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CAUCHY—SCHWARZ TYPE INEQUALITIES AND APPLICATIONS TO NUMERICAL RADIUS INEQUALITIES

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Abstract. We present new improvements of certain Cauchy–Schwarz type inequalities. As applications of the results obtained, we provide refinements of some numerical radius inequalities for Hilbert space operators. It is shown, among other inequalities, that if $A \in \mathbb{B}(\mathscr{H})$, then

$$\omega^2(A) \leqslant \frac{1}{6}\left\| |A|^2 + |A^*|^2 \right\| + \frac{1}{3}\omega(A) \left\| |A| + |A^*| \right\|.$$

1. Introduction

Let $\mathbb{B}(\mathscr{H})$ stand for the C^* algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$. In this context, an operator $A \in \mathbb{B}(\mathscr{H})$ is said to be normal if $A^*A = AA^*$, where A^* is the adjoint operator of A. For $A \in \mathbb{B}(\mathscr{H})$, the absolute value $|A|$ is defined by $|A| = (A^*A)^{\frac{1}{2}}$. Notice that $|A|$ is a positive semi-definite operator, in the sense that $\langle |A|x, x\rangle \geqslant 0$, for all $x \in \mathscr{H}$.

Among the most interesting numerical values associated with an operator $A \in \mathbb{B}(\mathscr{H})$ are the operator norm $\|A\|$ and the numerical radius $\omega(A)$ of A, defined respectively by

$$\|A\| = \sup_{\|x\|=1} \|Ax\| \quad \text{and} \quad \omega(A) = \sup_{\|x\|=1} |\langle Ax, x\rangle|.$$

It is easy to see that $\|A\| = \sup_{\|x\|=\|y\|=1} |\langle Ax, y\rangle|$. Also, it is well known that if A is normal, then $\|A\| = \omega(A)$.

There are some important properties of the numerical radius such as the power inequality

$$\omega(A^n) \leqslant \omega^n(A) \tag{1}$$

for $n = 1, 2, \ldots$.

It is well-known that if A is not normal, then $\|A\|$ and $\omega(A)$ are related via the inequalities

$$\frac{1}{2}\|A\| \leqslant \omega(A) \leqslant \|A\|. \tag{2}$$

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Various numerical radius inequalities improving these inequalities have been given in [2, 7, 8, 9].

In [7], the following inequality has been already shown

$$\omega(A) \leqslant \frac{1}{2}\| |A| + |A^*| \| \tag{3}$$

while in [8], the following inequalities have been shown

$$\frac{1}{4}\left\| |A|^2 + |A^*|^2 \right\| \leqslant \omega^2(A) \leqslant \frac{1}{2}\left\| |A|^2 + |A^*|^2 \right\|. \tag{4}$$

The inequality (3) is sharper than the second inequality in (2), and the inequalities (4) refine the inequalities (2). It should be mentioned here that the inequality (3) refines the second inequality in (4).

In [4], Dragomir showed the following numerical radius inequality involving the product of two operators:

$$\omega^r(B^*A) \leqslant \frac{1}{2}\left\| |A|^{2r} + |B|^{2r} \right\|, \ (r \geqslant 1). \tag{5}$$

The Cauchy–Schwarz inequality states that for all vectors x and y in an inner product space

$$|\langle x,y\rangle| \leqslant \|x\| \, \|y\|, \tag{6}$$

where $\langle \cdot,\cdot \rangle$ is the inner product and $\|x\|^2 = \langle x,x\rangle$.

Motivated by the inequality (6), we shall establish in this paper that

$$|\langle x,y\rangle| \leqslant \sqrt{\frac{1}{2}\left(\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2\right) + |\langle x,y\rangle| \, \|x\| \, \|y\|} \leqslant \|x\| \, \|y\|,$$

for any $x,y \in \mathscr{H}$. This inequality, which nicely improves the Cauchy–Schwarz inequality, enables us to get a new refinement of (5) for the case $r = 2$. We also establish a considerable improvement of the second inequality in (4).

2. Main results

In order to achieve our goal, we need the following three lemmas.

LEMMA 1. [5, Theorem 1.4] *Let $A \in \mathbb{B}(\mathscr{H})$ be a positive operator, and let $x \in \mathscr{H}$ be a unit vector. Then*

$$\langle Ax,x\rangle^r \leqslant \langle A^r x,x\rangle, \ (r \geqslant 1).$$

LEMMA 2. [1, Theorem 2.3] *Let f be a non-negative convex function on $[0,\infty)$, and let $A,B \in \mathbb{B}(\mathscr{H})$ be positive operators. Then*

$$\left\| f\left(\frac{A+B}{2}\right) \right\| \leqslant \left\| \frac{f(A)+f(B)}{2} \right\|.$$

LEMMA 3. *Let* $x,y \in \mathscr{H}$. *Then*

$$|\langle x,y\rangle|^2 \leqslant \frac{1}{2}\left(\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2\right) + |\langle x,y\rangle|\,\|x\|\,\|y\| \leqslant \|x\|^2\|y\|^2.$$

Proof. By applying the inequality (6) twice, we obtain

$$\sqrt{\frac{1}{2}\left(\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2\right) + |\langle x,y\rangle|\,\|x\|\,\|y\|} \geqslant \sqrt{|\langle x,y\rangle|\,\|x\|\,\|y\|} \geqslant |\langle x,y\rangle|\,. \tag{7}$$

On the other hand, by utilizing the arithmetic–geometric mean inequality, we infer that

$$\begin{aligned} &\sqrt{\frac{1}{2}\left(\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2\right) + |\langle x,y\rangle|\,\|x\|\,\|y\|} \\ \leqslant &\sqrt{\frac{1}{2}\left(\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2\right) + \frac{1}{2}\left(|\langle x,y\rangle|^2 + \|x\|^2\|y\|^2\right)} \\ = &\|x\|\,\|y\|\,. \end{aligned} \tag{8}$$

Combining (7) and (8), we get the desired result.

THEOREM 1. *Let* $A,B \in \mathbb{B}(\mathscr{H})$. *Then*

$$\omega^2(B^*A) \leqslant \frac{1}{6}\left\||A|^4 + |B|^4\right\| + \frac{1}{3}\omega(B^*A)\left\||A|^2 + |B|^2\right\|. \tag{9}$$

Proof. The first inequality in Lemma 3 implies

$$\frac{3}{2}|\langle x,y\rangle|^2 \leqslant \frac{1}{2}\|x\|^2\|y\|^2 + |\langle x,y\rangle|\,\|x\|\,\|y\|\,.$$

Therefore, for any $x,y \in \mathscr{H}$,

$$|\langle x,y\rangle|^2 \leqslant \frac{1}{3}\|x\|^2\|y\|^2 + \frac{2}{3}|\langle x,y\rangle|\,\|x\|\,\|y\|\,. \tag{10}$$

Assume $\|x\| = 1$ and replace x and y by Ax and Bx, respectively, in (10) to get

$$|\langle B^*Ax,x\rangle|^2 \leqslant \frac{1}{3}\|Ax\|^2\|Bx\|^2 + \frac{2}{3}|\langle B^*Ax,x\rangle|\,\|Ax\|\,\|Bx\|\,.$$

Therefore, by applying the arithmetic–geometric mean inequality and Lemma 1, we conclude that

$$\begin{aligned} |\langle B^*Ax,x\rangle|^2 &\leqslant \frac{1}{3}\left\langle |A|^2x,x\right\rangle\left\langle |B|^2x,x\right\rangle + \frac{2}{3}|\langle B^*Ax,x\rangle|\sqrt{\left\langle |A|^2x,x\right\rangle\left\langle |B|^2x,x\right\rangle} \\ &\leqslant \frac{1}{6}\left(\left\langle |A|^2x,x\right\rangle^2 + \left\langle |B|^2x,x\right\rangle^2\right) + \frac{1}{3}|\langle B^*Ax,x\rangle|\left\langle (|A|^2+|B|^2)x,x\right\rangle \\ &\leqslant \frac{1}{6}\left\langle (|A|^4+|B|^4)x,x\right\rangle + \frac{1}{3}|\langle B^*Ax,x\rangle|\left\langle (|A|^2+|B|^2)x,x\right\rangle \end{aligned}$$

i.e.,

$$|\langle B^*Ax,x\rangle|^2 \leqslant \frac{1}{6}\left\langle (|A|^4+|B|^4)x,x\right\rangle + \frac{1}{3}|\langle B^*Ax,x\rangle|\left\langle (|A|^2+|B|^2)x,x\right\rangle.$$

Taking the supremum over $x \in \mathscr{H}$ with $\|x\|=1$ in the above inequality, we deduce the desired inequality (9).

The following corollary shows that the inequality (9) is sharper than the inequality (5).

COROLLARY 1. *Let $A,B \in \mathbb{B}(\mathscr{H})$. Then*

$$\omega^2(B^*A) \leqslant \frac{1}{6}\left\| |A|^4+|B|^4 \right\| + \frac{1}{3}\omega(B^*A)\left\| |A|^2+|B|^2 \right\| \leqslant \frac{1}{2}\left\| |A|^4+|B|^4 \right\|.$$

Proof. The first inequality is clear by Theorem 1. For the second inequality, we have

$$\begin{aligned}
&\frac{1}{6}\left\| |A|^4+|B|^4 \right\| + \frac{1}{3}\omega(B^*A)\left\| |A|^2+|B|^2 \right\| \\
\leqslant&\frac{1}{6}\left\| |A|^4+|B|^4 \right\| + \frac{1}{6}\left\| |A|^2+|B|^2 \right\|^2 \quad \text{(by (5))} \\
=&\frac{1}{6}\left\| |A|^4+|B|^4 \right\| + \frac{1}{6}\left\| \left(\frac{2|A|^2+2|B|^2}{2}\right)^2 \right\| \\
\leqslant&\frac{1}{6}\left\| |A|^4+|B|^4 \right\| + \frac{1}{12}\left\| \left(2|A|^2\right)^2+\left(2|B|^2\right)^2 \right\| \quad \text{(by Lemma 2)} \\
=&\frac{1}{6}\left\| |A|^4+|B|^4 \right\| + \frac{1}{3}\left\| |A|^4+|B|^4 \right\| \\
=&\frac{1}{2}\left\| |A|^4+|B|^4 \right\|,
\end{aligned}$$

and so the proof is complete.

It follows from the mixed Schwarz inequality [6, pp. 75–76] that if $A \in \mathbb{B}(\mathscr{H})$, then for any vector $x \in \mathscr{H}$,

$$|\langle Ax,x\rangle| \leqslant \sqrt{\langle |A|x,x\rangle\langle |A^*|x,x\rangle}. \tag{11}$$

Applying the inequality (11), we have

$$\begin{aligned}
&\frac{1}{3}\langle |A|x,x\rangle\langle |A^*|x,x\rangle + \frac{2}{3}|\langle Ax,x\rangle|\sqrt{\langle |A|x,x\rangle\langle |A^*|x,x\rangle} \\
\geqslant&\frac{1}{3}|\langle Ax,x\rangle|^2 + \frac{2}{3}|\langle Ax,x\rangle||\langle Ax,x\rangle| \\
=&|\langle Ax,x\rangle|^2.
\end{aligned} \tag{12}$$

On the other hand,

$$\begin{aligned}
&\frac{1}{3}\langle|A|x,x\rangle\langle|A^*|x,x\rangle+\frac{2}{3}|\langle Ax,x\rangle|\sqrt{\langle|A|x,x\rangle\langle|A^*|x,x\rangle}\\
\leqslant&\frac{1}{3}\langle|A|x,x\rangle\langle|A^*|x,x\rangle+\frac{2}{3}\sqrt{\langle|A|x,x\rangle\langle|A^*|x,x\rangle}\sqrt{\langle|A|x,x\rangle\langle|A^*|x,x\rangle}\\
=&\langle|A|x,x\rangle\langle|A^*|x,x\rangle,
\end{aligned}\tag{13}$$

where we have used (11) again. Combining (12) and (13), we infer that

$$\begin{aligned}
|\langle Ax,x\rangle|^2&\leqslant\frac{1}{3}\langle|A|x,x\rangle\langle|A^*|x,x\rangle+\frac{2}{3}|\langle Ax,x\rangle|\sqrt{\langle|A|x,x\rangle\langle|A^*|x,x\rangle}\\
&\leqslant\langle|A|x,x\rangle\langle|A^*|x,x\rangle,
\end{aligned}\tag{14}$$

which can be considered as an improvement of (11). Based on the inequality (14), we have the following result.

THEOREM 2. *Let* $A\in\mathbb{B}(\mathscr{H})$. *Then*

$$\omega^2(A)\leqslant\frac{1}{6}\left\||A|^2+|A^*|^2\right\|+\frac{1}{3}\omega(A)\||A|+|A^*|\|.\tag{15}$$

Proof. Let $x\in\mathscr{H}$ be a unit vector. It follows from the first inequality in (14) that

$$\begin{aligned}
|\langle Ax,x\rangle|^2&\leqslant\frac{1}{3}\langle|A|x,x\rangle\langle|A^*|x,x\rangle+\frac{2}{3}|\langle Ax,x\rangle|\sqrt{\langle|A|x,x\rangle\langle|A^*|x,x\rangle}\\
&\leqslant\frac{1}{6}\left(\langle|A|x,x\rangle^2+\langle|A^*|x,x\rangle^2\right)+\frac{2}{6}|\langle Ax,x\rangle|\langle(|A|+|A^*|)x,x\rangle &(16)\\
&\leqslant\frac{1}{6}\left\langle(|A|^2+|A^*|^2)x,x\right\rangle+\frac{2}{6}|\langle Ax,x\rangle|\langle(|A|+|A^*|)x,x\rangle, &(17)
\end{aligned}$$

where (16) follows from the arithmetic–geometric mean inequality, and (17) obtained from Lemma 1. Now, by taking supremum over $x\in\mathscr{H}$ with $\|x\|$, we obtain

$$\omega^2(A)\leqslant\frac{1}{6}\left\||A|^2+|A^*|^2\right\|+\frac{1}{3}\omega(A)\||A|+|A^*|\|,$$

as required.

The following corollary shows that our inequality (15) is really stronger than Kittaneh's inequality (4).

COROLLARY 2. *Let* $A\in\mathbb{B}(\mathscr{H})$. *Then*

$$\omega^2(A)\leqslant\frac{1}{6}\left\||A|^2+|A^*|^2\right\|+\frac{1}{3}\omega(A)\||A|+|A^*|\|\leqslant\frac{1}{2}\left\||A|^2+|A^*|^2\right\|.$$

Proof. We have

$$
\begin{aligned}
\omega^2(A) &\leqslant \frac{1}{6}\left\||A|^2+|A^*|^2\right\|+\frac{1}{3}\omega(A)\,\||A|+|A^*|\| \\
&\leqslant \frac{1}{6}\left\||A|^2+|A^*|^2\right\|+\frac{1}{6}\||A|+|A^*|\|\,\||A|+|A^*|\| \qquad (18)\\
&= \frac{1}{6}\left\||A|^2+|A^*|^2\right\|+\frac{1}{6}\||A|+|A^*|\|^2 \\
&= \frac{1}{6}\left\||A|^2+|A^*|^2\right\|+\frac{1}{6}\left\|\left(\frac{2|A|+2|A^*|}{2}\right)^2\right\| \\
&\leqslant \frac{1}{6}\left\||A|^2+|A^*|^2\right\|+\frac{1}{12}\left\|(2|A|)^2+(2|A^*|)^2\right\| \qquad (19)\\
&= \frac{1}{6}\left\||A|^2+|A^*|^2\right\|+\frac{1}{3}\left\||A|^2+|A^*|^2\right\| \\
&= \frac{1}{2}\left\||A|^2+|A^*|^2\right\|,
\end{aligned}
$$

where (18) follows from (3), and (19) follows from Lemma 2.

We recall the following Cauchy–Schwarz type inequality obtained by Buzano [3], which says that

$$
|\langle x,e\rangle\langle e,y\rangle| \leqslant \frac{1}{2}(\|x\|\,\|y\|+|\langle x,y\rangle|), \qquad (20)
$$

where x, y, e are vectors in $\mathscr{H}$ and $\|e\|=1$. Utilizing (20), we may state that

$$
\begin{aligned}
&|\langle x,e\rangle\langle e,y\rangle|^2 \\
&\leqslant \frac{1}{12}\|x\|^2\|y\|^2+\frac{1}{12}|\langle x,y\rangle|^2+\frac{1}{6}\|x\|\,\|y\|\,|\langle x,y\rangle|+\frac{1}{3}|\langle x,e\rangle\langle e,y\rangle|(\|x\|\,\|y\|+|\langle x,y\rangle|).
\end{aligned} \qquad (21)
$$

Indeed,

$$
\begin{aligned}
\frac{1}{4}(\|x\|\,\|y\|+|\langle x,y\rangle|)^2 &= \frac{1}{12}(\|x\|\,\|y\|+|\langle x,y\rangle|)^2+\frac{1}{6}(\|x\|\,\|y\|+|\langle x,y\rangle|)^2 \\
&\geqslant \frac{1}{12}(\|x\|\,\|y\|+|\langle x,y\rangle|)^2+\frac{1}{3}|\langle x,e\rangle\langle e,y\rangle|(\|x\|\,\|y\|+|\langle x,y\rangle|) \\
&\geqslant \frac{4}{12}|\langle x,e\rangle\langle e,y\rangle|^2+\frac{2}{3}|\langle x,e\rangle\langle e,y\rangle|^2 \\
&= |\langle x,e\rangle\langle e,y\rangle|^2.
\end{aligned}
$$

Employing the above inequality, we prove the following theorem.

THEOREM 3. *Let* $A\in\mathbb{B}(\mathscr{H})$. *Then*

$$
\begin{aligned}
\omega^4(A) \leqslant& \frac{1}{24}\left\||A|^4+|A^*|^4\right\|+\frac{1}{12}\left(\omega^2(A^2)+\left\||A|^2+|A^*|^2\right\|\omega(A^2)\right) \\
&+\frac{1}{3}\omega^2(A)\left(\frac{1}{2}\left\||A|^2+|A^*|^2\right\|+\omega(A^2)\right).
\end{aligned} \qquad (22)
$$

Proof. Assume $\|x\| = 1$. Put $e = x$, and replace x and y by Ax and A^*x, respectively, in (21) to get

$$\begin{aligned} |\langle Ax,x\rangle|^4 \leqslant & \frac{1}{12}\|Ax\|^2\|A^*x\|^2 + \frac{1}{12}\left|\left\langle A^2x,x\right\rangle\right|^2 + \frac{1}{6}\,\|Ax\|\,\|A^*x\|\,\left|\left\langle A^2x,x\right\rangle\right| \\ & + \frac{1}{3}|\langle Ax,x\rangle|^2\left(\|Ax\|\,\|A^*x\| + \left|\left\langle A^2x,x\right\rangle\right|\right). \end{aligned}$$

Consequently,

$$\begin{aligned} & |\langle Ax,x\rangle|^4 \\ \leqslant & \frac{1}{12}\left\langle |A|^2x,x\right\rangle\left\langle |A^*|^2x,x\right\rangle + \frac{1}{12}\left|\left\langle A^2x,x\right\rangle\right|^2 + \frac{1}{6}\sqrt{\left\langle |A|^2x,x\right\rangle\left\langle |A^*|^2x,x\right\rangle}\left|\left\langle A^2x,x\right\rangle\right| \\ & + \frac{1}{3}|\langle Ax,x\rangle|^2\left(\sqrt{\left\langle |A|^2x,x\right\rangle\left\langle |A^*|^2x,x\right\rangle} + \left|\left\langle A^2x,x\right\rangle\right|\right) \\ \leqslant & \frac{1}{24}\left(\left\langle |A|^2x,x\right\rangle^2 + \left\langle |A^*|^2x,x\right\rangle^2\right) + \frac{1}{12}\left(\left|\left\langle A^2x,x\right\rangle\right|^2 + \left\langle (|A|^2+|A^*|^2)x,x\right\rangle\left|\left\langle A^2x,x\right\rangle\right|\right) \\ & + \frac{1}{3}|\langle Ax,x\rangle|^2\left(\frac{1}{2}\left\langle (|A|^2+|A^*|^2)x,x\right\rangle + \left|\left\langle A^2x,x\right\rangle\right|\right) \\ \leqslant & \frac{1}{24}\left\langle (|A|^4+|A^*|^4)x,x\right\rangle + \frac{1}{12}\left(\left|\left\langle A^2x,x\right\rangle\right|^2 + \left\langle (|A|^2+|A^*|^2)x,x\right\rangle\left|\left\langle A^2x,x\right\rangle\right|\right) \\ & + \frac{1}{3}|\langle Ax,x\rangle|^2\left(\frac{1}{2}\left\langle (|A|^2+|A^*|^2)x,x\right\rangle + \left|\left\langle A^2x,x\right\rangle\right|\right), \end{aligned}$$

where we have used the arithmetic–geometric mean inequality and Lemma 2, respectively. Taking the supremum over $x \in \mathscr{H}$ with $\|x\| = 1$ in the above inequality, we deduce the desired inequality (22).

COROLLARY 3. *Let $A \in \mathbb{B}(\mathscr{H})$. Then*

$$\begin{aligned} \omega^4(A) \leqslant & \frac{1}{24}\left\||A|^4+|A^*|^4\right\| + \frac{1}{12}\left(\omega^2\left(A^2\right) + \left\||A|^2+|A^*|^2\right\|\omega\left(A^2\right)\right) \\ & + \frac{1}{3}\omega^2(A)\left(\frac{1}{2}\left\||A|^2+|A^*|^2\right\| + \omega\left(A^2\right)\right) \\ \leqslant & \frac{1}{2}\left\||A|^4+|A^*|^4\right\|. \end{aligned}$$

Proof. We prove the second inequality. One can write

$$\begin{aligned} & \frac{1}{24}\left\||A|^4+|A^*|^4\right\| + \frac{1}{12}\left(\omega^2\left(A^2\right) + \left\||A|^2+|A^*|^2\right\|\omega\left(A^2\right)\right) \\ & + \frac{1}{3}\omega^2(A)\left(\frac{1}{2}\left\||A|^2+|A^*|^2\right\| + \omega\left(A^2\right)\right) \\ \leqslant & \frac{1}{24}\left\||A|^4+|A^*|^4\right\| + \frac{1}{12}\left(\omega^2\left(A^2\right) + \left\||A|^2+|A^*|^2\right\|\omega^2(A)\right) \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{3}\omega^2(A)\left(\frac{1}{2}\left\||A|^2+|A^*|^2\right\|+\omega^2(A)\right)\\
\leqslant&\frac{1}{24}\left\||A|^4+|A^*|^4\right\|+\frac{1}{12}\left(\frac{1}{2}\left\||A|^4+|A^*|^4\right\|+\left\||A|^2+|A^*|^2\right\|\omega^2(A)\right)\\
&+\frac{1}{3}\omega^2(A)\left(\frac{1}{2}\left\||A|^2+|A^*|^2\right\|+\omega^2(A)\right)\\
\leqslant&\frac{1}{24}\left\||A|^4+|A^*|^4\right\|+\frac{1}{12}\left(\frac{1}{2}\left\||A|^4+|A^*|^4\right\|+\frac{1}{2}\left\||A|^2+|A^*|^2\right\|^2\right)+\frac{1}{6}\left\||A|^2+|A^*|^2\right\|^2\\
=&\frac{1}{12}\left\||A|^4+|A^*|^4\right\|+\frac{5}{24}\left\||A|^2+|A^*|^2\right\|^2\\
=&\frac{1}{12}\left\||A|^4+|A^*|^4\right\|+\frac{5}{24}\left\|\left(\frac{2|A|^2+2|A^*|^2}{2}\right)^2\right\|\\
\leqslant&\frac{1}{12}\left\||A|^4+|A^*|^4\right\|+\frac{20}{48}\left\||A|^4+|A^*|^4\right\|\\
=&\frac{1}{2}\left\||A|^4+|A^*|^4\right\|,
\end{aligned}
$$

where the first inequality follows from (1), the second inequality follows from $\omega^2\left(A^2\right)\leqslant\frac{1}{2}\left\||A|^4+|A^*|^4\right\|$, the second inequality in (4) implies the third inequality above, and the last inequality is a consequence of Lemma 2.

We conclude the paper by pointing out that the numerical radius inequalities presented in this paper are sharp. In fact, the inequalities in Theorem 2, Corollary 2, Theorem 3, and Corollary 3 become equalities if A is normal. The inequalities in Theorem 1 and Corollary 1 become equalities if $A=B=\begin{bmatrix}0&1\\0&0\end{bmatrix}$.

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