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GENERALIZED TRIANGLE INEQUALITY OF THE SECOND TYPE IN QUASI NORMED SPACES

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Abstract. We investigate a generalized triangle inequality of the second type in the framework of quasi normed spaces. More precisely, by using the well-known Aoki-Rolewicz theorem and some quasi normed inequalities, we obtain some regions of $\mathbb{R}^n$ which contain the set of all n-tuples satisfying the mentioned inequality. Moreover, some reverse inclusions are also discussed. As applications, we deduce some new results associated with generalizations of the triangle inequality in p-normed spaces and we get some already known results in a new approach.

1. Introduction and preliminaries

The triangle inequality is considered to be one of the most fundamental inequalities in mathematics. There are many interesting generalizations, refinements and reverses of the triangle inequality in normed spaces, quasi normed spaces, inner product spaces, pre-Hilbert C^*- modules; see [14, 9, 8, 15, 16, 6, 4, 7] and references therein.
Some generalizations of the triangle inequality are profitable to study of the geometrical structure of Banach spaces [5]. Especially, some results based on the triangle inequality of the second type

$$\|x+y\|^2 \leqslant 2\left(\|x\|^2+\|y\|^2\right) \tag{1}$$

and its generalizations in normed spaces can be found in [2, 15]. The Euler-Lagrange type identity [13]

$$\frac{\|x\|^2}{\mu}+\frac{\|y\|^2}{\nu}-\frac{\|ax+by\|^2}{\lambda}=\frac{\|\nu bx-\mu ay\|^2}{\lambda\mu\nu} \quad (\lambda=\mu a^2+\nu b^2,\ \lambda\mu\nu>0)$$

yields the following more general triangle inequality of the second type

$$\frac{\|ax+by\|^2}{\lambda}\leqslant\frac{\|x\|^2}{\mu}+\frac{\|y\|^2}{\nu} \quad (\lambda=\mu a^2+\nu b^2,\ \lambda\mu\nu>0)$$

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in Hilbert spaces [15] that comprises inequality (1) as a special case. In addition, Takahasi et al. [15] obtained some conditions for which the inequality

$$\frac{\|ax+by\|^q}{\lambda} \leqslant \frac{\|x\|^q}{\mu} + \frac{\|y\|^q}{\nu}$$

holds for $q \geqslant 1$. In [3], the authors discussed the generalized triangle inequality of the second type and its reverse in normed spaces. Also Izumida et al. [9] presented another approach to characterizations of the generalized triangle inequality of the second type by using ψ-direct sums of Banach spaces.
The notion of quasi norm is a generalization of a norm that Hyers [12] introduced it under the names pseudo norm and absolute value. The label quasi norm was proposed by Bourgin in 1943. Tychonoff gave the first example of a quasi Banach space. In particular, he proved that $\|x+y\|_{\frac{1}{2}} \leqslant 2\left(\|x\|_{\frac{1}{2}} + \|y\|_{\frac{1}{2}}\right)$ for all $x,y \in l_{\frac{1}{2}}$ [12].

In this paper, we investigate the generalized triangle inequality of the second type

$$\|x_1+\ldots+x_n\|^q \leqslant \frac{\|x_1\|^q}{\mu_1} + \ldots + \frac{\|x_n\|^q}{\mu_n} \quad (q>0) \tag{2}$$

in the framework of quasi normed spaces. More precisely, by using the well-known Aoki-Rolewicz theorem and some quasi normed inequalities, we obtain some regions of $\mathbb{R}^n$ which contain the set of all n-tuples $(\mu_1,\ldots,\mu_n)$ satisfying inequality (2). Moreover, some reverse inclusions are discussed. As applications, we deduce some results associated with generalizations of the triangle inequality in p-normed spaces and get some already known results due to Belbachir et al. [2] and Dadipour et al. [3] in a new fashion.

In the remainder of this section, we recall some basic concepts, preliminary results and symbols that are used throughout the paper.
A quasi norm on a vector space X is a real valued function $\|\cdot\| : X \to \mathbb{R}$ satisfying the following properties:

(i) $\|x\| \geqslant 0$, for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$,

(ii) $\|\lambda x\| = |\lambda|\|x\|$, for all $\lambda \in \mathbb{R}$ and all $x \in X$,

(iii) There is a constant $C \geqslant 1$ such that $\|x+y\| \leqslant C(\|x\|+\|y\|)$, for all $x,y \in X$ (the quasi triangle inequality).

The smallest possible C in (iii) is called the modulus of concavity of $\|\cdot\|$ and the pair $(X,\|\cdot\|)$ is called a quasi normed space. If $C=1$, then obtain a norm. A quasi norm $\|\cdot\|$ is called a p-norm $(0<p\leqslant 1)$ if it satisfies

$$\|x+y\|^p \leqslant \|x\|^p + \|y\|^p \quad (x,y \in X).$$

In this case, a quasi normed space is called a p-normed space.
There are many different equivalent metrics on a quasi normed space one of which is

given by Aoki and Rolewicz. The Aoki-Rolewicz theorem [11] states that if $(X, \|.\|)$ is a quasi normed space with the modulus of concavity C, then there is a number $p \in (0,1]$ such that the functional

$$|||x||| := \inf\left\{ \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}} : n > 0, x_1, \ldots, x_n \in X, x = \sum_{i=1}^{n} x_i \right\},$$

defines a p-norm equivalent to the quasi norm $\|.\|$. Moreover $|||x||| \leqslant \|x\| \leqslant 2C\,|||\,x\,|||$ and $2^{\frac{1}{p}-1} \leqslant C$. So every quasi norm is equivalent to some p-norms $(0 < p \leqslant 1)$ and $d(x,y) := |||x-y|||^p$ defines a metric topology on X. A quasi normed space (p-normed space) is called a quasi Banach space (p-Banach space) if every Cauchy sequence converges. We refer the reader to [10, 12] for more information on quasi normed spaces. The notion of q-norm is a specification of a quasi norm that Belbachir et al. [2] introduced it as follows:
A real valued function $\|\cdot\|$ on a vector space X is called a q-norm $(q \geqslant 1)$ if it satisfies (i) and (ii) above and the following inequality

$$\|x+y\|^q \leqslant 2^{q-1}(\|x\|^q + \|y\|^q) \quad (x, y \in X). \tag{3}$$

Considering the inequality $\|x\|^q + \|y\|^q \leqslant (\|x\| + \|y\|)^q$, we infer that every q-norm is a quasi norm with the modulus of concavity $C \leqslant 2^{\frac{q-1}{q}}$.

Let $(X, \|.\|)$ be a quasi normed space and $q > 0$. We denote the set of all n-tuples $(\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ with positive coordinates for which inequality (2) holds for all $x_1, \ldots, x_n \in X$ by $F_{\|.\|}(q)$. When we deal with only one quasi norm in the underlying space, for simplicity, we use $F(q)$ instead of $F_{\|.\|}(q)$. We also call inequality (2) as the characteristic inequality of $F(q)$. It is noted that there is no n-tuple $(\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ with some negative coordinates satisfying inequality (2) (To see this, assume that there exists $(\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ such that $\mu_j < 0$ for some $j = 1, \ldots, n$ and inequality (2) holds for all $x_1, \ldots, x_n \in X$. One can take $x_j \in X \setminus \{0\}$ and $x_i = 0$ $(i = 1, \ldots, n, i \neq j)$ to get a contradiction.). So, our purpose is to investigate $F(q)$ for all $q > 0$. In this direction, the main results and consequences are prepared in the next two sections. In Section 2, we study $F(q)$ for $0 < q \leqslant 1$ by using the Aoki-Rolewicz theorem and the notion of equivalent p-norms. In Section 3, we deal with $F(q)$ for $q > 1$ by applying some sharp quasi norm inequalities. Some ideas of this note are inspired by [3, 15].

2. $F(q)$ for $0 < q \leqslant 1$

As mentioned in the preceding section, by the Aoki-Rolewicz theorem, it is known that every quasi norm is equivalent to some p-norm. Because of the simplicity of dealing with p-norms rather than quasi norms, we prepare some p-norm inequalities. In order to do this, in Theorem 1, we obtain some regions of $\mathbb{R}^n$ which contain $F(q)$ and vise versa. First we need the following lemma which is given in [3, Lemma 2.3].

LEMMA 1. [3] *Let* $0 < r \leqslant 1$, $\Omega \subseteq \{(s_1,\dots,s_n) : s_1,\dots,s_n \geqslant 0, \sum_{i=1}^n s_i \geqslant 1\}$ *and let* $D_r(\Omega) := \{(a_1,\dots,a_n) : a_1,\dots,a_n \geqslant 0, a_1 s_1^r + \dots + a_n s_n^r \geqslant 1 \quad \text{for all } (s_1,\dots,s_n) \in \Omega\}$. *Then the following hold:*

(i) $\{(a_1,\dots,a_n) : a_1 \geqslant 1,\dots,a_n \geqslant 1\} \subseteq D_r(\Omega)$;

(ii) *If* $\{(e_1,\dots,e_n)\} \subseteq \overline{\Omega}$ *where* $\{e_1,\dots,e_n\}$ *is the standard basis of* $\mathbb{R}^n$, *then*

$$D_r(\Omega) = \{(a_1,\dots,a_n) : a_1 \geqslant 1,\dots,a_n \geqslant 1\}.$$

THEOREM 1. *Let* $\|.\|$ *and* $|||.|||$ *be a quasi norm and a* p*-norm* $(0 < p \leqslant 1)$, *respectively, on a nonzero vector space* X *such that* $\|.\|$ *is equivalent to* $|||.|||$. *Let* $\hat{\alpha} := \inf\{\alpha > 0 : \|x\| \leqslant \alpha \, |||x||| \text{ for all } x \in X\}$, $\hat{\beta} := \inf\{\beta > 0 : |||x||| \leqslant \beta \|x\| \text{ for all } x \in X\}$ *and* $0 < q \leqslant p$. *Then the following hold:*

(i) $F_{\|.\|}(q) \subseteq (0, \hat{\alpha}^q \hat{\beta}^q] \times \dots \times (0, \hat{\alpha}^q \hat{\beta}^q]$;

(ii) $F_{\|.\|}(q) \supseteq (0, \frac{1}{\hat{\alpha}^q \hat{\beta}^q}] \times \dots \times (0, \frac{1}{\hat{\alpha}^q \hat{\beta}^q}]$;

(iii) *If* $\hat{\alpha}\hat{\beta} = 1$, *then* $F_{\|.\|}(q) = (0,1] \times \dots \times (0,1]$.

Proof. First we note that $\hat{\alpha}, \hat{\beta} > 0$. For this, let α be an arbitrary positive number such that $\|x\| \leqslant \alpha \, |||x|||$ for all $x \in X$. Since $X \neq 0$, we observe that $0 < \frac{\|x_0\|}{|||x_0|||} \leqslant \alpha$ for some $x_0 \neq 0$. It follows from the definition of $\hat{\alpha}$ that $\hat{\alpha} > 0$. Similarly, we have $\hat{\beta} > 0$. Let Ω be the set consisting of all n-tuples $\left(\frac{|||x_1|||^p}{|||\sum_{i=1}^n x_i|||^p}, \dots, \frac{|||x_n|||^p}{|||\sum_{i=1}^n x_i|||^p}\right)$ where $x_1,\dots,x_n \in X$ and $\sum_{i=1}^n x_i \neq 0$. Thus $\Omega \subseteq \{(s_1,\dots,s_n) : s_1,\dots,s_n \geqslant 0, \sum_{i=1}^n s_i \geqslant 1\}$ because $|||.|||$ is a p-norm. We also note that $0 < \frac{q}{p} \leqslant 1$ and Ω contains the standard basis of $\mathbb{R}^n$ and so $\overline{\Omega}$ contains too. From Lemma 1 we get $D_{\frac{q}{p}}(\Omega) = \{(a_1,\dots,a_n) : a_1 \geqslant 1,\dots,a_n \geqslant 1\}$.

(i) Let $(\mu_1,\dots,\mu_n) \in F_{\|.\|}(q)$. Thus $\mu_i > 0$ for all $i = 1,\dots,n$ and the following inequality holds

$$\|x_1 + \dots + x_n\|^q \leqslant \frac{\|x_1\|^q}{\mu_1} + \dots + \frac{\|x_n\|^q}{\mu_n} \quad (x_1,\dots,x_n \in X), \tag{4}$$

whence

$$\sum_{i=1}^n \frac{\|x_i\|^q}{\mu_i \|x_1 + \dots + x_n\|^q} \geqslant 1, \tag{5}$$

for all $x_1,\dots x_n \in X$ in which $\sum_{i=1}^n x_i \neq 0$. Since $\|x\| \leqslant \hat{\alpha} \, |||x|||$ and $|||x||| \leqslant \hat{\beta} \|x\|$ $(x \in X)$ by the definition of $\hat{\alpha}$ and $\hat{\beta}$, inequality (5) turns into

$$\sum_{i=1}^n \hat{\alpha}^q \hat{\beta}^q \frac{|||x_i|||^q}{\mu_i \, |||x_1 + \dots + x_n|||^q} \geqslant 1 \tag{6}$$

for all $x_1,\dots,x_n \in X$ in which $\sum_{i=1}^n x_i \neq 0$. From inequality (6) we infer that $(\frac{\hat{\alpha}^q \hat{\beta}^q}{\mu_1}, \dots, \frac{\hat{\alpha}^q \hat{\beta}^q}{\mu_n}) \in D_{\frac{q}{p}}(\Omega)$, or equivalently, we get $\frac{\hat{\alpha}^q \hat{\beta}^q}{\mu_i} \geqslant 1$, whence $\mu_i \leqslant \hat{\alpha}^q \hat{\beta}^q$ for all $i = 1,\dots,n$.

(ii) Let $\mu_1,\ldots,\mu_n$ be positive numbers such that $\mu_i \leqslant \frac{1}{\hat{\alpha}^q\hat{\beta}^q}$ for all $i=1,\ldots,n$. Thus we get $\frac{1}{\mu_i\hat{\alpha}^q\hat{\beta}^q} \geqslant 1$ $(i=1,\ldots,n)$ and so $\left(\frac{1}{\mu_1\hat{\alpha}^q\hat{\beta}^q},\ldots,\frac{1}{\mu_n\hat{\alpha}^q\hat{\beta}^q}\right) \in D_{\frac{q}{p}}(\Omega)$, or equivalently, the following inequality holds

$$\frac{|||x_1|||^q}{\mu_1\hat{\alpha}^q\hat{\beta}^q\,|||\sum_{i=1}^n x_i|||^q}+\ldots+\frac{|||x_n|||^q}{\mu_n\hat{\alpha}^q\hat{\beta}^q\,|||\sum_{i=1}^n x_i|||^q} \geqslant 1 \quad (x_1,\ldots,x_n \in X,\ \sum_{i=1}^n x_i \neq 0). \tag{7}$$

Due to the definition of $\hat{\alpha}$ and $\hat{\beta}$, we observe that $\|x\| \leqslant \hat{\alpha}\,|||\,x\,|||$ and $|||x||| \leqslant \hat{\beta}\|x\|$ for all $x\in X$. According to two last inequalites, inequality (7) turns into

$$\frac{\|x_1\|^q}{\mu_1\|\sum_{i=1}^n x_i\|^q}+\ldots+\frac{\|x_n\|^q}{\mu_n\|\sum_{i=1}^n x_i\|^q} \geqslant 1 \quad (x_1,\ldots,x_n \in X,\ \sum_{i=1}^n x_i \neq 0).$$

Hence inequality (4) holds.

(iii) It follows from (i) and (ii).

It is known that each p-norm is a quasi norm. Hence, a special case of Theorem 1 gives the following result. In Corollary 1, $F(q)$ is completely characterized in the setting of p-normed spaces.

COROLLARY 1. *Let* $(X,\|.\|)$ *be a* p*-normed space* $(0<p\leqslant 1)$ *and* $0<q\leqslant p$. *Then*

$$F(q)=(0,1]\times\ldots\times(0,1].$$

Proof. Applying Theorem 1 to $\|.\|$ as a p-norm and quasi norm we get $\hat{\alpha}=\hat{\beta}=1$. The result follows from parts (i) and (ii).

A special case of Corollary 1, where $p=1$ gives rise to the following result due to [3, Theorem 2.5 (i)].

COROLLARY 2. [3] *Let* $(X,\|.\|)$ *be a normed space and* $0<q\leqslant 1$. *Then*

$$F(q)=(0,1]\times\ldots\times(0,1].$$

3. $F(q)$ for $q>1$

We start this section with the following two lemmas which generalize the quasi triangle inequality for n vectors. The first lemma easily follows from the quasi triangle inequality by induction and the second lemma is given in [6, Lemma 2].

LEMMA 2. *Let* X *be a quasi normed space with the modulus of concavity* C *and* $n\geqslant 1$. *Then the following inequalities hold:*

$$\text{(i)}\ \left\|\sum_{i=1}^n x_i\right\| \leqslant C^{\frac{n}{2}}\sum_{i=1}^n \|x_i\| \qquad (\,n \text{ is even and } x_1,\ldots,x_n\in X); \tag{8}$$

$$\text{(ii)}\ \left\|\sum_{i=1}^n x_i\right\| \leqslant C^{\frac{n+1}{2}}\sum_{i=1}^n \|x_i\| \quad (\,n \text{ is odd and } x_1,\ldots,x_n\in X). \tag{9}$$

LEMMA 3. [6] *Let* $(X,\|.\|)$ *be a quasi normed space with the modulus of concavity* C *and* $n>1$. *Then*

$$\left\|\sum_{i=1}^{n} x_i\right\| \leqslant C^{1+\log_2(n-1)} \sum_{i=1}^{n} \|x_i\| \tag{10}$$

for all $x_1,\ldots,x_n \in X$.

REMARK 1. The preceding two lemmas provide some estimations of $\|x_1+\ldots+x_n\|$ in quasi normed spaces. According to sharp inequalities, some estimations of $\|x_1+\ldots+x_n\|$ is more accurate. In fact, inequality (10) gives a sharper bound for $\|x_1+\ldots+x_n\|$ (n is even except for $4,6$) than inequality (8) and inequality (8) gives us sharper bound than inequality (10) for $n=4,6$. The constant in inequality (10) is also sharper than that of inequality (9) where n is odd. To see these, it is enough to check that:

(i) $C^{1+\log_2(n-1)} \leqslant C^{\frac{n}{2}}$ $\quad$ (n is even except for $4,6$);

(ii) $C^{\frac{n}{2}} \leqslant C^{1+\log_2(n-1)}$ $\quad$ ($n=4,6$);

(iii) $C^{1+\log_2(n-1)} \leqslant C^{\frac{n+1}{2}}$ $\quad$ (n is odd).

To check (i) and (ii), we define the differentiable real valued function f as $f(x)=\frac{x}{2}-(1+\log_2(x-1))$ for all $x\in(1,\infty)$. One can show that the equation $f'(x)=0$ has exactly one solution $x_0=1+2\ln(2)^{-1}$, $f'(x)<0$ on $(1,x_0)$ and $f'(x)>0$ on (x_0,∞), whence the function f takes the absolute minimum at the point of x_0. We also get the logarithmic equation $\frac{x}{2}-(1+\log_2(x-1))=0$ has exactly two solutions, say, x_1 and x_2.
Obviously $x_1=2$ and by using the software MATLAB we observe that $x_2\in(7,8)$. Due to the facts that the function f increases on (x_0,∞), $x_2\in(x_0,\infty)$, $f(x_2)=0$ and $x_2<8$, we get $f(n)\geqslant 0$ for all $n=2k$ $(k\in\mathbb{N}\setminus\{1,2,3\})$. In addition, $f(2)=0$ therefore $f(n)\geqslant 0$ for all $n=2k$ $(k\in\mathbb{N}\setminus\{2,3\})$. We also notice that $f(x)\leqslant 0$ for all $x\in[x_1,x_2]$ and $4,6\in[x_1,x_2]$, whence $f(n)\leqslant 0$ for $n=4,6$. In a similar way, we can check (iii).

Applying Lemma 2 and Lemma 3 and according to the preceding remark, we obtain some regions of $\mathbb{R}^n$ which are contained in $F(q)$ with as accurate as possible. So we can state the following theorem.

THEOREM 2. *Let* $(X,\|.\|)$ *be a quasi normed space with the modulus of concavity* C *and* $q>1$. *Then*

(i) $F(q)\supseteq\left\{(\mu_1,\ldots,\mu_n):\mu_1,\ldots,\mu_n>0 \text{ and } \left(\sum_{i=1}^{n}\mu_i^{\frac{1}{q-1}}\right)^{q-1}\leqslant C^{-q(1+\log_2(n-1))}\right\}$;
(*the case where* $n\neq 4,6$);

(ii) $F(q)\supseteq\left\{(\mu_1,\ldots,\mu_n):\mu_1,\ldots,\mu_n>0 \text{ and } \left(\sum_{i=1}^{n}\mu_i^{\frac{1}{q-1}}\right)^{q-1}\leqslant C^{-\frac{nq}{2}}\right\}$;
(*the case where* $n=4,6$).

Proof. (i) Let $\mu_1,\ldots,\mu_n$ be positive numbers satisfying the following inequality

$$\left(\sum_{i=1}^{n}\mu_i^{\frac{1}{q-1}}\right)^{q-1} \leqslant C^{-q(1+\log_2(n-1))} \tag{11}$$

and $x_1,\ldots,x_n \in X$ be arbitrary. From Lemma 3 we deduce that

$$\left\|\sum_{i=1}^{n}x_i\right\|^q \leqslant C^{q(1+\log_2(n-1))}\left(\sum_{i=1}^{n}\|x_i\|\right)^q. \tag{12}$$

Applying the so-called Hölder inequality on the couple $\frac{q}{q-1}, q$ as conjugate exponents we get

$$\sum_{i=1}^{n}\|x_i\| \leqslant \left(\sum_{i=1}^{n}\mu_i^{\frac{1}{q-1}}\right)^{\frac{q-1}{q}}\left(\sum_{i=1}^{n}\frac{\|x_i\|^q}{\mu_i}\right)^{\frac{1}{q}},$$

whence inequality (12) turns into

$$\left\|\sum_{i=1}^{n}x_i\right\|^q \leqslant C^{q(1+\log_2(n-1))}\left(\sum_{i=1}^{n}\mu_i^{\frac{1}{q-1}}\right)^{q-1}\sum_{i=1}^{n}\frac{\|x_i\|^q}{\mu_i}. \tag{13}$$

From (11) and the last inequality we get $\|\sum_{i=1}^{n}x_i\|^q \leqslant \sum_{i=1}^{n}\frac{\|x_i\|^q}{\mu_i}$, or equivalently, $(\mu_1,\ldots,\mu_n)\in F(q)$.
(ii) With a similar argument to the proof of part (i) and by applying Lemma 2 (i), one can show that the desired inclusion holds. So we omit the details.

EXAMPLE 1. Here we provide some regions of $\mathbb{R}^2$ containing ordered pairs (μ_1,μ_2) that satisfy the conditions of Theorem 2. Let us make this in the sequence space l^p.
Let $p\in(0,1]$ be arbitrary and $X=l^p$. Then

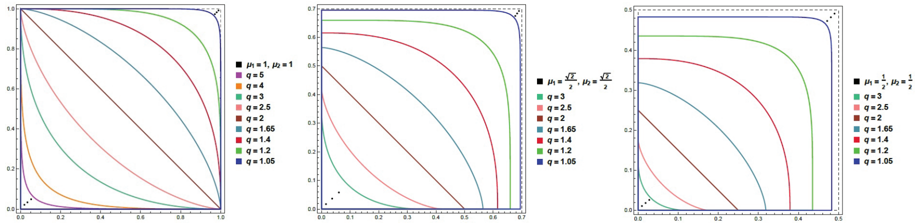
$$\|x\|_p = \left(\sum_{j=1}^{\infty}|\xi_j|^p\right)^{\frac{1}{p}} \qquad (x=(\xi_j)_{j=1}^{\infty}\in l^p)$$

is a quasi norm with the modulus of concavity $C_p = 2^{\frac{1}{p}-1}$ [1].
The inequality conditions in Theorem 2 equivalently turn into

$$\mu_1^{\frac{1}{q-1}}+\mu_2^{\frac{1}{q-1}} \leqslant 2^{\left(\frac{1}{p}-1\right)\left(\frac{q}{1-q}\right)} \qquad (q>1,\ \mu_1,\mu_2>0). \tag{14}$$

In Figure 1 that its diagrams have been drawn by the software Wolfram Mathematica, we can observe the regions of $\mathbb{R}^2$ which satisfy inequality (14) under the following assumptions:

a) $p=1,\ X=l^1,\ C_1=1,\quad q=1.05,\ 1.2,\ 1.4,\ 1.65,\ 2,\ 2.5,\ 3,\ 4,\ 5;$



(a) $C_1 = 1$ (b) $C_{\frac{2}{3}} = \sqrt{2}$ (c) $C_{\frac{1}{2}} = 2$

Figure 1: Plots of $\mu_1^{\frac{1}{q-1}} + \mu_2^{\frac{1}{q-1}} = 2^{\left(\frac{1}{p}-1\right)\left(\frac{q}{1-q}\right)}$

b) $p = \frac{2}{3}$, $X = l^{\frac{2}{3}}$, $C_{\frac{2}{3}} = \sqrt{2}$, $q = 1.05,\ 1.2,\ 1.4,\ 1.65,\ 2,\ 2.5,\ 3$;

c) $p = \frac{1}{2}$, $X = l^{\frac{1}{2}}$, $C_{\frac{1}{2}} = 2$, $q = 1.05,\ 1.2,\ 1.4,\ 1.65,\ 2,\ 2.5,\ 3$.

By Theorem 2, the above mentioned regions are contained in $F_{\|.\|_p}(q)$ and so the characteristic inequality of $F_{\|.\|_p}(q)$ (inequality (2)) holds for all ordered pairs (μ_1, μ_2) belonging to these regions.

Next, as a reverse inclusion of the last results, we can get a region of $\mathbb{R}^n$ which contains $F(q)$. Similar to the first half of the proof of [3, Theorem 2.4 (i)], one can easily check the following proposition by putting $x_i = \mu_i^{\frac{1}{q-1}} x$ (for some $x \neq 0$ and for all $i = 1, \ldots, n$) in the characteristic inequality of $F(q)$.

PROPOSITION 1. *Let $(X, \|.\|)$ be a quasi normed space and $q > 1$. Then the following inclusion holds:*

$$F(q) \subseteq \left\{ (\mu_1, \ldots, \mu_n) : \mu_1, \ldots, \mu_n > 0 \textit{ and } \sum_{i=1}^{n} \mu_i^{\frac{1}{q-1}} \leqslant 1 \right\}.$$

The results in the following corollaries are derived from Theorem 2 and Proposition 1 as some special cases.
Taking $C = 1$ and by using Theorem 2 and Proposition 1, we have the following corollary which was presented in [3, Theorem 2.4 (i)].

COROLLARY 3. [3] *Let $(X, \|.\|)$ be a normed space and $q > 1$. Then*

$$F(q) = \left\{ (\mu_1, \ldots, \mu_n) : \mu_1, \ldots, \mu_n > 0, \sum_{i=1}^{n} \mu_i^{\frac{1}{q-1}} \leqslant 1 \right\}.$$

Finally with connection to the notion of q-norms, we get the following result proved in [2, Proposition 2.1].

COROLLARY 4. [2] *Every norm in a usual sense is a q-norm for all $q > 1$.*

Proof. Let $q > 1$ be arbitrary. Applying Theorem 2 (i) by setting $n = 2$, one can observe that $(2^{1-q}, 2^{1-q}) \in F(q)$, or equivalently, $\|x+y\|^q \leqslant 2^{q-1}(\|x\|^q + \|y\|^q)$ for all $x, y \in X$.

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