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FURTHER INTERPOLATION INEQUALITIES RELATED TO ARITHMETIC–GEOMETRIC MEAN, CAUCHY–SCHWARZ AND HÖLDER INEQUALITIES FOR UNITARILY INVARIANT NORMS

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Abstract. An inequality for matrices that interpolates between the Cauchy-Schwarz and the arithmetic-geometric mean inequalities for unitarily invariant norms has been obtained by Audenaert. Alakhrass obtained a related result to Audenaert's inequality using a log-convex function g defined on $[0,1]$. Very recently, Zou obtained an inequality for matrices that unifies Hölder's inequality and the arithmetic-geometric mean inequality for unitarily invariant norms. A generalized version of Zou's inequality for unitarily invariant norms is given, and an alternative proof of Audenaert's inequality using a refined version of Alakhrass's function is presented.

1. Introduction

Let $M_n(\mathbb{C})$ be the space of all $n\times n$ complex matrices, and let $|||.|||$ on $M_n(\mathbb{C})$ denote the unitarily invariant norm. Let $s_1(A)\geqslant s_2(A)\geqslant \ldots \geqslant s_n(A)$ denote the eigenvalues of $|A|=(A^*A)^{\frac{1}{2}}$ arranged in a decreasing order, i.e., the singular values of A where $A\in M_n(\mathbb{C})$. If $s_i(A)\in\mathbb{R}$ for $i=1,2,\ldots,n$, then we write them as $\lambda_1(A)\geqslant \lambda_2(A)\geqslant\ldots\geqslant\lambda_n(A)$. It is known that $s_i(A^*)=s_i(|A|)=s_i(A)$ for $i=1,2,\ldots,n$ where $A\in M_n(\mathbb{C})$. The Ky Fan norm of $A\in M_n(\mathbb{C})$ is $||A||_{(k)}=\sum_{i=1}^{k}s_i(A)$, for $k=1,2,\ldots,n$. The Fan dominance theorem (see [5, p. 93]) asserts that $||A||_{(k)}\leqslant ||B||_{(k)}$ for $k=1,2,\ldots,n$ if and only if $|||A|||\leqslant|||B|||$ for every unitarily invariant norm. For more about the unitarily invariant norms, we refer the reader to [5, 14].

The arithmetic-geometric mean inequality for singular values of matrices, which has been obtained in [8], asserts that

$$s_j(A^*B)\leqslant \frac{1}{2}s_j(AA^*+BB^*) \text{ for } j=1,2,\ldots,n, \tag{1}$$

for all $A,B\in M_n(\mathbb{C})$. The unitarily invariant norm version of the inequality (1) says that

$$|||A^*B|||\leqslant\frac{1}{2}|||AA^*+BB^*|||. \tag{2}$$

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A generalization of the inequality (2), which has been obtained in [6, 12], asserts that

$$|||A^*XB||| \leqslant \frac{1}{2}|||AA^*X+XBB^*|||, \tag{3}$$

for all $A,B,X \in M_n(\mathbb{C})$.

The Cauchy-Schwarz inequality for unitarily invariant norms of matrices says that

$$|||A^*B|||^2 \leqslant |||AA^*|||\ |||BB^*|||, \tag{4}$$

for all $A,B \in M_n(\mathbb{C})$.

Here are some generalizations of the inequality (4),

$$|||\ |A^*B|^r|||^2 \leqslant |||(AA^*)^r|||\ |||(BB^*)^r|||, \tag{5}$$

$$|||\ |A^*B|^r||| \leqslant \left|\left|\left|(AA^*)^{\frac{rp}{2}}\right|\right|\right|^{\frac{1}{p}} \left|\left|\left|(BB^*)^{\frac{rq}{2}}\right|\right|\right|^{\frac{1}{q}}, \tag{6}$$

$$|||A^*XB|||^2 \leqslant |||AA^*X|||\ |||XBB^*|||, \tag{7}$$

$$|||\ |A^*XB|^r||| \leqslant \left|\left|\left||AA^*X|^{\frac{rp}{2}}\right|\right|\right|^{\frac{1}{p}} \left|\left|\left||XBB^*|^{\frac{rq}{2}}\right|\right|\right|^{\frac{1}{q}}, \tag{8}$$

for all $A,B,X \in M_n(\mathbb{C})$, $r \geqslant 0$, and $p,q > 1$ such that $\frac{1}{p}+\frac{1}{q}=1$. The inequality (5) has been obtained by Horn and Mathias in [9], while the inequality (6) has been obtained by Horn and Zhan in [11], and the inequality (8) firstly introduced by Kosaki in [13]. Note that the inequality (6) is the Hölder's inequality for unitarily invariant norms of matrices. A closer look at the Cauchy-Schwarz inequality and its applications can be found in [7].

Using some calculations, it can be found in [2] that the inequality (3) leads to the inequality (7).

Audenaert in [4] proved that

$$|||\ |A^*B|^r||| \leqslant \left|\left|\left|(\alpha AA^*+(1-\alpha)BB^*)^{\frac{rp}{2}}\right|\right|\right|^{\frac{1}{p}} \times \left|\left|\left|((1-\alpha)AA^*+\alpha BB^*)^{\frac{rq}{2}}\right|\right|\right|^{\frac{1}{q}}, \tag{9}$$

for all $A,B \in M_n(\mathbb{C})$, $r \geqslant 0$, $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, and for all $\alpha \in [0,1]$. The inequality (9) interpolates in a natural way between the inequality (2) and the inequality (6).

Very recently, a generalization of the inequality (9) has been given in [15]. This generalization asserts that

$$\left|\left|\left||A^*XB|^{2r}\right|\right|\right| \leqslant |||\ |\alpha AA^*X+(1-\alpha)XBB^*|^{rp}|||^{\frac{1}{p}} \times |||\ |(1-\alpha)AA^*X+\alpha XBB^*|^{rq}|||^{\frac{1}{q}}, \tag{10}$$

for all $A,B,X \in M_n(\mathbb{C})$, $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, $r \geqslant \max\left\{\frac{1}{p},\frac{1}{q}\right\}$, and for all $\alpha \in [0,1]$. The inequality (10) interpolates between the inequalities (3) and (8).

Alakhrass obtained in [1] a log-convex function g defined on $[0,1]$ and satisfies the following

$$
\begin{aligned}
|||A^*XB||| = g(\frac{1}{2}) &\leqslant g(\alpha) \\
&\leqslant |||\alpha AA^*X + (1-\alpha)XBB^*|||\ |||(1-\alpha)AA^*X + \alpha XBB^*|||,
\end{aligned} \tag{11}
$$

for all $\alpha \in [0,1]$.

In Section 2, we prove a generalization the inequality (10). In Section 3, we introduce an alternative proof of the inequality (9), this proof presents implicitly a refinement of (11) for the special case when $X = I$.

2. Main result

We will use the following lemmas, which can be found in [5], to start our work.

LEMMA 1. *If* $A \in M_n(\mathbb{C})$, *then*

$$
\prod_{j=1}^{k} |\lambda_j(A)| \leqslant \prod_{j=1}^{k} s_j(A),
$$

for $1 \leqslant k \leqslant n$.

LEMMA 2. *If* $A, B \in M_n(\mathbb{C})$, *then*

$$
\prod_{j=1}^{k} s_j(AB) \leqslant \prod_{j=1}^{k} s_j(A) \prod_{j=1}^{k} s_j(B),
$$

for $1 \leqslant k \leqslant n$.

And we need the following lemmas which can be found respectively in [3] and [15].

LEMMA 3. *Let* $A, B \in M_n(\mathbb{C})$ *be a positive semidefinite matrices and let* $0 \leqslant \alpha \leqslant 1$. *Then*

$$
s_j(A^{\alpha}B^{1-\alpha}) \leqslant s_j(\alpha A + (1-\alpha)B),
$$

for $1 \leqslant j \leqslant n$.

LEMMA 4. *Let* $A, B, X \in M_n(\mathbb{C})$ *where* X *is positive semidefinite,* $p, q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$, $r \geqslant \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$. *Then*

$$
\left|\left|\left|\left(\alpha X^{\frac{1}{2}}AA^*X^{\frac{1}{2}} + (1-\alpha)X^{\frac{1}{2}}BB^*X^{\frac{1}{2}}\right)^{rp}\right|\right|\right| \leqslant |||\ |\alpha AA^*X + (1-\alpha)XBB^*|^{rp}|||,
$$

and

$$
\left|\left|\left|\left((1-\alpha)X^{\frac{1}{2}}AA^*X^{\frac{1}{2}} + \alpha X^{\frac{1}{2}}BB^*X^{\frac{1}{2}}\right)^{rq}\right|\right|\right| \leqslant |||\ |(1-\alpha)AA^*X + \alpha XBB^*|^{rq}|||,
$$

for all $\alpha \in [0,1]$.

We start our results with the following theorem.

THEOREM 1. *Let $A,B \in M_n(\mathbb{C})$ and $\frac{1}{p}+\frac{1}{q}=1$, $p,q>1$, $r \geqslant 0$. If f_1,f_2,g_1,g_2 are non-negative continuous functions such that $f_1(t)f_2(t)=t$ and $g_1(t)g_2(t)=t$ for $t \geqslant 0$, then*

$$\begin{aligned} \|\| \, |A^*B|^r \|\| \leqslant & \left\|\left\| \left(\alpha f_1^{\frac{1}{\alpha}}(AA^*)+(1-\alpha)g_1^{\frac{1}{1-\alpha}}(BB^*)\right)^{\frac{rp}{2}} \right\|\right\|^{\frac{1}{p}} \\ & \times \left\|\left\| \left((1-\alpha)f_2^{\frac{1}{1-\alpha}}(AA^*)+\alpha g_2^{\frac{1}{\alpha}}(BB^*)\right)^{\frac{rq}{2}} \right\|\right\|^{\frac{1}{q}}, \end{aligned} \tag{12}$$

for all $\alpha \in [0,1]$.

Proof. For any k = 1,2, ...,n, using Lemmas 1, 2 and 3, we get that

$$\begin{aligned} & \prod_{j=1}^{k} s_j\left(|A^*B|^r\right)=\prod_{j=1}^{k} \lambda_j^{\frac{r}{2}}\left(B^*AA^*B\right) \\ = & \prod_{j=1}^{k} \lambda_j^{\frac{r}{2}}\left(AA^*BB^*\right) \\ = & \prod_{j=1}^{k} \lambda_j^{\frac{r}{2}}\left(f_2\left(AA^*\right) f_1\left(AA^*\right) g_1\left(BB^*\right) g_2\left(BB^*\right)\right) \\ = & \prod_{j=1}^{k} \lambda_j^{\frac{r}{2}}\left(f_1\left(AA^*\right) g_1\left(BB^*\right) g_2\left(BB^*\right) f_2\left(AA^*\right)\right) \\ \leqslant & \prod_{j=1}^{k} s_j^{\frac{r}{2}}\left(f_1\left(AA^*\right) g_1\left(BB^*\right) g_2\left(BB^*\right) f_2\left(AA^*\right)\right) \\ = & \prod_{j=1}^{k} s_j^{\frac{r}{2}}\left(\left(f_1^{\frac{1}{\alpha}}\left(AA^*\right)\right)^{\alpha}\left(g_1^{\frac{1}{1-\alpha}}\left(BB^*\right)\right)^{1-\alpha}\left(g_2^{\frac{1}{\alpha}}\left(BB^*\right)\right)^{\alpha}\left(f_2^{\frac{1}{1-\alpha}}\left(AA^*\right)\right)^{1-\alpha}\right) \\ \leqslant & \prod_{j=1}^{k} s_j^{\frac{r}{2}}\left(\left(f_1^{\frac{1}{\alpha}}\left(AA^*\right)\right)^{\alpha}\left(g_1^{\frac{1}{1-\alpha}}\left(BB^*\right)\right)^{1-\alpha}\right) s_j^{\frac{r}{2}}\left(\left(g_2^{\frac{1}{\alpha}}\left(BB^*\right)\right)^{\alpha}\left(f_2^{\frac{1}{1-\alpha}}\left(AA^*\right)\right)^{1-\alpha}\right) \\ \leqslant & \prod_{j=1}^{k} s_j^{\frac{r}{2}}\left(\alpha f_1^{\frac{1}{\alpha}}\left(AA^*\right)+(1-\alpha) g_1^{\frac{1}{1-\alpha}}\left(BB^*\right)\right) s_j^{\frac{r}{2}}\left(\alpha g_2^{\frac{1}{\alpha}}\left(BB^*\right)+(1-\alpha) f_2^{\frac{1}{1-\alpha}}\left(AA^*\right)\right), \end{aligned}$$

and so

$$\begin{aligned} \prod_{j=1}^{k} s_j\left(|A^*B|^r\right) \leqslant & \prod_{j=1}^{k} s_j^{\frac{r}{2}}\left(\alpha f_1^{\frac{1}{\alpha}}\left(AA^*\right)+(1-\alpha) g_1^{\frac{1}{1-\alpha}}\left(BB^*\right)\right) \\ & \times s_j^{\frac{r}{2}}\left((1-\alpha) f_2^{\frac{1}{1-\alpha}}\left(AA^*\right)+\alpha g_2^{\frac{1}{\alpha}}\left(BB^*\right)\right). \end{aligned} \tag{13}$$

Now let $L = X^{\frac{r}{2}}$ and $M = Y^{\frac{r}{2}}$ where

$$X = diag(s_1(\alpha f_1^{\frac{1}{\alpha}}(AA^*) + (1-\alpha)g_1^{\frac{1}{1-\alpha}}(BB^*)), \ldots, s_n(\alpha f_1^{\frac{1}{\alpha}}(AA^*) + (1-\alpha)g_1^{\frac{1}{1-\alpha}}(BB^*))),$$
$$Y = diag(s_1((1-\alpha)f_2^{\frac{1}{1-\alpha}}(AA^*) + \alpha g_2^{\frac{1}{\alpha}}(BB^*)), \ldots, s_n((1-\alpha)f_2^{\frac{1}{1-\alpha}}(AA^*) + \alpha g_2^{\frac{1}{\alpha}}(BB^*))).$$

Then, we get from the inequality (13) that

$$\prod_{j=1}^{k} s_j(|A^*B|^r) \leqslant \prod_{j=1}^{k} s_j(LM).$$

Since the weak majorization follows from the weak log-majorization, we conclude that

$$\sum_{j=1}^{k} s_j(|A^*B|^r) \leqslant \sum_{j=1}^{k} s_j\left(X^{\frac{r}{2}}Y^{\frac{r}{2}}\right).$$

By the Fan dominance theorem, we get that

$$|||\ |A^*B|^r||| \leqslant \left|\left|\left|X^{\frac{r}{2}}Y^{\frac{r}{2}}\right|\right|\right|. \tag{14}$$

Using inequality (6), leads to

$$\begin{aligned}\left|\left|\left|X^{\frac{r}{2}}Y^{\frac{r}{2}}\right|\right|\right| &\leqslant \left|\left|\left|X^{\frac{rp}{2}}\right|\right|\right|^{\frac{1}{p}}\left|\left|\left|Y^{\frac{rq}{2}}\right|\right|\right|^{\frac{1}{q}} \\ &= \left|\left|\left|\left(\alpha f_1^{\frac{1}{\alpha}}(AA^*) + (1-\alpha)g_1^{\frac{1}{1-\alpha}}(BB^*)\right)^{\frac{rp}{2}}\right|\right|\right|^{\frac{1}{p}} \\ &\quad \times \left|\left|\left|\left((1-\alpha)f_2^{\frac{1}{1-\alpha}}(AA^*) + \alpha g_2^{\frac{1}{\alpha}}(BB^*)\right)^{\frac{rq}{2}}\right|\right|\right|^{\frac{1}{q}}.\end{aligned} \tag{15}$$

Combining the inequalities (14) and (15) leads to the inequality (12).

COROLLARY 1. *Let* $A, B \in M_n(\mathbb{C})$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, $p, q > 1$, $r \geqslant 0$. *If* m, n, u *and* v *are real numbers such that* $m + n = u + v = 1$, *then*

$$\begin{aligned}|||\ |A^*B|^r||| \leqslant &\left|\left|\left|\left(\alpha(AA^*)^{\frac{m}{\alpha}} + (1-\alpha)(BB^*)^{\frac{u}{1-\alpha}}\right)^{\frac{rp}{2}}\right|\right|\right|^{\frac{1}{p}} \\ &\times \left|\left|\left|\left((1-\alpha)(AA^*)^{\frac{n}{1-\alpha}} + \alpha(BB^*)^{\frac{v}{\alpha}}\right)^{\frac{rq}{2}}\right|\right|\right|^{\frac{1}{q}},\end{aligned} \tag{16}$$

for all $\alpha \in [0, 1]$.

Proof. Let $f_1 = t^m$, $f_2 = t^n$, $g_1 = t^u$ and $g_2 = t^v$ in the inequality (12) to get the result.

Note that the inequality (16) generalizes the inequality (9).

COROLLARY 2. *Let $A,B,X \in M_n(\mathbb{C})$ and $\frac{1}{p}+\frac{1}{q}=1$, $p,q>1$, $r \geqslant 0$, $\alpha \in [0,1]$. If f_1,f_2,g_1,g_2 are non-negative continuous functions such that $f_1(t)f_2(t)=t$ and $g_1(t)g_2(t)=t$ for $t \geqslant 0$, then*

$$\left|\left|\left| \, |A^*XB|^{2r} \right|\right|\right| \leqslant \left|\left|\left| \left(\alpha f_1{}^{\frac{1}{\alpha}}\left(|X|^{\frac{1}{2}}U^*AA^*U|X|^{\frac{1}{2}}\right)+(1-\alpha)g_1{}^{\frac{1}{1-\alpha}}\left(|X|^{\frac{1}{2}}BB^*|X|^{\frac{1}{2}}\right)\right)^{rp} \right|\right|\right|^{\frac{1}{p}} \times \left|\left|\left| \left((1-\alpha)f_2{}^{\frac{1}{1-\alpha}}\left(|X|^{\frac{1}{2}}U^*AA^*U|X|^{\frac{1}{2}}\right)+\alpha g_2{}^{\frac{1}{\alpha}}\left(|X|^{\frac{1}{2}}BB^*|X|^{\frac{1}{2}}\right)\right)^{rq} \right|\right|\right|^{\frac{1}{q}}, \tag{17}$$

where $X=U|X|$ is the polar decomposition of X. In particular, if X is positive semidefinite, we have

$$\left|\left|\left| \, |A^*XB|^{2r} \right|\right|\right| \leqslant \left|\left|\left| \left(\alpha f_1{}^{\frac{1}{\alpha}}\left(X^{\frac{1}{2}}AA^*X^{\frac{1}{2}}\right)+(1-\alpha)g_1{}^{\frac{1}{1-\alpha}}\left(X^{\frac{1}{2}}BB^*X^{\frac{1}{2}}\right)\right)^{rp} \right|\right|\right|^{\frac{1}{p}} \times \left|\left|\left| \left((1-\alpha)f_2{}^{\frac{1}{1-\alpha}}\left(X^{\frac{1}{2}}AA^*X^{\frac{1}{2}}\right)+\alpha g_2{}^{\frac{1}{\alpha}}\left(X^{\frac{1}{2}}BB^*X^{\frac{1}{2}}\right)\right)^{rq} \right|\right|\right|^{\frac{1}{q}}.$$

Proof. Since $X=U|X|$ for some unitary matrix U, then

$$\left|\left|\left| \, |A^*XB|^{2r} \right|\right|\right| = \left|\left|\left| \, |A^*U|X|B|^{2r} \right|\right|\right| = \left|\left|\left| \, \left|\left(|X|^{\frac{1}{2}}U^*A\right)^*\left(|X|^{\frac{1}{2}}B\right)\right|^{2r} \right|\right|\right|.$$

The inequality (17) follows directly by replacing A, B and r in the inequality (12) respectively by $|X|^{\frac{1}{2}}U^*A$, $|X|^{\frac{1}{2}}B$ and $2r$.

REMARK 1. To see that the inequality (17) generalizes the inequality (10), let $f_1=g_2=t^{\alpha}$ and $f_2=g_1=t^{1-\alpha}$ in the inequality (17) to get that

$$\left|\left|\left| \, |A^*XB|^{2r} \right|\right|\right| \leqslant \left|\left|\left| \left(\alpha|X|^{\frac{1}{2}}U^*AA^*U|X|^{\frac{1}{2}}+(1-\alpha)|X|^{\frac{1}{2}}BB^*|X|^{\frac{1}{2}}\right)^{rp} \right|\right|\right|^{\frac{1}{p}} \times \left|\left|\left| \left((1-\alpha)|X|^{\frac{1}{2}}U^*AA^*U|X|^{\frac{1}{2}}+\alpha|X|^{\frac{1}{2}}BB^*|X|^{\frac{1}{2}}\right)^{rq} \right|\right|\right|^{\frac{1}{q}}.$$

If $r \geqslant \max\left\{\frac{1}{p},\frac{1}{q}\right\}$, using Lemma 4 leads to

$$\begin{aligned}
\left|\left|\left| \, |A^*XB|^{2r} \right|\right|\right| &\leqslant ||| \, |\alpha U^*AA^*U|X|+(1-\alpha)|X|BB^*|^{rp} |||^{\frac{1}{p}} \\
&\quad \times ||| \, |(1-\alpha)U^*AA^*U|X|+\alpha|X|BB^*|^{rq} |||^{\frac{1}{q}} \\
&= ||| \, |U^*(\alpha AA^*U|X|+(1-\alpha)U|X|BB^*)|^{rp} |||^{\frac{1}{p}} \\
&\quad \times ||| \, |U^*((1-\alpha)AA^*U|X|+\alpha U|X|BB^*)|^{rq} |||^{\frac{1}{q}} \\
&= ||| \, |U^*(\alpha AA^*X+(1-\alpha)XBB^*)|^{rp} |||^{\frac{1}{p}} \\
&\quad \times ||| \, |U^*((1-\alpha)AA^*X+\alpha XBB^*)|^{rq} |||^{\frac{1}{q}} \\
&= ||| \, |\alpha AA^*X+(1-\alpha)XBB^*|^{rp} |||^{\frac{1}{p}} \\
&\quad \times ||| \, |(1-\alpha)AA^*X+\alpha XBB^*|^{rq} |||^{\frac{1}{q}}.
\end{aligned}$$

3. Alternative proof of Audenaert's inequality

To introduce the alternative proof, we need the following lemma which is nothing but a direct consequence of Theorem 1.2 in [10].

LEMMA 5. *Let* $A,B,X \in M_n(\mathbb{C})$ *such that* A *and* B *are positive semidefinite and let* $r \geqslant 0$. *Then the function*

$$f(\alpha) = \left|\left|\left| |A^{\alpha}XB^{1-\alpha}|^{r} \right|\right|\right| \left|\left|\left| |A^{1-\alpha}XB^{\alpha}|^{r} \right|\right|\right| \tag{18}$$

is log-convex on $[0,1]$.

It is clear that the function $f(\alpha)$ defined in Lemma 5 is convex and satisfies the property that $f(\alpha) = f(1-\alpha)$ for $0 \leqslant \alpha \leqslant 1$. Moreover, it is decreasing on $[0,\frac{1}{2}]$, increasing on $[\frac{1}{2},1]$ and attains its minimum at $\frac{1}{2}$. We are now ready to present Audenaert's inequality alternative proof.

Proof of the inequality (9). Apply Lemma 5 to the positive semidefinite matrices AA^*, BB^*, and replace X by I, to get that the function

$$f(\alpha) = \left|\left|\left| |(AA^*)^{\alpha}(BB^*)^{1-\alpha}|^{r} \right|\right|\right| \left|\left|\left| |(AA^*)^{1-\alpha}(BB^*)^{\alpha}|^{r} \right|\right|\right| \tag{19}$$

is log-convex (and hence convex) on the interval $[0,1]$, decreasing on $[0,\frac{1}{2}]$, increasing on $[\frac{1}{2},1]$ and attains its minimum at $\frac{1}{2}$.
Using Lemma 3, we get that

$$s_j\left(\left|(AA^*)^{\alpha}(BB^*)^{1-\alpha}\right|\right) = s_j\left((AA^*)^{\alpha}(BB^*)^{1-\alpha}\right) \leqslant s_j\left(\alpha AA^* + (1-\alpha)BB^*\right),$$

and so

$$s_j{}^r\left(\left|(AA^*)^{\alpha}(BB^*)^{1-\alpha}\right|\right) \leqslant s_j{}^r\left(\alpha AA^* + (1-\alpha)BB^*\right),$$

therefore

$$\left|\left|\left| |(AA^*)^{\alpha}(BB^*)^{1-\alpha}|^{r} \right|\right|\right| \leqslant |||(\alpha AA^* + (1-\alpha)BB^*)^r|||. \tag{20}$$

Since $0 < \frac{1}{p} < 1$, and using the submultiplicativity of the unitarily invariant norm, we get that

$$\begin{aligned} |||(\alpha AA^* + (1-\alpha)BB^*)^r||| &= \left|\left|\left| \left((\alpha AA^* + (1-\alpha)BB^*)^{\frac{rp}{2}}\right)^{\frac{2}{p}} \right|\right|\right| \\ &\leqslant \left|\left|\left| \left((\alpha AA^* + (1-\alpha)BB^*)^{\frac{rp}{2}}\right)^{2} \right|\right|\right|^{\frac{1}{p}} \\ &\leqslant \left|\left|\left| (\alpha AA^* + (1-\alpha)BB^*)^{\frac{rp}{2}} \right|\right|\right|^{\frac{2}{p}}. \end{aligned} \tag{21}$$

Combining the inequalities (20) and (21) leads to the inequality

$$\left|\left|\left|\,\left|(AA^*)^{\alpha}(BB^*)^{1-\alpha}\right|^r\right|\right|\right| \leqslant \left|\left|\left|(\alpha AA^* + (1-\alpha)BB^*)^{\frac{rp}{2}}\right|\right|\right|^{\frac{2}{p}}. \tag{22}$$

Similarly, and since $0 < \frac{1}{q} < 1$, we get that

$$\left|\left|\left|\,\left|(AA^*)^{1-\alpha}(BB^*)^{\alpha}\right|^r\right|\right|\right| \leqslant \left|\left|\left|((1-\alpha)AA^* + \alpha BB^*)^{\frac{rq}{2}}\right|\right|\right|^{\frac{2}{q}}. \tag{23}$$

Combining the inequalities (22) and (23) with the equation (19) leads to the inequaltiy

$$f(\alpha) \leqslant \left|\left|\left|(\alpha AA^* + (1-\alpha)BB^*)^{\frac{rp}{2}}\right|\right|\right|^{\frac{2}{p}} \left|\left|\left|((1-\alpha)AA^* + \alpha BB^*)^{\frac{rq}{2}}\right|\right|\right|^{\frac{2}{q}}. \tag{24}$$

Now let $A^* = U\,|A^*|$ and $B^* = V\,|B^*|$ be the polar decompositions of A^* and B^* where U and V are unitary matrices. Then

$$f\left(\frac{1}{2}\right) = |||\ \left||A^*|\,|B^*|\right|^r|||^2 = |||\ |U\,|A^*|\,|B^*|\,V^*|^r|||^2 = |||\ |A^*B|^r|||^2. \tag{25}$$

Combining the inequality (24) with the latter equality in (25) gives

$$\begin{aligned} |||\ |A^*B|^r|||^2 &= f\left(\frac{1}{2}\right) \\ &\leqslant f(\alpha) \\ &\leqslant \left|\left|\left|(\alpha AA^* + (1-\alpha)BB^*)^{\frac{rp}{2}}\right|\right|\right|^{\frac{2}{p}} \left|\left|\left|((1-\alpha)AA^* + \alpha BB^*)^{\frac{rq}{2}}\right|\right|\right|^{\frac{2}{q}}. \end{aligned} \tag{26}$$

Defining $g(\alpha) = \sqrt{f(\alpha)}$ for all $\alpha \in [0,1]$, and taking the square root for all the sides in (26) leads to

$$\begin{aligned} |||\ |A^*B|^r||| &= g\left(\frac{1}{2}\right) \\ &\leqslant g(\alpha) \\ &\leqslant \left|\left|\left|(\alpha AA^* + (1-\alpha)BB^*)^{\frac{rp}{2}}\right|\right|\right|^{\frac{1}{p}} \left|\left|\left|((1-\alpha)AA^* + \alpha BB^*)^{\frac{rq}{2}}\right|\right|\right|^{\frac{1}{q}}. \quad \square \end{aligned} \tag{27}$$

Note that (27) represents a generalization of (11) for the special case when $X = I$.

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