_n$, we have $t_1(A,\frac{1}{2}e) = (p_1(A)q_1(A))^{\frac{1}{2}}$. Therefore by using (3.3) and (2.3), we have the following:

COROLLARY 3.4. *Let $f\in\mathscr{C}^1(J)$ and $A,B\in[\alpha I,\beta I]$, where $[\alpha,\beta]\subset J$. Suppose that D_t is the same as in (3.1). Then, for all unitarily invariant norm $|||\cdot|||$,*

$$\begin{aligned}|||f(A)-f(B)||| &\leqslant \sup_{0\leqslant t\leqslant 1} p_1\left(f^{[1]}(D_t)\right)|||A-B||| \leqslant \sup_{0\leqslant t\leqslant 1} r_1\left(f^{[1]}(D_t)\right)|||B||| \\ &\leqslant \sup_{0\leqslant t\leqslant 1} s_1\left(f^{[1]}(D_t)\right)|||A-B|||.\end{aligned}$$

Let Ω be the set of all unitary similarity invariant norms $\|\cdot\|$, such that $\|A\circ Z\|\leqslant d_1(A)\|Z\|$, whenever $A\geqslant 0$ and $Z\in\mathscr{M}_n$. By Corollary 2.4, all of the unitarily invariant norms are in Ω. In the next theorem, we present a bound for $\|f(A)-f(B)\|$, when $\|\cdot\|\in\Omega$ and f is an operator monotone.

COROLLARY 3.5. *Let f be an operator monotone on $[\alpha,\beta]$ and $A,B \in [\alpha I,\beta I]$ and $\|\cdot\| \in \Omega$. Then*

$$\|f(A)-f(B)\| \leqslant \max\{f'(\alpha),f'(\beta)\}\|A-B\|.$$

Proof. By using [2, Theorem V.3.6], we have $f \in \mathscr{C}^1(\alpha,\beta)$. Let $L_t = tA + (1-t)B$, $0 \leqslant t \leqslant 1$. Then $L_t \in [\alpha I,\beta I]$. Since f on $[\alpha,\beta]$ is operator monotone, by using [2, Theorem V.3.4], $f^{[1]}(L_t) \geqslant 0$. Hence $f^{[1]}(D_t) \geqslant 0$, where D_t is defined in (3.1). Since $\|\cdot\| \in \Omega$, by using (2.4), for all $Z \in M_n$ and $0 \leqslant t \leqslant 1$ we obtain that

$$\begin{aligned}\left\|f^{[1]}(D_t)\circ Z\right\| &\leqslant d_1\left(f^{[1]}(D_t)\right)\|Z\| \leqslant \max_{d_t\in\sigma(L_t)} f'(d_t)\|Z\| \\ &\leqslant \max_{\alpha\leqslant c\leqslant\beta} f'(c)\|Z\| = \max\left\{f'(\alpha),f'(\beta)\right\}\|Z\|.\end{aligned}$$

Therefore, for all $0 \leqslant t \leqslant 1$,

$$\left\|S_{f^{[1]}(D_t)}\right\| \leqslant \max\left\{f'(\alpha),f'(\beta)\right\}.$$

Hence

$$M = \sup_{0\leqslant t\leqslant 1}\left\|S_{f^{[1]}(D_t)}\right\| \leqslant \max\left\{f'(\alpha),f'(\beta)\right\}.$$

Using (3.2), proof is completed. □

REMARK 2. If f is an operator monotone on $[0,\infty)$ into itself, then by using [2, Theorem V.3.6], f on $[0,\infty)$ is continuously differentiable and by using [2, Theorem V.2.5], the operator f is concave and so $\max\{f'(\alpha),f'(\beta)\} = f'(\alpha)$. Therefore, by using Corollary 3.5, for all norm $\|\cdot\| \in \Omega$, we have

$$\|f(A)-f(B)\| \leqslant f'(\alpha)\|A-B\|,$$

where $A,B \geqslant \alpha I$, $\alpha > 0$ (see [2, Theorem X.3.8]).

Let Γ be the set of all unitary similarity invariant norms such that $\|S\circ Z\| \leqslant \max_{i,j}|s_{ij}|\|Z\|$, for all symmetric matrices $S \in M_n$ and $Z \in M_n$. By Remark 1, we see that $\|\cdot\|_F$, $\|\cdot\|_1$, and $\|\cdot\|_\infty$ are in Γ and $\|.\|_2$ is not in Γ.

In the following, we present a bound for $\|f(A)-f(B)\|$, when $\|\cdot\| \in \Gamma$.

THEOREM 3.6. *Let $f \in \mathscr{C}^1(J)$ and $A,B \in [\alpha I,\beta I]$, where $[\alpha,\beta] \subset J$. Then, for all $\|\cdot\| \in \Gamma$,*

$$\|f(A)-f(B)\| \leqslant \max_{\alpha\leqslant c\leqslant\beta}\left|f'(c)\right|\|A-B\|.$$

Proof. Let L_t and D_t, be the same as in (3.1), for all $0 \leqslant t \leqslant 1$. By assumptions,

$$\left\|f^{[1]}(D_t)\circ Z\right\| \leqslant \max_{i,j}\left|(f^{[1]}(D_t))_{ij}\right|\|Z\|$$

for all $Z \in M_n$. Using the mean value theorem, we have $(f^{[1]}(D_t))_{ij} = f'(c_{ij})$, where $\lambda_n(D_t) \leqslant c_{ij} \leqslant \lambda_1(D_t)$, for $1 \leqslant i, j \leqslant n$. Therefore

$$\left\| f^{[1]}(D_t) \circ Z \right\| \leqslant \max_{\lambda_n(D_t) \leqslant c \leqslant \lambda_1(D_t)} \left| f'(c) \right| \|Z\| \leqslant \max_{\alpha \leqslant c \leqslant \beta} \left| f'(c) \right| \|Z\|.$$

Hence,

$$M = \sup_{0 \leqslant t \leqslant 1} \left\| S_{f^{[1]}(D_t)} \right\| \leqslant \max_{\alpha \leqslant c \leqslant \beta} |f'(c)|.$$

Using (3.2), the proof is completed. □

We couldn't prove Theorem 3.6, for all unitary similarity invariant norms. But own conjecture is as following:

CONJECTURE 1. Let $f \in \mathscr{C}^1(J)$ and $A, B \in [\alpha I, \beta I]$, where $[\alpha, \beta] \subset J$. Then, for all unitary similarity invariant norms $\|\cdot\|$,

$$\|f(A) - f(B)\| \leqslant \max_{\alpha \leqslant c \leqslant \beta} \left| f'(c) \right| \|A - B\|.$$

PROPOSITION 3.7. *Let* $\|\cdot\| \in \Gamma$ *and* $f \in \mathscr{C}^1(J)$ *and let* $A, B \in [\alpha I, \beta I]$, *where* $[\alpha, \beta] \subset J$. *If* f *is an increasing and concave map on* $[\alpha, \beta]$, *then*

$$f'(\beta) \|A - B\| \leqslant \|f(A) - f(B)\| \leqslant f'(\alpha) \|A - B\|. \tag{3.4}$$

Proof. Let g be the inverse of f on $[\theta, \gamma] := [f(\alpha), f(\beta)]$ into $[\alpha, \beta]$. So $g \in \mathscr{C}^1(\theta, \gamma)$ and is a increasing and convex. Let $E = f(A)$ and $F = f(B)$. Therefore $E, F \in [\theta I, \gamma I]$.

Since f is increasing and concave on $[\alpha, \beta]$, using Theorem 3.6,

$$\|f(A) - f(B)\| \leqslant \max_{\alpha \leqslant c \leqslant \beta} \left| f'(c) \right| \|A - B\| = f'(\alpha) \|A - B\|. \tag{3.5}$$

Since g is increasing and convex on $[\theta, \gamma]$, by (3.5),

$$\begin{aligned} \|A - B\| = \|g(E) - g(F)\| &\leqslant g'(\gamma) \|E - F\| \\ &= \frac{1}{f'(\beta)} \|f(A) - f(B)\|. \end{aligned}$$

Therefore (3.4) holds. □

If f is a decreasing and convex map, then $-f$ is an increasing and concave map and if f is an increasing and convex map, then f^{-1}, where f^{-1} denoted inverse of map f, is an increasing and concave map. Therefore we have the following :

COROLLARY 3.8. *If* f *is a decreasing and convex map on* $[\alpha, \beta]$, *then*

$$-f'(\beta) \|A - B\| \leqslant \|f(A) - f(B)\| \leqslant -f'(\alpha) \|A - B\|, \tag{3.6}$$

and if f *is an increasing and convex map on* $[\alpha, \beta]$, *then*

$$f'(\alpha) \|A - B\| \leqslant \|f(A) - f(B)\| \leqslant f'(\beta) \|A - B\|. \tag{3.7}$$

EXAMPLE 1. Let $A,B \in [\alpha I, \beta I]$, for $\alpha > 0$. If $\|.\| \in \Gamma$, then

$$-r\beta^{r-1}\|A-B\| \leqslant \|A^r - B^r\| \leqslant -r\alpha^{r-1}\|A-B\|;\ r \in (-\infty, 0),$$
$$r\beta^{r-1}\|A-B\| \leqslant \|A^r - B^r\| \leqslant r\alpha^{r-1}\|A-B\|;\ 0 < r < 1,$$
$$r\alpha^{r-1}\|A-B\| \leqslant \|A^r - B^r\| \leqslant r\beta^{r-1}\|A-B\|;\ r > 1.$$

See [10, inequality (2.9)], [11, p. 86 and p. 87], [14, p. 29], and [15, inequalities (2.14) and (2.15)].

THEOREM 3.9. *Let $f(x) = \sum_{i=-\infty}^{\infty} a_i x^i$ with $a_i \geqslant 0$, for all $-\infty \leqslant i \leqslant \infty$. If (r,R) is the convergence interval of Laurent series of f and $A,B \in [mI, MI] \subset (rI, RI)$, then for all matrix norm $\|\cdot\|$ on $\mathscr{M}_n$, we have*

$$\|f(A) - f(B)\| \leqslant \left(g'(M) - h'(m)\right)\|A-B\|$$

where $g(x) = \sum_{i=0}^{\infty} a_i x^i$ and $h(x) = \sum_{i=-\infty}^{-1} a_i x^i$.

Proof. Suppose that $f_{p,q}(x) = g_p(x) + h_q(x)$, where $g_p(x) = \sum_{i=0}^{p} a_i x^i$ and $h_q(x) = \sum_{i=-q}^{-1} a_i x^i$, with $a_i \geqslant 0$ for all $-q \leqslant i \leqslant p$. Hence $f_{p,q}(x) = \sum_{i=-q}^{p} a_i x^i$. Let $A,B \in [mI,\ MI]$ and $0 \notin [m,\ M]$. We show that for all matrix norm $\|\cdot\|$ on $\mathscr{M}_n$,

$$\left\|f_{p,q}(A) - f_{p,q}(B)\right\| \leqslant \left(g_p'(M) - h_q'(m)\right)\|A-B\|.$$

By using Lemma 3.2,

$$\left\|f_{p,q}(A) - f_{p,q}(B)\right\| \leqslant \sup_{0\leqslant t\leqslant 1}\left\|\mathscr{D}f_{p,q}(L_t)\right\|\|A-B\|.$$

Since $\mathscr{D}$ is a linear map, we have

$$\begin{aligned}\left\|\mathscr{D}f_{p,q}(L_t)\right\| &= \sup_{\|X\|=1}\left\|\mathscr{D}f_{p,q}(L_t)(X)\right\| = \sup_{\|X\|=1}\left\|\mathscr{D}\Big(\sum_{i=-q}^{p} a_i L_t^i\Big)(X)\right\| \\ &= \sup_{\|X\|=1}\left\|\sum_{i=-q}^{p} a_i \mathscr{D}L_t^i(X)\right\| \leqslant \sum_{i=-q}^{p} a_i \sup_{\|X\|=1}\left\|\mathscr{D}L_t^i(X)\right\|.\end{aligned}$$

Let $0 \leqslant t \leqslant 1$ and $L_t = tA + (1-t)B$. Therefore $m \leqslant \|L_t\| \leqslant M$.

If $1 \leqslant i \leqslant p$, then

$$\begin{aligned}\left\|\mathscr{D}L_t^i(X)\right\| &= \left\|\sum_{j=1}^{i} L_t^{i-j} X L_t^{j-1}\right\| \leqslant \sum_{j=1}^{i} \|L_t\|^{i-j}\|X\|\|L_t\|^{j-1} \\ &= i\|L_t\|^{i-1}\|X\| \leqslant iM^{i-1}\|X\|,\end{aligned}$$

and if $-q \leqslant i \leqslant -1$, we have

$$\begin{aligned}\left\|\mathscr{D}L_t^i(X)\right\| &= \left\|\sum_{j=i}^{-1} -L_t^{j} X L_t^{i-j-1}\right\| \leqslant \sum_{j=i}^{-1} \|L_t\|^{j}\|X\|\|L_t\|^{i-j-1} \\ &= -i\|L_t\|^{i-1}\|X\| \leqslant -im^{i-1}\|X\|.\end{aligned}$$

Therefore

$$\begin{aligned}\left\|\mathscr{D}f_{p,q}(L_t)\right\| &\leqslant \sum_{i=-q}^{p} a_i \sup_{\|X\|=1}\left\|\mathscr{D}L_t^i(X)\right\| \\ &\leqslant \sum_{i=1}^{p} ia_iM^{i-1} - \sum_{i=-q}^{-1} ia_im^{i-1} = g_p'(M) - h_q'(m).\end{aligned}$$

Since $f = \lim_{(p,q)\to(\infty,\infty)} f_{p,q}$, we have

$$\begin{aligned}\|f(A)-f(B)\| &\leqslant \lim_{(p,q)\to(\infty,\infty)}\left\|f_{p,q}(A)-f_{p,q}(B)\right\| \leqslant \lim_{(p,q)\to(\infty,\infty)}\left(g_p'(M)-h_q'(m)\right)\|A-B\| \\ &= \left(g'(M)-h'(m)\right)\|A-B\|. \qquad \square\end{aligned}$$

EXAMPLE 2. Let $A,B \in [\alpha I, \beta I]$ where $\alpha > 0$. Then for all matrix norm $\|\cdot\|$,

$$\left\|e^{\frac{1}{A}} - e^{\frac{1}{B}}\right\| \leqslant \frac{1}{\alpha^2}e^{\frac{1}{\alpha}}\|A-B\|.$$

$$\|\sinh(A)-\sinh(B)\| \leqslant \cosh(\beta)\|A-B\|.$$

$$\|\ln(I-A)-\ln(I-B)\| \leqslant \frac{1}{1-\beta}\|A-B\|, \text{whenever } \beta < 1.$$

EXAMPLE 3. Consider the diferential equation (1.1) with nonsingular matrices A and $\tilde{A}$ and let y and $\tilde{y}$ be solutions of these equations, respectively. By using Theorem 3.6, for all $\|\cdot\| \in \Gamma$, we obtain that

$$\begin{aligned}&\|y(t)-\tilde{y}(t)\| \\ &\leqslant \left\|\cos(\sqrt{A}t) - \cos(\sqrt{\tilde{A}}t)\right\|\|y_0\| + \left\|\sqrt{A}^{-1}\sin(\sqrt{A}t) - \sqrt{\tilde{A}}^{-1}\sin(\sqrt{\tilde{A}}t)\right\|\|y_0'\| \\ &\leqslant \left(\max_{\alpha\leqslant c\leqslant\beta}\left|\frac{t}{2\sqrt{c}}\sin(\sqrt{c}t)\right|\|y_0\| + \max_{\alpha\leqslant c\leqslant\beta}\left|\frac{tc\cos(\sqrt{c}t)-\sin(\sqrt{c}t)}{2c\sqrt{c}}\right|\|y_0'\|\right)\|A-\tilde{A}\|.\end{aligned}$$

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