Mathematical
Inequalities
& Applications
Volume 23, Number 3 (2020), 1145–1154
doi:10.7153/mia-2020-23-88

COVERING FUNCTIONALS OF MINKOWSKI SUMS AND DIRECT SUMS OF CONVEX BODIES

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(*Communicated by H. Martini*)

Abstract. We prove a series of inequalities concerning covering functionals of convex bodies having the form $K+L$, where K is a convex body and L is a segment. Several estimations of covering functionals of direct sums of convex bodies are also presented.

1. Introduction

A compact convex set $K \subseteq \mathbb{R}^n$ having interior points is called a *convex body*. The *interior* and *boundary* of K is denoted by $\operatorname{int} K$ and $\operatorname{bd} K$, respectively. We denote by $\mathscr{K}^n$ the set of convex bodies in $\mathbb{R}^n$ and by o the *origin* of $\mathbb{R}^n$. Concerning the least number $c(K)$ of translates of $\operatorname{int} K$ needed to cover a convex body K, there is a long standing conjecture:

CONJECTURE 1. (Hadwiger's covering conjecture) For each $K \in \mathscr{K}^n$, $c(K)$ is bounded from above by 2^n, and this upper bound is attained only by parallelotopes.

See e.g., [6], [10], [7], [1], and [3] for more information and references about this conjecture. There are good estimations of $c(K)$ for special classes of convex bodies. A convex body is called a *zonotope* if it is the Minkowski sum of a finite number of segments, and is called a *zonoid* if it is the limit (with respect to the Hausdorff metric) of a converging sequence of zonotopes. Martini proved that

$$c(K) \leqslant \frac{3}{4} \cdot 2^n \tag{1}$$

holds for each n-dimensional zonotope distinct from a parallelotope (cf. [9]). The same estimation holds also for n-dimensional zonoids, belt polytopes, and belt bodies (cf. [4] for the definition) that are not parallelotopes, see, e.g., p. 339 and p. 341 in [6] and [4].

Mathematics subject classification (2010): 52C17, 52A20.
Keywords and phrases: Convex body, covering, Hadwiger's covering conjecture.
Senlin Wu is supported by the Natural Science Foundation of Shanxi Province of China (grant number 201901D111141). Chan He is supported by Science Foundation of North University of China (grant number XJJ201812).
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Note that, for each $K \in \mathscr{K}^n$, $c(K)$ equals the least number of smaller homothetic copies of K needed to cover K (see, e.g., [6, p. 262, Theorem 34.3]). Therefore, $c(K) \leqslant m$ for some $m \in \mathbb{Z}^+$ if and only if $\Gamma_m(K) < 1$, where $\Gamma_m(K)$ is defined by

$$\Gamma_m(K) = \min\left\{\gamma > 0 \mid \exists\, \{x_i \mid i = 1,\cdots,m\} \subseteq \mathbb{R}^n \text{ s.t. } K \subseteq \bigcup_{i=1}^{m}(x_i + \gamma K)\right\},$$

and is called *the covering functional of K with respect to m*. Clearly, for each $m \in \mathbb{Z}^+$, $\Gamma_m(\cdot)$ is an affine invariant. More precisely,

$$\Gamma_m(K) = \Gamma_m(T(K)),\ \forall T \in \mathscr{A}^n,$$

where $\mathscr{A}^n$ is the set of non-degenerate affine transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$. Thus we identify convex bodies that are affinely equivalent, and when writing $\mathscr{K}^n$ we are actually referring to the quotient space of $\mathscr{K}^n$ with respect to affine equivalence.

For each pair of convex bodies K_1 and K_2 in $\mathscr{K}^n$, the *Banach-Mazur distance* $d_{BM}(K_1,K_2)$ (also called the *Asplund metric*, cf. [16]) between them is defined by

$$d_{BM}(K_1,K_2) := \ln\min\{\gamma \geqslant 1 \mid K_1 \subseteq T(K_2) \subseteq \gamma K_1 + x,\ x \in \mathbb{R}^n,\ T \in \mathscr{A}^n\}.$$

Then $(\mathscr{K}^n, d_{BM})$ is a compact metric space (cf. [8] and [16]). Zong (cf. [15]) proved that $\Gamma_m(\cdot)$ is uniformly continuous on $\mathscr{K}^n$. Bezdek and Khan improved this result by showing that $\Gamma_m(\cdot)$ is Lipschitz continuous on $\mathscr{K}^n$ with $(n^2-1)/(2\ln n)$ as a Lipschitz constant (cf. [2]). These results show that each $K \in \mathscr{K}^n$ can be covered by at most 2^n smaller homothetic copies of K if and only if

$$c(n) := \sup\{\Gamma_{2^n}(K) \mid K \in \mathscr{K}^n\} < 1.$$

Based on these results, Zong proposed a quantitative program for attacking Hadwiger's covering conjecture (cf. [15] for more details), in which estimating the supremum of $\Gamma_{2^n}(K)$ over special classes of n-dimensional convex bodies plays an important role.

As we have mentioned, we already have good knowledge about $c(K)$ for the classes of zonotopes and zonoids. Thus it is natural to try to obtain good estimations of the following number:

$$c_z(n) := \sup\{\Gamma_{2^n}(K) \mid K \subset \mathbb{R}^n \text{ is a zonoid}\}.$$

Let $\delta \in (0,\ln 2)$. By Corollary 20 and Remark 21 in [13], we have

$$\sup\{\Gamma_{2^n}(K) \mid K \subset \mathbb{R}^n \text{ is a zonoid satisfying } d_{BM}(K,[0,1]^n) < \delta\} < 1.$$

Also, we claim that

$$\sup\{\Gamma_{2^n}(K) \mid K \subset \mathbb{R}^n \text{ is a zonoid satisfying } d_{BM}(K,[0,1]^n) \geqslant \delta\} < 1.$$

Otherwise there exists a sequence of zonoids $\{K_i\}_{i=1}^{\infty}$ converging to a convex body K_0 and a sequence of zonotopes $\{L_i\}_{i=1}^{\infty}$ such that

$$\lim_{i\to\infty} d_{BM}(K_i,L_i) = 0 \quad \text{and} \quad \lim_{i\to\infty}\Gamma_{2^n}(K_i) = 1.$$

Then $\lim_{i\to\infty} L_i = K_0$ is a zonoid distinct from a parallelotope and $\Gamma_{2^n}(K_0) = 1$. This is in contradiction to the estimation (1) which is also valid for zonoids. Hence, we have the following

PROPOSITION 1. *For each $n \geqslant 2$, $c_z(n) < 1$.*

Getting the value of $c_z(n)$ is not an easy task. One possible starting point for this is to study covering functionals of convex bodies having the form $K+L$, where K is a convex body and L is a segment. We prove a series of results in this direction in Section 2. We note that, in general, the relation between $\Gamma_m(K)$ and $\Gamma_m(K+L)$ could be complicated. For example, when $K = [-1,1]^2$, $u_1 = (1,1)$, and $u_2 = (1,0)$, we have

$$\Gamma_4(K+[-u_1,u_1]) > \Gamma_4(K) = \frac{1}{2},$$
$$\Gamma_4(K+[-u_2,u_2]) = \Gamma_4(K),$$
$$\Gamma_3(K+[-u_1,u_1]) < \Gamma_3(K) = 1.$$

In Section 3 we present an estimation of $\Gamma_m(K \oplus L)$, which can be viewed as a quantitative version of Theorem 1 in [5].

In the sequel, for each $m \in \mathbb{Z}^+$, we denote by $[m]$ the set of positive integers not greater than m.

2. The sum of K and a segment

Let K be a convex body in $\mathbb{R}^n$, $u \in \mathbb{S}^{n-1}$ be a direction, $[b,t]$ be an *affine diameter* of K in the direction of u (cf. e.g., [11] for the definition and properties of affine diameters), and H be an $(n-1)$-dimensional linear subspace of $\mathbb{R}^n$ such that the two supporting hyperplanes H_b and H_t of K parallel to H contains b and t, respectively. By taking a suitable affine transformation if necessary, we may assume that $u = e_n$, $H = \mathbb{R}^{n-1} \times \{0\}$, $b = -e_n$ and $t = e_n$. Let Π_u be the orthogonal projection of $\mathbb{R}^n$ onto H. For each $\lambda \geqslant 0$, put $K_\lambda = K + \lambda[-u,u]$. For each compact convex set L, we denote by $\operatorname{relint} L$ the *relative interior* of L. We note that, when $\operatorname{int} L \neq \emptyset$, we have $\operatorname{int} L = \operatorname{relint} L$. When L is not a convex body in $\mathbb{R}^n$, it can be viewed as a convex body having smaller dimension. In this case $c(L)$ is the least number of translates of $\operatorname{relint} L$ needed to cover L.

PROPOSITION 2. *For each $\lambda \geqslant 0$, $c(K_\lambda) \geqslant c(\Pi_u(K))$.*

Proof. Suppose that $c(K_\lambda) = m$. Then there exists a set $C = \{c_i \mid i \in [m]\}$ such that $K_\lambda \subseteq C + \operatorname{int} K_\lambda$.

It is clear that $\Pi_u(K) \subseteq \Pi_u(K_\lambda)$ holds for each $\lambda \geqslant 0$. For each point $w \in \Pi_u(K_\lambda)$, there exist a point $z \in K$ and a number $\alpha \in [-\lambda,\lambda]$ such that $w = \Pi_u(z+$

$\alpha u) = \Pi_u(z) \in \Pi_u(K)$. Thus $\Pi_u(K_\lambda) \subseteq \Pi_u(K)$. It follows that $\Pi_u(K) = \Pi_u(K_\lambda)$. By Theorem 2.34 in [12], we have

$$\begin{aligned}\Pi_u(K) = \Pi_u(K_\lambda) &\subseteq \Pi_u(C + \operatorname{int} K_\lambda)\\ &= \Pi_u(C) + \Pi_u(\operatorname{int} K_\lambda)\\ &= \Pi_u(C) + \operatorname{relint} \Pi_u(K_\lambda)\\ &= \Pi_u(C) + \operatorname{relint} \Pi_u(K).\end{aligned}$$

Hence $c(\Pi_u(K)) \leqslant m$.

REMARK 1. It is not difficult to find a convex body K and a direction u such that $c(K) = c(\Pi_u(K)) + 1$. We are not sure whether the following is true:

$$c(K_\lambda) > c(\Pi_u(K)),\ \forall K \in \mathscr{K}^n,\ \forall u \in \mathbb{S}^{n-1}.$$

PROPOSITION 3. *We have*

$$|\Gamma_m(K_\lambda) - \Gamma_m(K)| \leqslant \lambda,\ \forall \lambda \geqslant 0. \tag{2}$$

Proof. For each $\lambda \geqslant 0$, we have

$$K \subseteq K_\lambda = K + \lambda[-u,u] \subseteq K + \lambda K = (1+\lambda)K,$$

which, together with the proof of Theorem A in [15], shows the desired inequality.

The inequality (2) shows that

$$\Gamma_m(K) - \lambda \leqslant \Gamma_m(K_\lambda) \leqslant \lambda + \Gamma_m(K). \tag{3}$$

This estimation can be improved.

THEOREM 1. *We have*

$$\frac{1}{1+\lambda}\Gamma_m(K) \leqslant \Gamma_m(K_\lambda) \leqslant 1 - \frac{1}{1+\lambda} + \frac{1}{1+\lambda}\Gamma_m(K),\ \forall \lambda \geqslant 0. \tag{4}$$

Proof. Let $\lambda \geqslant 0$. Put $\gamma = \Gamma_m(K)$. There exists a set $C = \{c_i \mid i \in [m]\}$ of m points such that $K \subseteq C + \gamma K$. We have

$$\begin{aligned}K_\lambda = K + \lambda[-u,u] &\subseteq C + \gamma K + \lambda[-u,u]\\ &= C + \gamma(K + \lambda[-u,u]) + (\lambda - \gamma\lambda)[-u,u]\\ &\subseteq C + \gamma K_\lambda + \frac{\lambda - \gamma\lambda}{1+\lambda}K_\lambda\\ &= C + \frac{\lambda+\gamma}{1+\lambda}K_\lambda.\end{aligned}$$

It follows that

$$\Gamma_m(K_\lambda) \leqslant \frac{\lambda+\gamma}{1+\lambda} = 1 - \frac{1}{1+\lambda} + \frac{1}{1+\lambda}\Gamma_m(K).$$

Now we prove the inequality on the left. Put $\gamma_\lambda = \Gamma_m(K_\lambda)$. Then there exists a set C_λ of m points such that $K_\lambda \subseteq C_\lambda + \gamma_\lambda K_\lambda$. We have

$$K \subseteq K_\lambda \subseteq C_\lambda + \gamma_\lambda K_\lambda = C_\lambda + \gamma_\lambda K + \gamma_\lambda \lambda\,[-u,u] \subseteq C_\lambda + \gamma_\lambda (1+\lambda)K.$$

It follows that

$$\Gamma_m(K) \leqslant (1+\lambda)\Gamma_m(K_\lambda),$$

which completes the proof.

COROLLARY 1. *For each integer $m > 0$, we have*

$$\lim_{\lambda\to 0}\Gamma_m(K_\lambda) = \Gamma_m(K).$$

When λ tends to infinity, (4) provides less information on the upper bound of $\Gamma_m(K_\lambda)$. The next result gives a better estimation in this situation.

THEOREM 2. *For each $m \in \mathbb{Z}^+$ and each $\lambda > 1$, we have*

$$|\Gamma_m(K_\lambda) - \Gamma_m(\Pi_u(K)+[-u,u])| \leqslant \frac{2}{\lambda-1}, \tag{5}$$

and

$$\lim_{\lambda\to\infty}\Gamma_m(K_\lambda) = \Gamma_m(\Pi_u(K)+[-u,u]). \tag{6}$$

Proof. We only need to show (5). It is clear that

$$K \subseteq \Pi_u(K) + [-u,u].$$

Therefore, for each $\lambda \geqslant 0$, we have

$$K_\lambda = K + \lambda\,[-u,u] \subseteq \Pi_u(K) + [-u,u] + \lambda\,[-u,u] = \Pi_u(K) + (\lambda+1)\,[-u,u].$$

For each $x \in \Pi_u(K)$, there exist a point $y \in K$ and a number $\mu \in [-1,1]$ such that

$$x = y + \mu u \in K + [-u,u] = K_1.$$

Hence $\Pi_u(K) \subseteq K_1$. Since $\lambda > 1$ and $o \in \Pi_u(K)$, we have

$$\begin{aligned}\frac{\lambda-1}{\lambda+1}(\Pi_u(K)+(\lambda+1)\,[-u,u]) &= \frac{\lambda-1}{\lambda+1}\Pi_u(K) + (\lambda-1)\,[-u,u]\\ &\subseteq \Pi_u(K) + (\lambda-1)\,[-u,u]\\ &\subseteq K_1 + (\lambda-1)\,[-u,u]\\ &= K + [-u,u] + (\lambda-1)\,[-u,u] = K_\lambda.\end{aligned}$$

It follows that, when $\lambda > 1$,

$$K_\lambda \subseteq \Pi_u(K) + (\lambda+1)[-u,u] \subseteq \frac{\lambda+1}{\lambda-1}K_\lambda = \left(1+\frac{2}{\lambda-1}\right)K_\lambda. \tag{7}$$

From the proof of Theorem A in [15] and (7), we have

$$|\Gamma_m(K_\lambda) - \Gamma_m(\Pi_u(K) + (\lambda+1)[-u,u])| \leqslant \frac{2}{\lambda-1},$$

which, together with the fact that $\Pi_u(K) + (\lambda+1)[-u,u]$ is affinely equivalent to $\Pi_u(K) + [-u,u]$, implies (5).

COROLLARY 2. *If $m \in \mathbb{Z}^+$ and $m < 2 \cdot c(\Pi_u(K))$, then*

$$\lim_{\lambda\to\infty} \Gamma_m(K_\lambda) = 1.$$

Proof. By Corollary 3.10 in [14] or Theorem 2 in [5],

$$c(\Pi_u(K) + [-u,u]) = 2 \cdot c(\Pi_u(K)).$$

Thus, if $m < 2 \cdot c(\Pi_u(K))$ then

$$\lim_{\lambda\to\infty} \Gamma_m(K_\lambda) = \Gamma_m(\Pi_u(K) + [-u,u]) = 1.$$

LEMMA 1. *Let K be a convex body. Suppose that a and b are two points in $\mathbb{R}^n$ such that $(a+K)\cap(b+K) \neq \emptyset$. Then $(a+K)\cup(b+K)$ is contained in a translate of $2K$.*

Proof. We only need to consider the case when $a \neq b$. Let $[u,v]$ be an affine diameter of K parallel to $\langle a,b\rangle$. Without loss of generality we may assume that

$$\frac{u-v}{\|u-v\|} = \frac{a-b}{\|a-b\|}.$$

Put $c = \frac{u+v}{2}$, $K' = K - c$, $a' = a + c$, and $b' = b + c$. Then

$$(a'+K')\cap(b'+K') \neq \emptyset,$$

from which it follows that $a' - b' = a - b \in K' - K' = K - K$. Therefore we have two points $s,t \in K$ such that $a' - b' = s - t$. It follows that $[s,t]$ is a segment contained in K and parallel to $\langle a,b\rangle$. Therefore $\|a-b\| = \|s-t\| \leqslant \|u-v\|$. It suffices to show that $(a'+K')\cup(b'+K')$ is contained in a translate of $2K'$.

Let x be an arbitrary point in $a'+K'$. Then $x - a' \in K'$. This yields

$$x - \frac{a'+b'}{2} = x - a' + \frac{a'-b'}{2}$$

$$\begin{aligned}
&= x - a' + \frac{a-b}{2} \\
&= x - a' + \frac{1}{2}\cdot\frac{\|a-b\|}{\|u-v\|}(u-v) \\
&= x - a' + \frac{\|a-b\|}{\|u-v\|}(u-c) \\
&\in K' + \frac{\|a-b\|}{\|u-v\|}K' \\
&\subseteq K' + K' = 2K'.
\end{aligned}$$

It follows that $a' + K' \subseteq \frac{a'+b'}{2} + 2K'$. In a similar way we can show that $b' + K' \subseteq \frac{a'+b'}{2} + 2K'$.

LEMMA 2. *Let K be a convex body and $m = c(K)$. Then $\Gamma_m(K) \geqslant \frac{1}{2}$.*

Proof. Otherwise, $\gamma := \Gamma_m(K) < \frac{1}{2}$. Let $C = \{c_i \mid i \in [m]\}$ be a set of points such that $K \subseteq C + \gamma K$. Since $m = c(K)$, for each $i \in [m]$, $(c_i + \gamma K) \cap K$ is a nonempty closed convex subset of K. Since K is connected, there are two members of $\{c_i + \gamma K \mid i \in [m]\}$ having nonempty intersection. Assume without loss of generality that $(c_1 + \gamma K) \cap (c_2 + \gamma K) \neq \emptyset$. By Lemma 1, there exists a translate of $2\gamma K$ containing $(c_1 + \gamma K) \cup (c_2 + \gamma K)$, which yields a contradiction to the fact that $m = c(K)$.

PROPOSITION 4. *Let $m = c(\Pi_u(K))$. Then*

$$\Gamma_{2m}(\Pi_u(K) + [-u,u]) = \Gamma_m(\Pi_u(K)).$$

Proof. Put $\gamma = \Gamma_m(\Pi_u(K))$. By Lemma 2, $\gamma \geqslant \frac{1}{2}$. There exists a set $C = \{c_i \mid i \in [m]\}$ of points such that

$$\Pi_u(K) \subseteq C + \gamma\Pi_u(K).$$

Then

$$\begin{aligned}
\Pi_u(K) + [-u,u] &\subseteq C + \gamma\Pi_u(K) + [-u,u] \\
&\subseteq C + \gamma\Pi_u(K) + \left\{\frac{1}{2}u, -\frac{1}{2}u\right\} + \gamma[-u,u] \\
&= \left(C + \frac{1}{2}u\right) \cup \left(C - \frac{1}{2}u\right) + \gamma(\Pi_u(K) + [-u,u]),
\end{aligned}$$

which implies that $\Gamma_{2m}(\Pi_u(K) + [-u,u]) \leqslant \gamma$.

Suppose that $\gamma' \in (0,\gamma)$ and that $c + \gamma'(\Pi_u(K) + [-u,u])$ is a smaller homothetic copy of $\Pi_u(K) + [-u,u]$ that intersects $\Pi_u(K) + u$. Then there exists $\alpha < 0$ such that $\Pi_u(c) - c = \alpha u$, and

$$(c + \gamma'(\Pi_u(K) + [-u,u])) \cap (\Pi_u(K) - u) = \emptyset.$$

It is not difficult to show that

$$\begin{aligned}(c+\gamma'(\Pi_u(K)+[-u,u]))\cap(\Pi_u(K)+u) &= (\Pi_u(c)+u+\gamma'\Pi_u(K))\cap(\Pi_u(K)+u)\\ &= ((\Pi_u(c)+\gamma'\Pi_u(K))\cap\Pi_u(K))+u.\end{aligned}$$

Thus, to cover $\Pi_u(K)+u$, one needs at least $m+1$ translates of $\gamma'(\Pi_u(K)+[-u,u])$. Similarly, to cover $\Pi_u(K)-u$, one needs at least further $m+1$ translates of $\gamma'(\Pi_u(K)+[-u,u])$. Therefore $\Gamma_{2m}(\Pi_u(K)+[-u,u])\geqslant\gamma$. This completes the proof.

PROPOSITION 5. *If $m\in\mathbb{Z}^+$ and $m\geqslant 2\cdot c(\Pi_u(K))$, then*

$$\lim_{\lambda\to\infty}\Gamma_m(K_\lambda)\leqslant\Gamma_{c(\Pi_u(K))}(\Pi_u(K)).$$

Proof. By Theorem 2 and Proposition 4 we have

$$\lim_{\lambda\to\infty}\Gamma_m(K_\lambda)=\Gamma_m(\Pi_u(K)+[-u,u])\leqslant\Gamma_{2\cdot c(\Pi_u(K))}(\Pi_u(K)+[-u,u])=\Gamma_{c(\Pi_u(K))}(\Pi_u(K)).$$

PROPOSITION 6. *If $K=\Pi_u(K)+\gamma[-u,u]$, where $\gamma>0$, then*

$$\Gamma_m(K)=\Gamma_m(K_\lambda),\ \forall m\in\mathbb{Z}^+,\lambda\geqslant 0.$$

3. Covering functionals of direct sum of convex bodies

PROPOSITION 7. *Suppose that $\mathbb{R}^n$ is the direct vector sum of two of its subspaces L_1 and L_2, and K_1 and K_2 are convex bodies in L_1 and L_2, respectively. Moreover, we assume that K_1 and K_2 contains the origin of L_1 and L_2, respectively. For each pair of positive integers m_1 and m_2, we have*

$$\Gamma_{m_1\times m_2}(K_1\oplus K_2)\leqslant\max\{\Gamma_{m_1}(K_1),\Gamma_{m_2}(K_2)\}.$$

Moreover, if $m_1=c(K_1)$ and $m_2=c(K_2)$, we have

$$\Gamma_{m_1\times m_2}(K_1\oplus K_2)=\max\{\Gamma_{m_1}(K_1),\Gamma_{m_2}(K_2)\}. \tag{8}$$

Proof. Put $\gamma_1=\Gamma_{m_1}(K_1)$ and $\gamma_2=\Gamma_{m_2}(K_2)$. There exist a set $C_1\subset L_1$ of m_1 points and a set $C_2\subset L_2$ of m_2 points such that

$$K_1\subseteq C_1+\gamma_1K_1\quad\text{and}\quad K_2\subseteq C_2+\gamma_2K_2.$$

For each point $x\in K_1\oplus K_2$, there exists a unique pair of points $x_1\in K_1$ and $x_2\in K_2$ such that $x=x_1+x_2$. Then there exist two points $c_1\in C_1$ and $c_2\in C_2$ such that

$$\begin{aligned}x=x_1+x_2 &\in (c_1+\gamma_1K_1)\oplus(c_2+\gamma_2K_2)\\ &\subseteq (c_1+\max\{\gamma_1,\gamma_2\}K_1)\oplus(c_2+\max\{\gamma_1,\gamma_2\}K_2)\\ &= c_1+c_2+\max\{\gamma_1,\gamma_2\}(K_1\oplus K_2)\end{aligned}$$

$$\subseteq C_1 \oplus C_2 + \max\{\gamma_1, \gamma_2\}(K_1 \oplus K_2).$$

Since $C_1 \oplus C_2$ consists of $m_1 \times m_2$ points, we have

$$\Gamma_{m_1 \times m_2}(K_1 \oplus K_2) \leqslant \max\{\Gamma_{m_1}(K_1), \Gamma_{m_2}(K_2)\}.$$

Suppose that $m_1 = c(K_1)$ and $m_2 = c(K_2)$. By the definition of $\Gamma_{m_1 \times m_2}(K_1 \oplus K_2)$, there exists a set $C = \{c_i \mid i \in [m_1 \times m_2]\}$ of $m_1 \times m_2$ points in $\mathbb{R}^n$ such that

$$K_1 \oplus K_2 \subseteq C + \Gamma_{m_1 \times m_2}(K_1 \oplus K_2) K_1 \oplus K_2.$$

Since $\mathbb{R}^n = L_1 \oplus L_2$, for each $i \in [m_1 \times m_2]$, there exists a unique pair of points $p_i \in L_1$ and $q_i \in L_2$ such that $c_i = p_i + q_i$. Note that, for distinct $i, j \in [m_1 \times m_2]$, p_i (q_i, resp.) might coincide with p_j (q_j, resp.).

Let x_1 be an arbitrary point in K_1 and x_2 be an arbitrary in K_2. Then there exists $i \in [m_1 \times m_2]$ such that

$$x_1 + x_2 \in c_i + \Gamma_{m_1 \times m_2}(K_1 \oplus K_2) K_1 \oplus K_2 = p_i + q_i + \Gamma_{m_1 \times m_2}(K_1 \oplus K_2) K_1 \oplus K_2,$$

which implies that there exist points $y_1 \in K_1$ and $y_2 \in K_2$ such that

$$x_1 + x_2 = p_i + q_i + \Gamma_{m_1 \times m_2}(K_1 \oplus K_2)(y_1 + y_2).$$

Thus

$$x_1 = p_i + \Gamma_{m_1 \times m_2}(K_1 \oplus K_2) y_1 \quad \text{and} \quad x_2 = q_i + \Gamma_{m_1 \times m_2}(K_1 \oplus K_2) y_2.$$

It follows that

$$K_1 \subseteq \{p_i \mid i \in [m_1 \times m_2]\} + \Gamma_{m_1 \times m_2}(K_1 \oplus K_2) K_1$$

and

$$K_2 \subseteq \{q_i \mid i \in [m_1 \times m_2]\} + \Gamma_{m_1 \times m_2}(K_1 \oplus K_2) K_2.$$

Therefore

$$\operatorname{card}\{p_i \mid i \in [m_1 \times m_2]\} \geqslant m_1 \quad \text{and} \quad \operatorname{card}\{q_i \mid i \in [m_1 \times m_2]\} \geqslant m_2,$$

which, together with the fact that

$$m_1 \times m_2 = \operatorname{card} C = \operatorname{card}\{p_i \mid i \in [m_1 \times m_2]\} \times \operatorname{card}\{q_i \mid i \in [m_1 \times m_2]\},$$

shows that

$$\operatorname{card}\{p_i \mid i \in [m_1 \times m_2]\} = m_1 \quad \text{and} \quad \operatorname{card}\{q_i \mid i \in [m_1 \times m_2]\} = m_2.$$

Finally we have

$$\max\{\Gamma_{m_1}(K_1), \Gamma_{m_2}(K_2)\} \leqslant \Gamma_{m_1 \times m_2}(K_1 \oplus K_2).$$

This completes the proof.

REMARK 2. We remark that (8) can be viewed as an extension of Proposition 4. In general, (8) is not true. When $c(K_1 \oplus K_2) \leqslant m_1 \times m_2$ and $c(K_1) > m_1$, we have

$$\Gamma_{m_1 \times m_2}(K_1 \oplus K_2) < 1 = \max\{\Gamma_{m_1}(K_1), \Gamma_{m_2}(K_2)\}.$$

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(Received May 6, 2020)

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