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WEAK TYPE ENDPOINT ESTIMATES FOR THE COMMUTATORS OF ROUGH SINGULAR INTEGRAL OPERATORS

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Abstract. Let Ω be homogeneous of degree zero and have mean value zero on the unit sphere S^{n-1}, T_Ω be the convolution singular integral operator with kernel $\frac{\Omega(x)}{|x|^n}$. For $b \in \mathrm{BMO}(\mathbb{R}^n)$, let $T_{\Omega,b}$ be the commutator of T_Ω. In this paper, by establishing suitable sparse dominations, the authors establish some weak type endpoint estimates of $L\log L$ type for $T_{\Omega,b}$ when $\Omega \in L^q(S^{n-1})$ for some $q \in (1,\infty]$.

1. Introduction

We will work on $\mathbb{R}^n$, $n \geqslant 2$. Let Ω be homogeneous of degree zero, integrable and have mean value zero on the unit sphere S^{n-1}. Define the singular integral operator T_Ω by

$$T_\Omega f(x) = \mathrm{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y|^n} f(x-y)\,dy, \tag{1.1}$$

where and in the following, $y' = y/|y|$ for $y \in \mathbb{R}^n$. This operator was introduced by Calderón and Zygmund [2], and then studied by many authors in the last sixty years. Calderón and Zygmund [3] proved that if $\Omega \in L\log L(S^{n-1})$, then T_Ω is bounded on $L^p(\mathbb{R}^n)$ for $p \in (1,\infty)$. Ricci and Weiss [23] improved the result of Calderón-Zygmund, and showed that $\Omega \in H^1(S^{n-1})$ guarantees the $L^p(\mathbb{R}^n)$ boundedness on $L^p(\mathbb{R}^n)$ for $p \in (1,\infty)$. Seeger [25] showed that $\Omega \in L\log L(S^{n-1})$ is a sufficient condition such that T_Ω is bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. For other works about the $L^p(\mathbb{R}^n)$ boundedness and weak type endpoint estimates for T_Ω, we refer the papers see [4, 7, 8, 9, 12, 23, 27] and the references therein.

Now let T be a linear operator from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}'(\mathbb{R}^n)$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. The commutator of T with symbol b, is defined by

$$T_b f(x) = b(x)Tf(x) - T(bf)(x).$$

A celebrated result of Coifman, Rochberg and Weiss [6] states that if T is a Calderón-Zygmund operator, then T_b is bounded on $L^p(\mathbb{R}^n)$ for every $p \in (1,\infty)$ and also a converse result in terms of the Riesz transforms. Pérez [21] considered the weak type endpoint estimate for the commutator of Calderón-Zygmund operator, and proved the following result.

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THEOREM 1.1. *Let T be a Calderón-Zygmund operator and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then for any $\lambda > 0$,*

$$|\{x \in \mathbb{R}^n : |T_b f(x)| > \lambda\}| \lesssim_n \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log\Big(\mathrm{e} + \frac{|f(x)|}{\lambda}\Big) dx.$$

By Theorem 1.1, we know that if $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$ with $\alpha \in (0,1]$, then for $b \in \mathrm{BMO}(\mathbb{R}^n)$, $T_{\Omega,b}$, the commutator of T_Ω, satisfies that,

$$|\{x \in \mathbb{R}^n : |T_{\Omega,b} f(x)| > \lambda\}| \lesssim_n \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log\Big(\mathrm{e} + \frac{|f(x)|}{\lambda}\Big) dx. \tag{1.2}$$

Let $p \in [1,\infty)$ and w be a nonnegative, locally integrable function on $\mathbb{R}^n$. We say that $w \in A_p(\mathbb{R}^n)$ if the A_p constant $[w]_{A_p}$ is finite, with

$$[w]_{A_p} := \sup_Q \Big(\frac{1}{|Q|}\int_Q w(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q w^{1-p'}(x)dx\Big)^{p-1}, \ p \in (1,\infty),$$

the supremum is taken over all cubes in $\mathbb{R}^n$, $p' = p/(p-1)$ and

$$[w]_{A_1} := \sup_{x\in\mathbb{R}^n} \frac{Mw(x)}{w(x)},$$

see [11] for the properties of $A_p(\mathbb{R}^n)$. For a weight $w \in A_\infty(\mathbb{R}^n) = \cup_{p\geqslant 1} A_p(\mathbb{R}^n)$, define $[w]_{A_\infty}$, the A_∞ constant of w, by

$$[w]_{A_\infty} = \sup_{Q\subset\mathbb{R}^n} \frac{1}{w(Q)} \int_Q M(w\chi_Q)(x)dx,$$

see [28]. By the result of Duandikoetxea and Rubio de Francia [8], and the result in [7], we know that if $\Omega \in L^q(S^{n-1})$ for some $q \in (1,\infty]$, then for $p \in (q',\infty)$ and $w \in A_{p/q'}(\mathbb{R}^n)$

$$\|T_\Omega f\|_{L^p(\mathbb{R}^n,w)} \lesssim_{n,p,w} \|f\|_{L^p(\mathbb{R}^n,w)}.$$

This, together with Theorem 2.13 in [1], tells us that if $\Omega \in L^q(S^{n-1})$ for $q \in (1\infty]$, then for $b \in \mathrm{BMO}(\mathbb{R}^n)$,

$$\|T_{\Omega,b} f\|_{L^p(\mathbb{R}^n,w)} \lesssim_{n,p,w} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n,w)}, \ p \in (q',\infty), \ w \in A_{p/q'}(\mathbb{R}^n).$$

Hu [13] proved that $\Omega \in L(\log L)^2(S^{n-1})$ is a sufficient condition such that $T_{\Omega,b}$ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1,\infty)$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. However, as far as we know, there is no result concerning the weak type endpoint estimate for $T_{\Omega,b}$ when Ω only satisfies size condition. In this paper, we consider this question. Our first result can be stated as follows.

THEOREM 1.2. *Let Ω be homogeneous of degree zero and have mean value zero on S^{n-1}, $b \in \mathrm{BMO}(\mathbb{R}^n)$. Suppose that $\Omega \in L^q(S^{n-1})$ for some $q \in (1,\infty)$, then for any $\lambda > 0$ and weight w such that $w^{q'} \in A_1(\mathbb{R}^n)$,*

$$w(\{x \in \mathbb{R}^n : |T_{\Omega,b} f(x)| > \lambda\}) \lesssim_{n,w} \int_{\mathbb{R}^n} \frac{D|f(x)|}{\lambda} \log\Big(\mathrm{e} + \frac{D|f(x)|}{\lambda}\Big) w(x)dx,$$

with $D = \|\Omega\|_{L^q(S^{n-1})} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$.

In the last several years, considerable attention has been paid to the quantitative weighted bounds for T_Ω when $\Omega \in L^\infty(S^{n-1})$. The first result in this area was established by Hytönen, Roncal and Tapiola [16], who proved that for $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^n)$,

$$\|T_\Omega f\|_{L^p(\mathbb{R}^n,w)} \lesssim_{n,p} \|\Omega\|_{L^\infty(S^{n-1})}[w]^{2\max\{1,\frac{1}{p-1}\}}\|f\|_{L^p(\mathbb{R}^n,w)}. \tag{1.3}$$

Li, Pérez, Rivera-Rios and Roncal [19] improved (1.3) and showed that for $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^n)$

$$\|T_\Omega f\|_{L^p(\mathbb{R}^n,w)} \lesssim_{n,p} [w]_{A_p}^{\frac{1}{p}}\big([w]_{A_\infty}^{\frac{1}{p'}}+[\sigma]_{A_\infty}^{\frac{1}{p}}\big)\min\{[\sigma]_{A_\infty},[w]_{A_\infty}\}\|f\|_{L^p(\mathbb{R}^n,w)}, \tag{1.4}$$

where and in the following, for $w \in A_p(\mathbb{R}^n)$, $\sigma = w^{1-p'}$. The estimate (1.4), via the method in [5], implies the following quantitative weighted estimate

$$\begin{aligned}\|T_{\Omega,b} f\|_{L^p(\mathbb{R}^n,w)} \lesssim_{n,p}\ & [w]_{A_p}^{\frac{1}{p}}\big([w]_{A_\infty}^{\frac{1}{p'}}+[\sigma]_{A_\infty}^{\frac{1}{p}}\big)\min\{[\sigma]_{A_\infty},[w]_{A_\infty}\}\\ &\times([w]_{A_\infty}+[\sigma]_{A_\infty})\|f\|_{L^p(\mathbb{R}^n,w)}.\end{aligned}$$

Rivera-Ríos [24] established the sparse domination for $T_{\Omega,b}$ when $\Omega \in L^\infty(S^{n-1})$, and proved that for $p \in (1,\infty)$ and $w \in A_1(\mathbb{R}^n)$,

$$\|T_{\Omega,b} f\|_{L^p(\mathbb{R}^n,w)} \lesssim_{n,p} \|\Omega\|_{L^\infty(S^{n-1})} p'^3 p^2 [w]_{A_1}^{\frac{1}{p}}[w]_{A_\infty}^{1+\frac{1}{p'}}\|f\|_{L^p(\mathbb{R}^n,w)}.$$

Our second result is the following quantitative weighted weak type estimate for $T_{\Omega,b}$.

THEOREM 1.3. *Let Ω be homogeneous of degree zero and have mean value zero on S^{n-1}, $b \in \mathrm{BMO}(\mathbb{R}^n)$. Suppose that $\Omega \in L^\infty(S^{n-1})$ and $w \in A_1(\mathbb{R}^n)$, then for any $\lambda > 0$,*

$$\begin{aligned}& w(\{x \in \mathbb{R}^n : |T_{\Omega,b} f(x)| > \lambda\})\\ &\lesssim_n [w]_{A_1}[w]_{A_\infty}^2 \log(\mathrm{e}+[w]_{A_\infty})\int_{\mathbb{R}^n}\frac{D_\infty|f(x)|}{\lambda}\log\Big(\mathrm{e}+\frac{D_\infty|f(x)|}{\lambda}\Big)w(x)dx,\end{aligned}$$

with $D_\infty = \|\Omega\|_{L^\infty(S^{n-1})}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$.

REMARK 1.4. Proofs of Theorem 1.2 and Theorem 1.3 depend essentially on the weak type endpoint estimates for the maximal operator defined by

$$\mathscr{M}_{r,T_\Omega} f(x) = \sup_{Q \ni x}\Big(\frac{1}{|Q|}\int_Q |T_\Omega(f\chi_{\mathbb{R}^n\setminus 3Q})(\xi)|^r d\xi\Big)^{1/r}, \tag{1.5}$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x. This operator was introduced by Lerner [18], who proved that for any $r \in (1,\infty)$,

$$\|M_{r,T_\Omega} f\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim r\|\Omega\|_{L^\infty(S^{n-1})}\|f\|_{L^1(\mathbb{R}^n)}, \tag{1.6}$$

see [18, Lemma 3.3]. Although we can show that

$$\|M_{r,T_\Omega} f\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim_r \|\Omega\|_{L^q(S^{n-1})} \|f\|_{L^1(\mathbb{R}^n)},$$

we do not know if there exists a $\alpha \in (0,\infty)$ such that the estimate

$$\|M_{r,T_\Omega} f\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim r^\alpha \|\Omega\|_{L^q(S^{n-1})} \|f\|_{L^1(\mathbb{R}^n)}$$

holds true when $\Omega \in L^q(S^{n-1})$ for some $q \in (1,\infty)$. This is the main difficult which prevent us obtaining a desired quantitative weighted weak type endpoint estimates for $T_{\Omega,b}$ when $\Omega \in L^q(S^{n-1})$ for $q \in (1,\infty)$.

In what follows, C always denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line. We use the symbol $A \lesssim B$ to denote that there exists a positive constant C such that $A \leqslant CB$. Specially, we use $A \lesssim_{n,p} B$ to denote that there exists a positive constant C depending only on n,p such that $A \leqslant CB$. Constant with subscript such as c_1, does not change in different occurrences. For any set $E \subset \mathbb{R}^n$, χ_E denotes its characteristic function. For a cube $Q \subset \mathbb{R}^n$ and $\lambda \in (0,\infty)$, we use λQ to denote the cube with the same center as Q and whose side length is λ times that of Q. For a fixed cube Q, denote by $\mathscr{D}(Q)$ the set of dyadic cubes with respect to Q, that is, the cubes from $\mathscr{D}(Q)$ are formed by repeating subdivision of Q and each of descendants into 2^n congruent subcubes. For a function f and cube Q, $\langle f\rangle_Q$ denotes the mean value of f on Q, and $\langle |f|\rangle_{Q,r} = (\langle |f|^r\rangle_Q)^{1/r}$ for $r \in (0,\infty)$.

For a cube Q, $\beta \in (0,\infty)$ and suitable function f, define $\|f\|_{L(\log L)^\beta,Q}$ by

$$\|f\|_{L(\log L)^\beta,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \frac{|f(y)|}{\lambda} \log^\beta\left(\mathrm{e} + \frac{|f(y)|}{\lambda}\right) dy \leqslant 1\right\}.$$

Also, we define $\|h\|_{\exp L,Q}$ as

$$\|h\|_{\exp L,Q} = \inf\left\{t > 0 : \frac{1}{|Q|}\int_Q \exp\left(\frac{|h(y)|}{t}\right) \mathrm{d}y \leqslant 2\right\}.$$

By the generalization of Hölder's inequality (see [22, p. 64]), we know that for any cube Q and suitable functions f and h,

$$\int_Q |f(x)h(x)|dx \lesssim \|f\|_{L\log L,Q} \|h\|_{\exp L,Q} |Q|. \tag{1.7}$$

2. Proof of theorems

Given an operator T, define the maximal operator $M_{\lambda,T}$ by

$$M_{\lambda,T} f(x) = \sup_{Q \ni x} \Big(T(f\chi_{\mathbb{R}^n \setminus 3Q})\chi_Q\Big)^*(\lambda|Q|),\ (0<\lambda<1),$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x, and h^* denotes the non-increasing rearrangement of h. This operator was introduced by Lerner [18] and is useful in the study of weighted bounds for rough operators, see [18, 24].

LEMMA 2.1. *Let Ω be homogeneous of degree zero, have mean value zero and $\Omega \in L^\infty(S^{n-1})$. Then for any $\lambda \in (0,1)$,*

$$\|M_{\lambda, T_\Omega} f\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim_n \|\Omega\|_{L^\infty(S^{n-1})}\left(1+\log\left(\frac{1}{\lambda}\right)\right)\|f\|_{L^1(\mathbb{R}^n)}.$$

Lemma 2.1 is Theorem 1.1 in [18].

For a function Ω on S^{n-1}, define $\|\Omega\|^*_{L\log L(S^{n-1})}$ by

$$\|\Omega\|^*_{L\log L(S^{n-1})} = \inf\left\{\lambda>0: \int_{S^{n-1}} \frac{|\Omega(\theta)|}{\lambda}\log\left(\mathrm{e}+\frac{|\Omega(\theta)|}{\lambda}\right)d\theta \leqslant 1\right\}.$$

LEMMA 2.2. *Let Ω be homogeneous of degree zero, have mean value zero and $\|\Omega\|^*_{L\log L(S^{n-1})}<\infty$, then*

$$\|T_\Omega f\|_{L^{1,\infty}(S^{n-1})} \lesssim \|\Omega\|^*_{L\log L(S^{n-1})}\|f\|_{L^1(\mathbb{R}^n)}.$$

Proof. This lemma is essentially a corollary of estimate (3.1) in [25]. At first, we claim that

$$\int_{S^{n-1}} |\Omega(\theta)|\log\left(\mathrm{e}+\frac{|\Omega(\theta)|}{\|\Omega\|_{L^1(S^{n-1})}}\right)d\theta \lesssim \|\Omega\|^*_{L\log L(S^{n-1})}. \tag{2.1}$$

In fact, by homogeneity, it suffices to prove (2.1) for the case $\|\Omega\|_{L^1(S^{n-1})}=1$. Let

$$\lambda_0 = \int_{S^{n-1}} |\Omega(\theta)|\log(\mathrm{e}+|\Omega(\theta)|)d\theta.$$

We consider the following two cases.

Case I. $\lambda_0 > \mathrm{e}^{10}$. Let $S_0=\{\theta\in S^{n-1}: |\Omega(\theta)|\leqslant 2\}$, and

$$S_k = \left\{\theta\in S^{n-1}: 2^k<|\Omega(\theta)|\leqslant 2^{k+1}\right\},\ k\in\mathbb{N}.$$

Set $k_0\in\mathbb{N}$ such that $2^{k_0-1}<\lambda_0\leqslant 2^{k_0}$. Then $k_0\leqslant \lambda_0/8$

$$\begin{aligned}\int_{S^{n-1}} \frac{|\Omega(\theta)|}{\lambda_0}\log\left(\mathrm{e}+\frac{|\Omega(\theta)|}{\lambda_0}\right)d\theta &> \lambda_0^{-1}\sum_{k=k_0+1}^{\infty}|S_k|2^k(k-k_0)+\lambda_0^{-1}\sum_{k\leqslant k_0}|S_k|2^k\\ &> \lambda_0^{-1}\left(\sum_{k=1}^{\infty}2^k k|S_k|+|S_0|\right)\\ &\quad -\lambda_0^{-1}\left(k_0\sum_{k\geqslant k_0+1}2^k|S_k|+\sum_{1\leqslant k\leqslant k_0}k2^k|S_k|\right).\end{aligned}$$

Obviously,

$$\sum_{k=1}^{\infty}2^k k|S_k|+|S_0| \geqslant \frac{1}{4}\int_{S^{n-1}}|\Omega(\theta)|\log(\mathrm{e}+|\Omega(\theta)|)d\theta = \frac{\lambda_0}{4},$$

and

$$k_0 \sum_{k \geqslant k_0+1} 2^k |S_k| + \sum_{1 \leqslant k \leqslant k_0} k 2^k |S_k| \leqslant k_0 \sum_{k \geqslant 1} 2^k |S_k| \leqslant k_0 \|\Omega\|_{L^1(S^{n-1})}.$$

Recall that $\|\Omega\|_{L^1(S^{n-1})} = 1$. It then follows that

$$\int_{S^{n-1}} \frac{|\Omega(\theta)|}{\lambda_0} \log\left(\mathrm{e} + \frac{|\Omega(\theta)|}{\lambda_0}\right) d\theta > \frac{1}{8}.$$

This in turn leads to that

$$\|\Omega\|^*_{L\log L(S^{n-1})} > \lambda_0/8.$$

Case II. $\lambda_0 \leqslant \mathrm{e}^{10}$. Let $\lambda > 0$ satisfies that

$$\int_{S^{n-1}} \frac{|\Omega(\theta)|}{\lambda} \log\left(\mathrm{e} + \frac{|\Omega(\theta)|}{\lambda}\right) d\theta \leqslant 1. \tag{2.2}$$

If $10\mathrm{e}^{10}\lambda < \lambda_0$, we then have that

$$\int_{S^{n-1}} \frac{|\Omega(\theta)|}{\lambda_0} \log\left(\mathrm{e} + \frac{|\Omega(\theta)|}{\lambda_0}\right) d\theta \leqslant \int_{Q} \frac{|\Omega(\theta)|}{10\mathrm{e}^{10}\lambda} \log\left(\mathrm{e} + \frac{|\Omega(\theta)|}{10\mathrm{e}^{10}\lambda}\right) d\theta \leqslant \left(10\mathrm{e}^{10}\right)^{-1}.$$

On the other hand, a trivial computation gives us that

$$\begin{aligned}\int_{S^{n-1}} \frac{|\Omega(\theta)|}{\lambda_0} \log\left(\mathrm{e} + \frac{|\Omega(\theta)|}{\lambda_0}\right) d\theta &> \int_{S^{n-1}} \frac{|\Omega(\theta)|}{\mathrm{e}^{10}} \log\left(\mathrm{e} + \frac{|\Omega(\theta)|}{\mathrm{e}^{10}}\right) d\theta \\ &> \int_{S^{n-1}} |\Omega(\theta)| \log(\mathrm{e} + |\Omega(\theta)|) d\theta \left(10\mathrm{e}^{10}\right)^{-1} \\ &> \left(10\mathrm{e}^{10}\right)^{-1},\end{aligned}$$

where the last inequality follows from the fact that $\lambda_0 \geqslant \|\Omega\|_{L^1(S^{n-1})} = 1$ (recall that $\|\Omega\|_{L^1(S^{n-1})} = 1$). This is a contradiction. Thus, the positive numbers λ in (2.2) satisfy $\lambda \geqslant (10\mathrm{e}^{10})^{-1}\lambda_0$. Inequality (2.1) holds true in this case.

We now conclude the proof of Lemma 2.2. By the result of Seeger (see inequality (3.1) in [25]), we know that if $\Omega \in L\log L(S^{n-1})$, then

$$\begin{aligned}\|T_\Omega f\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim_n \Big[&\|T_\Omega\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} + \|\Omega\|_{L^1(S^{n-1})} \\ &+ \int_{S^{n-1}} |\Omega(\theta)| \left(1 + \log^+\left(|\Omega(\theta)|/\|\Omega\|_{L^1(S^{n-1})}\right)\right) d\theta\Big] \|f\|_{L^1(\mathbb{R}^n)},\end{aligned}$$

where $\log^+ s = \log s$ if $s > 1$ and $\log^+ s = 0$ if $s \in (0, 1]$. Thus by (2.1),

$$\|T_\Omega f\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim_n \left[\|T_\Omega\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} + \|\Omega\|_{L^1(S^{n-1})} + \|\Omega\|^*_{L\log L(S^{n-1})}\right] \|f\|_{L^1(\mathbb{R}^n)}.$$

On the other hand, we know that

$$\|T_\Omega f\|_{L^2(\mathbb{R}^n)} \lesssim \Big[1+\|\Omega\|_{L\log L(S^{n-1})}\Big]\,\|f\|_{L^2(\mathbb{R}^n)},$$

with

$$\|\Omega\|_{L\log L(S^{n-1})} = \int_{S^{n-1}} |\Omega(\theta)|\big(1+\log^+|\Omega(\theta)|\big)d\theta.$$

see [10, Theorem 4.2.10]. The last two inequality, along with homogeneity, yields

$$\|T_\Omega f\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim_n \|\Omega\|^*_{L\log L(S^{n-1})}\|f\|_{L^1(\mathbb{R}^n)},$$

and completes the proof of Lemma 2.2. □

LEMMA 2.3. *Let Ω be homogeneous of degree zero, have mean value zero and $\Omega\in L^q(S^{n-1})$ for some $q\in(1,\infty)$. Then for any $\lambda\in(0,1)$ and $\varepsilon\in(0,\min\{1,q-1\})$,*

$$\|M_{\lambda,T_\Omega} f\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim_{q,\varepsilon} \|\Omega\|_{L^q(S^{n-1})}\left(\frac{1}{\lambda}\right)^{\frac{1+2\varepsilon}{q}}\|f\|_{L^1(\mathbb{R}^n)}.$$

Proof. For $\lambda\in(0,1)$, let $M_{0,\lambda}$ be the operator

$$M_{0,\lambda}h(x) = \sup_{Q\ni x}(h\chi_Q)^*(\lambda|Q|),$$

see [17, 26]. It is well known that for $\alpha>0$,

$$|\{x\in\mathbb{R}^n: M_{0,\lambda}f(x)>\alpha\}| \lesssim \lambda^{-1}|\{x\in\mathbb{R}^n: |f(x)|>\alpha\}|.$$

Let S be a linear operator which is bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$ with bound 1. We claim that the operator $S^\star_\lambda$ defined by

$$S^\star_\lambda f(x) = \sup_{Q\ni x}\big(S(f\chi_Q)\big)^*(\lambda|Q|)$$

is bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$ with bound $C_n\lambda^{-1}$. To prove this, let

$$E_\alpha = \{x\in\mathbb{R}^n: S^\star_\lambda f(x)>\alpha\}.$$

For each $x\in E_\alpha$, we can choose a cube Q such that $Q\ni x$ and

$$|\{y\in Q: |S(f\chi_Q)(y)|>\alpha\}| > \lambda|Q|.$$

This, via the weak type $(1,1)$ boundedness of S, tells us that

$$|Q| \leqslant \frac{1}{\alpha\lambda}\int_Q |f(y)|dy,$$

and so $Mf(x)\geqslant \alpha\lambda$. Therefore,

$$|E_\alpha| \leqslant |\{x\in\mathbb{R}^n: Mf(x)>\lambda\alpha\}| \lesssim \frac{1}{\lambda\alpha}\|f\|_{L^1(\mathbb{R}^n)}.$$

This verifies our claim.

We now conclude the proof of Lemma 2.3. Using the estimate $\log t \leqslant t^{\varepsilon}/\varepsilon$ when $t>1$ and $\varepsilon>0$, we can verify by homogeneity that

$$\|\Omega\|^*_{L\log L(S^{n-1})} \lesssim_{\varepsilon} \|\Omega\|_{L^{1+\varepsilon}(S^{n-1})}.$$

This, along with Lemma 2.2, tells us that for $\varepsilon>0$,

$$\|T_{\Omega}f\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim_{n,\varepsilon} \|\Omega\|_{L^{1+\varepsilon}(S^{n-1})}\|f\|_{L^1(\mathbb{R}^n)}.$$

Observe that

$$M_{\lambda,T_{\Omega}}f(x) \leqslant M_{0,\frac{\lambda}{2}}T_{\Omega}f(x) + \sup_{Q\ni x}\left(T_{\Omega}(f\chi_{3Q})\chi_Q\right)^*\left(\frac{\lambda}{2}|Q|\right),$$

and

$$\sup_{Q\ni x}\left(T_{\Omega}(f\chi_{3Q})\chi_Q\right)^*\left(\frac{\lambda}{2}|Q|\right) \leqslant \sup_{Q\ni x}\left(T_{\Omega}(f\chi_{Q})\chi_Q\right)^*\left(\frac{1}{3^n}\frac{\lambda}{2}|Q|\right).$$

Our claim states that

$$\|M_{\lambda,T_{\Omega}}f\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim_{\varepsilon} \frac{1}{\lambda}\|\Omega\|_{L^{1+\varepsilon}(S^{n-1})}\|f\|_{L^1(\mathbb{R}^n)}. \tag{2.3}$$

Now let $\Omega\in L^q(S^{n-1})$, have mean value zero on S^{n-1}. Without loss of generality, we assume that $\|\Omega\|_{L^q(S^{n-1})}=1$. Set

$$t_0 = \left(\frac{1}{\lambda}\right)^{\frac{1+\varepsilon}{q}}\left[1+\log\left(\frac{1}{\lambda}\right)\right]^{-\frac{1+\varepsilon}{q}}.$$

Let

$$\Omega^{t_0}(\theta) = \Omega(\theta)\chi_{\{|\Omega(\theta)|>t_0\}}(\theta),\ \Omega_{t_0}(\theta) = \Omega(\theta)\chi_{\{|\Omega(\theta)|\leqslant t_0\}}(\theta),$$

and

$$\widetilde{\Omega}^{t_0}(\theta) = \Omega^{t_0}(\theta) - A^{t_0},\ \widetilde{\Omega}_{t_0}(\theta) = \Omega_{t_0}(\theta) - A_{t_0},$$

where

$$A^{t_0} = \frac{1}{|S^{n-1}|}\int_{S^{n-1}}\Omega^{t_0}(\theta)d\theta,\ A_{t_0} = \frac{1}{|S^{n-1}|}\int_{S^{n-1}}\Omega_{t_0}(\theta)d\theta.$$

Both of $\widetilde{\Omega}^{t_0}$ and $\widetilde{\Omega}_{t_0}$ have mean value zero. Moreover,

$$\|\widetilde{\Omega}^{t_0}\|_{L^{1+\varepsilon}(S^{n-1})} \lesssim t_0^{1-\frac{q}{1+\varepsilon}},\ \|\widetilde{\Omega}_{t_0}\|_{L^{\infty}(S^{n-1})} \lesssim t_0,$$

and $\Omega(\theta) = \widetilde{\Omega}^{t_0}(\theta) + \widetilde{\Omega}_{t_0}(\theta)$. Applying Lemma 2.1 and (2.3), we deduce that

$$\begin{aligned}\|M_{\lambda,T_{\Omega}}f\|_{L^{1,\infty}(\mathbb{R}^n)} &\lesssim \|M_{\lambda,T_{\widetilde{\Omega}^{t_0}}}f\|_{L^{1,\infty}(\mathbb{R}^n)} + \|M_{\lambda,T_{\widetilde{\Omega}_{t_0}}}f\|_{L^{1,\infty}(\mathbb{R}^n)}\\ &\lesssim_{\varepsilon} \frac{1}{\lambda}\|\widetilde{\Omega}^{t_0}\|_{L^{1+\varepsilon}(S^{n-1})}\|f\|_{L^1(\mathbb{R}^n)}\end{aligned}$$

$$+\left[1+\log\left(\frac{1}{\lambda}\right)\right]\|\widetilde{\Omega}_{t_0}\|_{L^\infty(S^{n-1})}\|f\|_{L^1(\mathbb{R}^n)}$$
$$\lesssim_{q,\varepsilon}\left(\frac{1}{\lambda}\right)^{\frac{1+\varepsilon}{q}}\left[1+\log\left(\frac{1}{\lambda}\right)\right]^{1-\frac{1+\varepsilon}{q}}\|f\|_{L^1(\mathbb{R}^n)}$$
$$\lesssim_{q,\varepsilon}\left(\frac{1}{\lambda}\right)^{\frac{1+2\varepsilon}{q}}\|f\|_{L^1(\mathbb{R}^n)},$$

where in the last inequality, we again invoked the fact that $\log t \leqslant t^\alpha/\alpha$ for all $t>1$ and $\alpha>0$. This completes the proof of Lemma 2.3. □

LEMMA 2.4. *Let $r\in(1,\infty)$ and w be a weight. The following two statements are equivalent.*

(i) *$w\in A_1(\mathbb{R}^n)$ and $w^{1-p'}\in A_{p'/r}(\mathbb{R}^n)$ for some $p\in(1,r')$;*

(ii) *$w^r\in A_1(\mathbb{R}^n)$.*

Proof. Let $w\in A_1(\mathbb{R}^n)$ and $w^{1-p'}\in A_{p'/r}(\mathbb{R}^n)$ for some $p\in(1,r')$, then for any cube $Q\subset\mathbb{R}^n$,

$$\left(\frac{1}{|Q|}\int_Q w^{1-p'}(x)dx\right)\left(\frac{1}{|Q|}\int_Q w^{r\frac{p'-1}{p'-r}}(x)dx\right)^{\frac{p'}{r}-1}\leqslant[w^{1-p'}]_{A_{p'/r}},$$

and so

$$\begin{aligned}\frac{1}{|Q|}\int_Q w^{r\frac{p'-1}{p'-r}}(x)dx &\leqslant [w^{1-p'}]_{A_{p'/r}}^{\frac{1}{\frac{p'}{r}-1}}\left(\frac{1}{|Q|}\int_Q w^{1-p'}(x)dx\right)^{-\frac{1}{\frac{p'}{r}-1}}\\ &\leqslant [w^{1-p'}]_{A_{p'/r}}^{\frac{1}{\frac{p'}{r}-1}}[w]_{A_1}^{\frac{1}{\frac{p'}{r}-1}\frac{1}{p-1}}\left(\frac{1}{|Q|}\int_Q w(x)dx\right)^{\frac{1}{\frac{p'}{r}-1}\frac{1}{p-1}}\\ &\leqslant [w^{1-p'}]_{A_{p'/r}}^{\frac{1}{\frac{p'}{r}-1}}[w]_{A_1}^{\frac{1}{\frac{p'}{r}-1}\frac{1}{p-1}}(\operatorname{essinf}_{y\in Q}w(y))^{\frac{p'-1}{\frac{p'}{r}-1}},\end{aligned}$$

where the second inequality follows from the fact that

$$\left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w^{1-p'}(x)dx\right)^{p-1}\geqslant 1.$$

We thus deduce that $w^r\in A_1(\mathbb{R}^n)$, with $[w^r]_{A_1}\leqslant[w^{1-p'}]_{A_{p'/r}}^{\frac{1}{p'-1}}[w]_{A_1}^r$.

Let $w^r\in A_1(\mathbb{R}^n)$. By the reverse Hölder inequality, we know that $w^{r\frac{p'-1}{p'-r}}\in A_1(\mathbb{R}^n)$ for some $p\in(1,r')$, and $[w]_{A_1}\leqslant[w^r]_{A_1}$, $[w^{r\frac{p'-1}{p'-r}}]_{A_1}\leqslant[w^r]_{A_1}^{(p'-1)/(p'-r)}$. Thus for any cube $Q\subset\mathbb{R}^n$,

$$\left(\frac{1}{|Q|}\int_Q w^{1-p'}(x)dx\right)\left(\frac{1}{|Q|}\int_Q w^{r\frac{p'-1}{p'-r}}(x)dx\right)^{\frac{p'}{r}-1}$$

$$\leqslant \left[\operatorname{essinf}_{y\in Q} w(y)\right]^{1-p'} \left[w^{r\frac{p'-1}{p'-r}}\right]_{A_1}^{\frac{p'}{r}-1} \left[\operatorname{essinf}_{y\in Q} w(y)\right]^{p'-1} \leqslant [w^r]_{A_1}^{\frac{p'-1}{r}}.$$

This shows that $w^{1-p'} \in A_{p'/r}(\mathbb{R}^n)$. □

LEMMA 2.5. *Let T be a sublinear operator. Suppose that there exists a constant $\tau \in (0,1)$, such that for all $\lambda \in (0,1/2)$,*

$$\|M_{\lambda,T} f\|_{L^{1,\infty}(\mathbb{R}^n)} \leqslant \lambda^{-\tau} \|f\|_{L^1(\mathbb{R}^n)}.$$

Then for $p_0 \in (1, 1/\tau)$,

$$\|\mathscr{M}_{p_0,T} f\|_{L^{1,\infty}(\mathbb{R}^n)} \leqslant 2^{2+\frac{4}{1-\tau p_0}} \|f\|_{L^1(\mathbb{R}^n)},$$

where $\mathscr{M}_{p_0,T}$ is the maximal operator defined as (1.5).

Proof. We employ the argument used in the proof of Lemma 3.3 in [18]. As it was proved in [18],

$$\mathscr{M}_{p_0,T} f(x) \leqslant \left(\int_0^1 \left(M_{\lambda,T} f(x)\right)^{p_0} d\lambda\right)^{\frac{1}{p_0}}.$$

For $N > 0$, denote

$$G_{p_0,T,N} f(x) = \left(\int_0^1 \left(\min\{M_{\lambda,T} f(x), N\}\right)^{p_0} d\lambda\right)^{\frac{1}{p_0}},$$

and

$$\mu_f(\alpha, R) = |\{x \in \mathbb{R}^n : |x| \leqslant R, |f(x)| > \alpha\}|, \ \alpha, R > 0.$$

Let $p_0 \in (1,\infty)$ such that $\tau p_0 \in (0,1)$, $k = \lfloor \frac{4}{1-\tau p_0} \rfloor + 1$, where and in the following, for $a \in \mathbb{R}$, $\lfloor a \rfloor$ denotes the integer part of a. By Hölder's inequality,

$$\begin{aligned} G_{p_0,T,N} f(x) &\leqslant \left(\int^{\frac{1}{2^{kp_0}}} \left(\min\{M_{\lambda,T} f(x), N\}\right)^{p_0} d\lambda\right)^{\frac{1}{p_0}} + M_{1/2^{kp_0},T} f(x) \\ &\leqslant \frac{1}{2^{k-1}} G_{kp_0,T,N} f(x) + M_{1/2^{kp_0},T} f(x). \end{aligned}$$

Therefore,

$$\begin{aligned} \mu_{G_{p_0,T,N}f}(\alpha, R) &\leqslant \mu_{G_{kp_0,T,N}f}(2^{k-2}\alpha, R) + \mu_{M_{1/2^{kp_0},T}f}(\alpha/2, R) \\ &\leqslant \mu_{G_{kp_0,T,N}f}(2^{k-2}\alpha, R) + \frac{1}{\alpha} 2^{\tau k p_0 + 1} \|f\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

Repeating the last inequality j times, we have that

$$\mu_{G_{p_0,T,N}f}(\alpha, R) \leqslant \mu_{G_{k^j p_0,T,N}f}(2^{j(k-2)}\alpha, R) + \frac{2^{k-2}}{\alpha} \textstyle\sum_{l=1}^{j} \left(\frac{2^{\tau k p_0 + 1}}{2^{k-2}}\right)^l \|f\|_{L^1(\mathbb{R}^n)}.$$

Since $G_{p_0,T,N}f$ is uniformly bounded in p_0, we obtain that $\mu_{G_{k^j p_0,T,N}f}(\alpha, R) \to 0$ as $j \to \infty$. We finally deduce that

$$\mu_{G_{p_0,T,N}f}(\alpha, R) \leqslant 2^{2+\frac{4}{1-\tau p_0}} \frac{1}{\alpha} \|f\|_{L^1(\mathbb{R}^n)}.$$

This completes the proof of Lemma 2.5. □

Let $\eta \in (0,1)$ and $\mathscr{S} = \{Q_j\}$ be a family of cubes. We say that $\mathscr{S}$ is η-sparse, if for each fixed $Q \in \mathscr{S}$, there exists a measurable subset $E_Q \subset Q$, such that $|E_Q| \geqslant \eta |Q|$ and E_Q's are pairwise disjoint. For sparse family $\mathscr{S}$ and constants β, $r \in [0,\infty)$, we define the bilinear sparse operator $\mathscr{A}_{\mathscr{S};L(\log L)^\beta,L^r}$ by

$$\mathscr{A}_{\mathscr{S};L(\log L)^\beta,L^r}(f,g) = \sum_{Q\in\mathscr{S}} |Q| \|f\|_{L(\log L)^\beta,Q} \langle |g| \rangle_{Q,r}.$$

We denote $\mathscr{A}_{\mathscr{S};L(\log L)^1,L^r}$ by $\mathscr{A}_{\mathscr{S};L\log L,L^r}$ for simplicity, and $\mathscr{A}_{\mathscr{S};L(\log L)^0,L^r}$ by $\mathscr{A}_{\mathscr{S};L,L^r}$.

LEMMA 2.6. *Let $\alpha, \beta \in \mathbb{N} \cup \{0\}$ and U be an operator. Suppose that for any $r \in (1, 3/2)$, and bounded function f with compact support, there exists a sparse family of cubes $\mathscr{S}$, such that for any function $g \in L^1(\mathbb{R}^n)$,*

$$\Big| \int_{\mathbb{R}^n} Uf(x)g(x)dx \Big| \leqslant r'^{\alpha} \mathscr{A}_{\mathscr{S};L(\log L)^\beta,L^r}(f,g). \tag{2.4}$$

Then for any $u \in A_1(\mathbb{R}^n)$ and bounded function f with compact support,

$$\begin{aligned} &w(\{x \in \mathbb{R}^n : |Uf(x)| > \lambda\}) \\ &\lesssim_{n,\alpha,\beta} [w]_{A_\infty}^{\alpha} \log^{1+\beta}(\mathrm{e} + [w]_{A_\infty})[w]_{A_1} \int_{\mathbb{R}^d} \frac{|f(x)|}{\lambda} \log^\beta \left(\mathrm{e} + \frac{|f(x)|}{\lambda} \right) w(x)dx. \end{aligned}$$

Lemma 2.6 is Corollary 3.6 in [14].

THEOREM 2.7. *Let $p_0 \in (1,\infty)$, $r \in (1,\infty)$, $b \in \mathrm{BMO}(\mathbb{R}^n)$, T be a linear operator and T_b be the commutator of T. Suppose that both of operators T and $\mathscr{M}_{p_0,T}$ are bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$ with bound 1. Then for bounded functions f with compact supports, there exists a $\frac{1}{2}\frac{1}{3^n}$-sparse family $\mathscr{S}$ and functions $\mathrm{H}_1 f$, $\mathrm{H}_2 f$, such that for each function $g \in L^{rp_0'}_{\mathrm{loc}}(\mathbb{R}^n)$,*

$$\Big| \int_{\mathbb{R}^n} \mathrm{H}_1 f(x)g(x)dx \Big| \lesssim_n \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} r' p_0' \mathscr{A}_{\mathscr{S};L^1,L^{rp_0'}}(f,g), \tag{2.5}$$

$$\Big| \int_{\mathbb{R}^n} \mathrm{H}_2 f(x)g(x)dx \Big| \lesssim_n \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \mathscr{A}_{\mathscr{S};L\log L,L^{p_0'}}(f,g), \tag{2.6}$$

and for a. e. $x \in \mathbb{R}^n$,

$$T_b f(x) = \mathrm{H}_1 f(x) + \mathrm{H}_2 f(x).$$

Proof. We will employ the ideas in [18], see also the proof of Theorem 3.2 in [14]. Without loss of generality, we may assume that $\|b\|_{\mathrm{BMO}(R^n)} = 1$. For a fixed cube Q_0, define the local analogy of $\mathscr{M}_{p_0,T}$ by

$$\mathscr{M}_{p_0,T;Q_0}f(x) = \sup_{Q\ni x,\, Q\subset Q_0}\left(\frac{1}{|Q|}\int_Q |T(f\chi_{3Q_0\setminus 3Q})(y)|^{p_0}dy\right)^{\frac{1}{p_0}}.$$

Let $E = \cup_{j=1}^4 E_j$ with

$$\begin{aligned}
E_1 &= \big\{x\in Q_0 : |T(f\chi_{3Q_0})(x)| > D\langle|f|\rangle_{3Q_0}\big\},\\
E_2 &= \big\{x\in Q_0 : |T\big((b-\langle b\rangle_{Q_0})f\chi_{3Q_0}\big)(x)| > D\langle|(b-\langle b\rangle_{Q_0})f|\rangle_{3Q_0}\big\},\\
E_3 &= \{x\in Q_0 : \mathscr{M}_{p_0,T;Q_0}f(x) > D\langle|f|\rangle_{3Q_0}\},
\end{aligned}$$

and

$$E_4 = \big\{x\in Q_0 : \mathscr{M}_{p_0,T_\Omega;Q_0}\big((b-\langle b\rangle_{Q_0})f\big)(x) > D\langle|b-\langle b\rangle_{Q_0}||f|\rangle_{Q_0}\big\},$$

where D is a positive constant. If we choose D large enough, it then follows from the weak type $(1,1)$ boundedness of T and $\mathscr{M}_{p_0,T}$ that

$$|E| \leqslant \frac{1}{2^{n+2}}|Q_0|.$$

Now on the cube Q_0, we apply the Calderón-Zygmund decomposition to χ_E at level $\frac{1}{2^{n+1}}$, and obtain pairwise disjoint cubes $\{P_j\}\subset\mathscr{D}(Q_0)$, such that

$$\frac{1}{2^{n+1}}|P_j| \leqslant |P_j\cap E| \leqslant \frac{1}{2}|P_j|$$

and $|E\setminus\cup_j P_j| = 0$. Observe that $\sum_j |P_j| \leqslant \frac{1}{2}|Q_0|$. Let

$$G^1_{Q_0}(x) = (b(x)-\langle b\rangle_{Q_0})T(f\chi_{3Q_0})\chi_{Q_0\setminus\cup_l P_l}(x) + \sum_l (b(x)-\langle b\rangle_{Q_0})T(f\chi_{3Q_0\setminus 3P_l})\chi_{P_l}(x),$$

$$G^2_{Q_0}(x) = T\big((b-\langle b\rangle_{Q_0})f\chi_{3Q_0}\big)\chi_{Q_0\setminus\cup_l P_l}(x) + \sum_l T\big((b-\langle b\rangle_{Q_0})f\chi_{3Q_0\setminus 3P_l}\big)\chi_{P_l}(x).$$

It then follows that

$$T_b(f\chi_{3Q_0})(x)\chi_{Q_0}(x) = G^1_{Q_0}(x) + G^2_{Q_0}(x) + \sum_l T_b(f\chi_{3P_l})(x)\chi_{P_l}(x).$$

We now estimate $G^1_{Q_0}$ and $G^2_{Q_0}$. By (1.7) and the John-Nirenberg inequality (see [11, p.128]), we know that

$$\begin{aligned}
\int_{Q_0}|b(x)-\langle b\rangle_{Q_0}||h(x)|dx &\lesssim |Q_0|\|b-\langle b\rangle_{Q_0}\|_{\exp L,Q}\|h\|_{L\log L,Q_0}\\
&\lesssim |Q_0|\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|h\|_{L\log L,Q_0}.
\end{aligned}$$

This, along with the fact that $|E\backslash\cup_j P_j| = 0$, implies that

$$\Big|\int_{Q_0\backslash\cup_l P_l}(b(x)-\langle b\rangle_{Q_0})T(f\chi_{3Q_0})(x)g(x)dx\Big| \lesssim \langle|f|\rangle_{3Q_0}\|g\|_{L\log L,Q_0}|Q_0|,$$

and

$$\Big|\int_{Q_0\backslash\cup_l P_l}T\big((b-\langle b\rangle_{Q_0})f\chi_{3Q_0}\big)(x)g(x)dx\Big| \lesssim \langle|f|\rangle_{L\log L,3Q_0}\langle|g|\rangle_{Q_0}|Q_0|.$$

On the other hand, the fact that $P_j\cap E^c\neq\emptyset$ tells us that

$$\begin{aligned}
&\sum_l\Big|\int_{P_l}(b(x)-\langle b\rangle_{Q_0})T(f\chi_{3Q_0\backslash 3P_l})(x)g(x)dx\Big|\\
&\lesssim\sum_l\Big(\int_{P_l}|b(x)-\langle b\rangle_{Q_0}|^{p_0'}|g(x)|^{p_0'}dx\Big)^{\frac{1}{p_0'}}\Big(\int_{P_l}|T(f\chi_{3Q_0\backslash 3P_l})(x)|^{p_0}dx\Big)^{p_0}\\
&\lesssim\sum_l\Big(\int_{P_l}|b(x)-\langle b\rangle_{Q_0}|^{p_0'r'}\Big)^{\frac{1}{p_0'r'}}|P_l|^{\frac{1}{p_0'r}+\frac{1}{p_0}}\langle|g|\rangle_{P_l,p_0'r}\inf_{y\in P_l}\mathscr{M}_{T,p_0,Q_0}f(y)\\
&\lesssim r'p_0'\langle|f|\rangle_{3Q_0}\sum_l|P_l|\langle|g|\rangle_{P_l,rp_0'}\lesssim r'p_0'\langle|f|\rangle_{3Q_0}\langle|g|\rangle_{Q_0,rp_0'}|Q_0|,
\end{aligned}$$

here we have invoked the following estimate

$$\Big(\int_{Q_0}|b(x)-\langle b\rangle_{Q_0}|^{p_0'r'}dx\Big)^{\frac{1}{p_0'r'}}\lesssim r'p_0'|Q_0|^{\frac{1}{p_0'r'}},$$

see [11, p. 128]. Similarly, we can deduce that

$$\begin{aligned}
&\sum_l\Big|\int_{P_l}T\big((b-\langle b\rangle_{Q_0})f\chi_{3Q_0\backslash 3P_l}\big)(x)g(x)dx\Big|\\
&\lesssim\sum_l|P_l|\langle|g|\rangle_{P_l,p_0'}\inf_{y\in P_l}\mathscr{M}_{p_0,T;Q_0}\big(b-\langle b\rangle_{Q_0}\big)f(y)\\
&\lesssim\langle|f|\rangle_{3Q_0}\sum_l|P_l|\langle|g|\rangle_{P_l,p_0'}\lesssim\langle|f|\rangle_{3Q_0}\langle|g|\rangle_{Q_0,p_0'}|Q_0|.
\end{aligned}$$

Therefore, for function $g\in L^r_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\Big|\int_{\mathbb{R}^n}G^1_{Q_0}(x)g(x)dx\Big|\lesssim r'p_0'\langle|f|\rangle_{3Q_0}\langle|g|\rangle_{Q_0,rp_0'}|Q_0|. \tag{2.7}$$

and

$$\Big|\int_{\mathbb{R}^n}G^2_{Q_0}(x)g(x)dx\Big|\lesssim\|f\|_{L\log L,3Q_0}\langle|g|\rangle_{Q_0,p_0'}|Q_0|. \tag{2.8}$$

We repeat argument above with $T(f\chi_{3Q_0})(x)\chi_{Q_0}$ replaced by $T(\chi_{3P_l})(x)\chi_{P_l}(x)$, and so on. Let $Q_0^{j_0}=Q_0$, $Q_0^{j_1}=P_j$, and for fixed $j_1,\dots,j_{m-1}$, $\{Q_0^{j_1\cdots j_{m-1}j_m}\}_{j_m}$ be the cubes obtained at the m-th stage of the decomposition process to the cube $Q_0^{j_1\cdots j_{m-1}}$. Set $\mathscr{F}=\{Q_0\}\cup_{m=1}^{\infty}\cup_{j_1,\dots,j_m}\{Q_0^{j_1\cdots j_m}\}$. Then $\mathscr{F}\subset\mathscr{D}(Q_0)$ is a $\frac{1}{2}$-sparse family. We define the functions H_{1,Q_0} and H_{2,Q_0} by

$$H_{1,Q_0}(x)=\sum_{m=1}^{\infty}\sum_{j_1\dots j_{m-1}}\big(b(x)-\langle b\rangle_{Q_0^{j_1,\dots,j_{m-1}}}\big)\times T\big(f\chi_{3Q_0^{j_1\cdots j_{m-1}}}\big)(x)\chi_{Q_0^{j_1,\dots,j_{m-1}}\setminus\cup_{j_m}Q_0^{j_1,\dots,j_m}}(x)$$
$$+\sum_{m=1}^{\infty}\sum_{j_1\dots j_m}\big(b(x)-\langle b\rangle_{Q_0^{j_1,\dots,j_{m-1}}}\big)\times T\big(f\chi_{3Q_0^{j_1\cdots j_{m-1}}\setminus\cup_{j_m}3Q_0^{j_1\cdots j_m}}\big)(x)\chi_{Q_0^{j_1\cdots j_m}}(x),$$

and

$$H_{2,Q_0}(x)=\sum_{m=1}^{\infty}\sum_{j_1\dots j_{m-1}}T\big((b(x)-\langle b\rangle_{Q_0^{j_1,\dots,j_{m-1}}})f\chi_{3Q_0^{j_1\cdots j_{m-1}}}\big)(x)$$
$$\times\chi_{Q_0^{j_1,\dots,j_{m-1}}\setminus\cup_{j_m}Q_0^{j_1,\dots,j_m}}(x)$$
$$+\sum_{m=1}^{\infty}\sum_{j_1\dots j_m}T\big((b(x)-\langle b\rangle_{Q_0^{j_1,\dots,j_{m-1}}})f\chi_{3Q_0^{j_1\cdots j_m}\setminus\cup_{j_{m+1}}3Q_0^{j_1\cdots j_{m-1}}}\big)(x)$$
$$\times\chi_{Q_0^{j_1\cdots j_{m-1}}}(x).$$

Then for a. e. $x\in Q_0$,

$$T_b(f\chi_{3Q_0})(x)=H_{1,Q_0}(x)+H_{2,Q_0}(x).$$

Moreover, as in inequalities (2.7)-(2.8), the process of producing $\{Q_0^{j_1\cdots j_m}\}$ leads to that

$$\Big|\int_{Q_0}g(x)H_{1,Q_0}(x)dx\Big|\lesssim r'p_0'\sum_{Q\in\mathscr{F}}|Q|\langle|f|\rangle_{3Q}\langle|g|\rangle_{Q,rp_0'},$$

and

$$\Big|\int_{Q_0}g(x)H_{2,Q_0}(x)dx\Big|\lesssim\sum_{Q\in\mathscr{F}}|Q|\|f\|_{L\log L,3Q}\langle|g|\rangle_{Q,p_0'}.$$

We can now conclude the proof of Theorem 2.7. In fact, as in [18], we decompose $\mathbb{R}^n$ by cubes $\{R_l\}$, such that $\mathrm{supp}f\subset 3R_l$ for each l, and R_l's have disjoint interiors. Then for a. e. $x\in\mathbb{R}^n$,

$$T_bf(x)=\sum_l H_{1,R_l}f(x)+\sum_l H_{2,R_l}f(x)=:\mathrm{H}_1f(x)+\mathrm{H}_2f(x).$$

Obviously, H_1, H_2 satisfies (2.5) and (2.6). Our desired conclusion then follows directly. □

LEMMA 2.8. *Let $\gamma\in\mathbb{N}\cup\{0\}$, $r\in[1,\infty)$, and U be an operator. Suppose that for any bounded function f with compact support, there exists a sparse family of cubes $\mathscr{S}$, such that for any function $g\in L^r_{\mathrm{loc}}(\mathbb{R}^n)$,*

$$\Big|\int_{\mathbb{R}^n}Uf(x)g(x)dx\Big|\leqslant\mathscr{A}_{\mathscr{S};L(\log L)^\gamma,L^r}(f,g).\tag{2.9}$$

Then for any w with $w^r\in A_1(\mathbb{R}^n)$, $\alpha>0$ and bounded function f with compact support,

$$w(\{x\in\mathbb{R}^n:\,|Uf(x)|>\alpha\})\lesssim_{n,\,\gamma,w}\int_{\mathbb{R}^d}\frac{|f(x)|}{\alpha}\log^{\gamma}\Big(\mathrm{e}+\frac{|f(x)|}{\alpha}\Big)w(x)dx.$$

Proof. By Theorem 3.2 in [14], we know that U satisfies the following estimate:

$$\begin{aligned}w(\{x\in\mathbb{R}^d:\,|Uf(x)|>1\})\lesssim&\left(1+\left\{p_1'^{1+\gamma}\Big(\frac{p_1'}{r}\Big)'\Big(t\frac{p_1'/r-1}{p_1'-1}\Big)'^{\frac{1}{p_1'}}\right\}^{p_1}\right)\\&\times\int_{\mathbb{R}^n}|f(y)|\log^{\gamma}(\mathrm{e}+|f(y)|)M_tw(y)dy,\end{aligned}\tag{2.10}$$

where $t\in[1,\infty)$, $p_1\in(1,r')$ such that $t\frac{p_1'/r-1}{p_1'-1}>1$, and M_t is defined by

$$M_rf(x)=\big[M(|f|^r)(x)\big]^{1/r}.$$

Let $w^r\in A_1(\mathbb{R}^n)$. We choose $\varepsilon>0$ such that $w^{r(1+\varepsilon)}\in A_1(\mathbb{R}^n)$. Set $t=r(1+\varepsilon)$ and $p_1'=2(r-1)\frac{1+\varepsilon}{\varepsilon}+1$. Then $t\frac{p_1'/r-1}{p_1'-1}=1+\frac{\varepsilon}{2}$. We obtain from (2.10) that

$$w(\{x\in\mathbb{R}^d:\,|Uf(x)|>1\})\lesssim_{n,\gamma,w}\int_{\mathbb{R}^n}|f(y)|\log^{\gamma}(\mathrm{e}+|f(y)|)w(y)dy.$$

This, via homogeneity, leads to our desired conclusion. □

Proof of Theorem 1.2. By homogeneity, we may assume that $\|\Omega\|_{L^q(S^{n-1})}=1=\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$. Let $w^{q'}\in A_1(\mathbb{R}^n)$. We choose $\varepsilon>0$ such that $\varepsilon\in(0,\min\{1,(q-1)/3\})$ and $w^{q'(1+\varepsilon)}\in A_1(\mathbb{R}^n)$. On the other hand, by Lemma 2.3 and Lemma 2.5, we know that for any $p_0\in(0,q/(1+2\varepsilon))$,

$$\|\mathscr{M}_{p_0,T_\Omega}f\|_{L^1(\mathbb{R}^n)}\lesssim 2^{4^{\frac{1}{1-p_0\frac{1+2\varepsilon}{q}}}}\|f\|_{L^1(\mathbb{R}^n)}.$$

Take $p_0=q/(1+3\varepsilon)$ and $r=\frac{q-(1+3\varepsilon)}{q-1}(1+\varepsilon)$, then $rp_0'=(1+\varepsilon)q'$. Applying Theorem 2.7 with such indices p_0 and r, we see that for any bounded function f with compact support, there exists a sparse family of cubes $\mathscr{S}$, such that for any $g\in L^{q'(1+\varepsilon)}_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\Big|\int_{\mathbb{R}^n}T_bf(x)g(x)dx\Big|\lesssim p_0'r'2^{4^{\frac{1+3\varepsilon}{\varepsilon}}}\mathscr{A}_{\mathscr{S};L\log L,L^{q'(1+\varepsilon)}}(f,g).$$

Theorem 1.2 now follows from Lemma 2.8 immediately. □

Proof of Theorem 1.3. Again we assume that $\|\Omega\|_{L^\infty(S^{n-1})}=1=\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$. Let $s\in(1,\infty)$. Applying (1.6) and Theorem 2.7 (with $p_0=(\sqrt{s})'$ and $r=\sqrt{s}$), we know that for bounded function f with compact support, there exists a $\frac{1}{2}\frac{1}{3^n}$-sparse family of cubes $\mathscr{S}=\{Q\}$, and functions $\mathrm{H}_1 f$, $\mathrm{H}_2 f$, such that for each function $g\in L^s_{\rm loc}(\mathbb{R}^n)$,

$$\Big|\int_{\mathbb{R}^n}\mathrm{H}_1f(x)g(x)dx\Big|\lesssim(\sqrt{s})'^2\mathscr{A}_{\mathscr{S};L^1,L^s}(f,g)\lesssim s'^2\mathscr{A}_{\mathscr{S};L^1,L^s}(f,g),$$
$$\Big|\int_{\mathbb{R}^n}\mathrm{H}_2f(x)g(x)dx\Big|\lesssim(\sqrt{s})'\mathscr{A}_{\mathscr{S};L\log L,L^{\sqrt{s}}}(f,g)\lesssim s'\mathscr{A}_{\mathscr{S};L\log L,L^s}(f,g),$$

and for a. e. $x\in\mathbb{R}^n$,

$$T_{\Omega,b}f(x)=\mathrm{H}_1f(x)+\mathrm{H}_2f(x).$$

Let $w\in A_1(\mathbb{R}^n)$, $\lambda>0$, f be a bounded function with compact support. It follows from Lemma 2.6 that

$$\begin{aligned}&w(\{x\in\mathbb{R}^n:|T_{\Omega,b}f(x)|>\lambda\})\\ \leqslant\;& w(\{x\in\mathbb{R}^n:|\mathrm{H}_1f(x)|>\lambda/2\})+w(\{x\in\mathbb{R}^n:|\mathrm{H}_2f(x)|>\lambda/2\})\\ \lesssim\;&[w]_{A_1}[w]_{A_\infty}^2\log(\mathrm{e}+[w]_{A_\infty})\int_{\mathbb{R}^n}\frac{|f(x)|}{\lambda}w(x)dx\\ &+[w]_{A_1}[w]_{A_\infty}\log^2(\mathrm{e}+[w]_{A_\infty})\int_{\mathbb{R}^n}\frac{|f(x)|}{\lambda}\log\Big(\mathrm{e}+\frac{|f(x)|}{\lambda}\Big)w(x)dx\\ \lesssim\;&[w]_{A_1}[w]_{A_\infty}^2\log(\mathrm{e}+[w]_{A_\infty})\int_{\mathbb{R}^n}\frac{|f(x)|}{\lambda}\log\Big(\mathrm{e}+\frac{|f(x)|}{\lambda}\Big)w(x)dx.\end{aligned}$$

This completes the proof of Theorem 1.3. □

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