Mathematical
Inequalities
& Applications
Volume 23, Number 4 (2020), 1197–1203 doi:10.7153/mia-2020-23-92

AN INEQUALITY INVOLVING THE CONSTANT e AND A GENERALIZED CARLEMAN–TYPE INEQUALITY

CHAO-PING CHEN AND RICHARD B. PARIS

(*Communicated by I. Perić*)

Abstract. In this paper, we establish a double inequality involving the constant e. As an application, we give a generalized Carleman-type inequality.

1. Introduction

Let $a_n \geqslant 0$ for $n \in \mathbb{N} := \{1,2,\ldots\}$ and $0 < \sum_{n=1}^{\infty} a_n < \infty$. Then

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n. \tag{1.1}$$

The constant e is the best possible. The inequality (1.1) was presented in 1922 in [3] by the Swedish mathematician Torsten Carleman and it is called Carleman's inequality. Carleman discovered this inequality during his important work on quasi-analytical functions.

Carleman's inequality (1.1) was generalized by Hardy [12] (see also [13, p. 256]) as follows: If $a_n \geqslant 0$, $\lambda_n > 0$, $\Lambda_n = \sum_{m=1}^{n} \lambda_m$ for $n \in \mathbb{N}$, and $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$, then

$$\sum_{n=1}^{\infty} \lambda_n \left(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n}\right)^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \lambda_n a_n. \tag{1.2}$$

Note that inequality (1.2) is usually referred to as a Carleman-type inequality or weighted Carleman-type inequality. In his original paper [12], Hardy himself said that it was Pólya who pointed out this inequality to him. For information about the history of Carleman-type inequalities, please refer to [15, 16, 18, 24].

In [4, 5, 6, 9, 10, 11, 14, 19, 20, 21, 22, 23, 26, 27, 28, 29, 30, 31], some strengthened and generalized results of (1.1) and (1.2) have been given by estimating the weight coefficient $(1+1/n)^n$. For example, Mortici and Jang [23] proved that for $0 < x \leqslant 1$,

$$e\left(1 - \frac{1}{2}x + \frac{11}{24}x^2 - \frac{7}{16}x^3 + \frac{2447}{5760}x^4 - \frac{959}{2304}x^5\right) < (1+x)^{1/x} < e\left(1 - \frac{1}{2}x + \frac{11}{24}x^2 - \frac{7}{16}x^3 + \frac{2447}{5760}x^4\right). \tag{1.3}$$

Mathematics subject classification (2010): 26D15.
Keywords and phrases: Carleman's inequality, weight coefficient.
This work was supported by Key Science Research Project in Universities of Henan (20B110007).

According to Pólya's proof of (1.1) in [25],

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} \leqslant \sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n a_n, \tag{1.4}$$

and then the following strengthened Carleman's inequality can be derived directly from the right-hand side of (1.3):

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty}\left(1-\frac{1}{2n}+\frac{11}{24n^2}-\frac{7}{16n^3}+\frac{2447}{5760n^4}\right)a_n. \tag{1.5}$$

In this paper, we develop the double inequality (1.3) to produce a general result. As an application, we give a generalized Carleman-type inequality.

2. A double inequality involing the constant e

Brothers and Knox [2] (see also [17, 7]) derived, without a formula for the general term, the following expansion:

$$\left(1+\frac{1}{x}\right)^x = e\left(1-\frac{1}{2x}+\frac{11}{24x^2}-\frac{7}{16x^3}+\frac{2447}{5760x^4}-\frac{959}{2304x^5}+\frac{238043}{580608x^6}-\cdots\right) \tag{2.1}$$

for $x < -1$ or $x \geqslant 1$. Chen and Choi [7] gave an explicit formula for successively determining the coefficients. More precisely, these authors proved that

$$\left(1+\frac{1}{x}\right)^x \sim e\sum_{j=0}^{\infty}(-1)^j b_j x^{-j} \qquad (x\to\infty), \tag{2.2}$$

where the coefficients b_j are given by

$$b_0 = 1 \quad \text{and} \quad b_j = \sum_{k_1+2k_2+\cdots+jk_j=j}\frac{\left(\frac{1}{2}\right)^{k_1}\left(\frac{1}{3}\right)^{k_2}\cdots\left(\frac{1}{j+1}\right)^{k_j}}{k_1!k_2!\cdots k_j!} \qquad (j\geqslant 1) \tag{2.3}$$

summed over all nonnegative integers k_j satisfying the equation $k_1+2k_2+\cdots+jk_j = j$.

A recurrence relation for the coefficients b_j can be obtained by use of the result given in [8, Lemma 3]. This states that for a function $A(x)$ with asymptotic expansion $A(x)\sim\sum_{n=1}^{\infty}\alpha_n x^{-n}$ as $x\to\infty$, the composition $B(x)=\exp[A(x)]$ has the expansion $B(x)\sim\sum_{n=1}^{\infty}\beta_n x^{-n}$ as $x\to\infty$, where $\beta_0=1$ and

$$\beta_n = \frac{1}{n}\sum_{k=1}^{n}k\alpha_k\beta_{n-k} \qquad (n\geqslant 1).$$

From the Maclaurin expansion

$$\frac{1}{x}\ln(1+x) = 1+\sum_{j=1}^{\infty}\frac{(-1)^j x^j}{j+1} \qquad (-1<x\leqslant 1),$$

it therefore follows (upon replacing x by $1/x$) that the coefficients b_j in (2.2) are given by the recurrence relation

$$b_0 = 1 \quad \text{and} \quad b_j = \frac{1}{j}\sum_{k=1}^{j}\frac{k}{k+1}b_{j-k} \qquad (j \geqslant 1). \tag{2.4}$$

Use of (2.4) is easily seen to generate the values

$$b_1 = \frac{1}{2}, \quad b_2 = \frac{11}{24}, \quad b_3 = \frac{7}{16}, \quad b_4 = \frac{2447}{5760}, \quad b_5 = \frac{959}{2304}, \quad b_6 = \frac{238043}{580608}, \ \ldots,$$

which are the same coefficients as in (2.1). The representation using a recursive algorithm for the coefficients b_j is more practical for numerical evaluation than the expression in (2.3).

The above result immediately shows that $b_j > 0$ so that (2.2) is an alternating series for positive x. Replacement of x by $1/x$ in (2.1) and (2.2) then enables us to write

$$(1+x)^{1/x} = e\sum_{j=0}^{\infty}(-1)^j b_j x^j \qquad (-1 < x \leqslant 1). \tag{2.5}$$

We now establish a monotonicity property satisfied by the coefficients b_j.

LEMMA 2.1. *The sequence $\{b_j\}_{j=0}^{\infty}$ in (2.5) is monotonically decreasing.*

Proof. By Cauchy's theorem it follows from (2.5) that

$$b_j = \frac{(-1)^j}{2\pi i e}\oint_C (1+t)^{1/t}\frac{dt}{t^{j+1}},$$

where C is a closed loop surrounding $t=0$ described in the positive sense. Define

$$\Delta_j = b_j - b_{j+1}.$$

Then

$$\Delta_j = \frac{(-1)^j}{2\pi i e}\oint_C (1+t)^{1/t}\left(1+\frac{1}{t}\right)\frac{dt}{t^{j+1}} = \frac{(-1)^j}{2\pi i e}\oint_C (1+t)^{1+1/t}\frac{dt}{t^{j+2}}.$$

In the t-plane there is a branch cut along $(-\infty,-1]$. Now expand C to be a large circle of radius R that is indented to pass along the upper and lower sides of the branch cut. The contribution from the large circle tends to zero as $R \to \infty$. Similarly, the contribution round the branch point $t = -1+\rho e^{i\theta}$, $-\pi \leqslant \theta \leqslant \pi$ vanishes as $\rho \to 0$. Then we have upon putting $t = xe^{\pm\pi i}$ on the upper and lower sides of the branch cut

$$\begin{aligned}\Delta_j &= \frac{1}{2\pi i e}\int_{\infty}^{1}(x-1)^{1-1/x}e^{-\pi i/x}\frac{dx}{x^{j+2}} + \frac{1}{2\pi i e}\int_{1}^{\infty}(x-1)^{1-1/x}e^{\pi i/x}\frac{dx}{x^{j+2}}\\ &= \frac{1}{\pi e}\int_{1}^{\infty}(x-1)^{1-1/x}\sin(\pi/x)\frac{dx}{x^{j+2}}. \end{aligned}\tag{2.6}$$

Now on the interval $x \in [1,\infty)$ the function $\sin(\pi/x) \geqslant 0$ so that the integrand in (2.6) is non-negative on $[1,\infty)$. Hence $\Delta_j > 0$ and the sequence $\{b_j\}_{j=0}^{\infty}$ is monotonically decreasing. This completes the proof.

REMARK 2.1. We thank a referee for providing the literature [1]. It was proved in [1, Lemma 1] that

$$(x+1)\left[e-\left(1+\frac{1}{x}\right)^x\right] = \frac{e}{2} + \int_0^1 \frac{g(s)}{x+s}\mathrm{d}s \qquad (x>0), \tag{2.7}$$

where

$$g(s) = \frac{1}{\pi}s^s(1-s)^{1-s}\sin(\pi s) \qquad (0 \leqslant s \leqslant 1). \tag{2.8}$$

By (2.7), we here give an integral representation of the coefficients b_j in (2.5), and then use it to prove Lemma 2.1.

Write (2.7) as

$$\left(1+\frac{1}{x}\right)^x = e - \frac{e}{2(x+1)} - \int_0^1 \frac{g(s)}{(x+1)(x+s)}\mathrm{d}s \qquad (x>0). \tag{2.9}$$

Replacing x by $1/t$ in (2.9) yields, for $t>0$,

$$\begin{aligned} f(t) := (1+t)^{1/t} &= \frac{e}{2} + \frac{e}{2(t+1)} - \int_0^1 \frac{g(s)}{s}\frac{t^2}{(t+1)(t+\frac{1}{s})}\mathrm{d}s \\ &= \frac{e}{2} + \frac{e}{2(t+1)} - \int_0^1 \frac{g(s)}{s}\left\{1 + \frac{s}{(1-s)(t+1)} - \frac{1}{s(1-s)(t+\frac{1}{s})}\right\}\mathrm{d}s. \end{aligned} \tag{2.10}$$

Clearly,

$$eb_0 = f(0) = e.$$

Differentiating the expression in (2.10), we find that for $n \geqslant 1$,

$$\frac{(-1)^n f^{(n)}(t)}{n!} = \frac{e}{2(t+1)^{n+1}} - \int_0^1 \frac{g(s)}{s}\left\{\frac{s}{(1-s)(t+1)^{n+1}} - \frac{1}{s(1-s)\left(t+\frac{1}{s}\right)^{n+1}}\right\}\mathrm{d}s,$$

we then obtain the following integral representation of the coefficients b_j in (2.5):

$$b_n = \frac{(-1)^n f^{(n)}(0)}{n!e} = \frac{1}{2} - \frac{1}{e}\int_0^1 \frac{1-s^{n-1}}{1-s}g(s)\mathrm{d}s$$

for $n \geqslant 1$, and we have

$$\Delta_j = b_j - b_{j+1} = \frac{1}{e}\int_0^1 s^{j-1}g(s)\mathrm{d}s > 0 \qquad (j \geqslant 1). \tag{2.11}$$

Noting that $b_0 = 1 > \frac{1}{2} = b_1$ holds, we see that the sequence $\{b_j\}_{j=0}^{\infty}$ in (2.5) is monotonically decreasing.

In fact, by an elementary change of variable $x = 1/s$ $(0 \leqslant s \leqslant 1)$, we see that (2.6) $\Longleftrightarrow$ (2.11).

From (2.5) and Lemma 2.1 we obtain the following theorem that develops the double inequality (1.3) to produce a general result.

THEOREM 2.1. *For all integers* $m \geqslant 0$,

$$e\sum_{j=0}^{2m+1}(-1)^j b_j x^j < (1+x)^{1/x} < e\sum_{j=0}^{2m}(-1)^j b_j x^j \qquad (0<x\leqslant 1), \tag{2.12}$$

or alternatively

$$e\sum_{j=0}^{2m+1}\frac{(-1)^j b_j}{x^j} < \left(1+\frac{1}{x}\right)^x < e\sum_{j=0}^{2m}\frac{(-1)^j b_j}{x^j} \qquad (x\geqslant 1), \tag{2.13}$$

where the coefficients b_j *are given by the recursive relation* (2.4).

3. A generalized Carleman-type inequality

THEOREM 3.1. *Let* $0<\lambda_{n+1}\leqslant\lambda_n$, $\Lambda_n=\sum_{m=1}^{n}\lambda_m$ $(\Lambda_n\geqslant 1)$, $a_n\geqslant 0$ $(n\in\mathbb{N})$ *and* $0<\sum_{n=1}^{\infty}\lambda_n a_n<\infty$. *Then for* $0<p\leqslant 1$,

$$\sum_{n=1}^{\infty}\lambda_{n+1}(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{1/\Lambda_n} < \frac{e^p}{p}\sum_{n=1}^{\infty}\left(\sum_{j=0}^{2m}\frac{(-1)^j b_j}{(\Lambda_n/\lambda_n)^j}\right)^p \lambda_n a_n^p\Lambda_n^{p-1}\left(\sum_{k=1}^{n}\lambda_k(c_k a_k)^p\right)^{(1-p)/p}, \tag{3.1}$$

where b_j *is given by* (2.4), *and*

$$c_n^{\lambda_n}=\frac{(\Lambda_{n+1})^{\Lambda_n}}{(\Lambda_n)^{\Lambda_{n-1}}}.$$

Proof. The following inequality:

$$\sum_{n=1}^{\infty}\lambda_{n+1}(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{1/\Lambda_n} \leqslant \frac{1}{p}\sum_{m=1}^{\infty}\left(1+\frac{1}{\Lambda_m/\lambda_m}\right)^{p\Lambda_m/\lambda_m}\lambda_m a_m^p\Lambda_m^{p-1}\left(\sum_{k=1}^{m}\lambda_k(c_k a_k)^p\right)^{(1-p)/p} \tag{3.2}$$

has been proved in Theorem 2.2 of [11] (see also [21, p. 96]). From (3.2) and the right-hand side of (2.13), we obtain (3.1). The proof is complete.

REMARK 3.1. In Theorem 2.2 of [11], $c_k^{\lambda_n}=\frac{(\Lambda_{n+1})^{\Lambda_n}}{(\Lambda_n)^{\Lambda_{n-1}}}$ should be $c_n^{\lambda_n}=\frac{(\Lambda_{n+1})^{\Lambda_n}}{(\Lambda_n)^{\Lambda_{n-1}}}$; see [11, p. 44, line 3]. Likewise, $c_s^{\lambda_n}=\frac{(\Lambda_{n+1})^{\Lambda_n}}{(\Lambda_n)^{\Lambda_{n-1}}}$ in Theorem 3.1 of [21] should be $c_n^{\lambda_n}=\frac{(\Lambda_{n+1})^{\Lambda_n}}{(\Lambda_n)^{\Lambda_{n-1}}}$; see [21, p. 96, Eq. (9)].

The choice $p = 1$ in (3.1) yields

$$\sum_{n=1}^{\infty} \lambda_{n+1}(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \left(\sum_{j=0}^{2m} \frac{(-1)^j b_j}{(\Lambda_n/\lambda_n)^j} \right) \lambda_n a_n. \tag{3.3}$$

Taking $\lambda_n \equiv 1$ in (3.3) we obtain

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(\sum_{j=0}^{2m} \frac{(-1)^j b_j}{n^j} \right) a_n. \tag{3.4}$$

When $m = 2$ in (3.4) we recover (1.5).

REFERENCES

[1] H. ALZER AND C. BERG, *Some classes of completely monotonic functions*, Ann. Acad. Sci. Fenn. Math. **27** (2002), 445–460,

[2] H. J. BROTHERS AND J. A. KNOX, *New closed-form approximations to the logarithmic constant e*, Math. Intelligencer **20** (1998), 25–29.

[3] T. CARLEMAN, *Sur les fonctions quasi-analytiques, Comptes rendus du V^e Congres des Mathematiciens Scandinaves*, Helsingfors (1922), 181–196.

[4] H.-W. CHEN, *On an infinite series for $(1+1/x)^x$ and its application*, Int. J. Math. Math. Sci. **29** (2002), 675–680.

[5] C.-P. CHEN, *Generalization of weighted Carleman-type inequality*, East J. Approx. **12** (2006), 63–69.

[6] C.-P. CHEN, W.-S. CHEUNG AND F. QI, *Note on weighted Carleman type inequality*, Int. J. Math. Math. Sci. **3** (2005), 475–481.

[7] C.-P. CHEN AND J. CHOI, *Asymptotic formula for $(1+1/x)^x$ based on the partition function*, Amer. Math. Monthly **121** (2014), 338–343.

[8] C.-P. CHEN, N. ELEZOVIĆ AND L. VUKŠIĆ, *Asymptotic formulae associated with the Wallis power function and digamma function*, J. Classical Anal. **2**, 2 (2013), 151–166.

[9] C.-P. CHEN AND F. QI, *Generalization of Hardy's inequality*, Proc. Jangjeon Math. Soc. **7** (2004), 57–61.

[10] A. ČIŽMEŠIJA, J. PEČARIĆ AND L. E. PERSSON, *On strengtened weighted Carleman's inequality*, Bull. Austral. Math. Soc. **68** (2003), 481–490.

[11] S. S. DRAGOMIR AND Y.-H. KIM, *The strengthened Hardy inequalities and their new generalizations*, Fac. Sci. Math. **20**, 2 (2006), 39–49.

[12] G. H. HARDY, *Notes on some points in the integral calculus*, Messenger of Math. **54** (1925), 150–156.

[13] G. H. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, 2nd ed., Cambridge Univ. Press, Cambridge, 1952.

[14] Y. HU, *A strengthened Carleman's inequality*, Commun. Math. Anal. 1 (2006) 115–119.

[15] M. JOHANSSON, L. E. PERSSON AND A. WEDESTIG, *Carleman's inequality–history, proofs and some new generalizations*, J. Inequal. Pure Appl. Math. **4**, 3 (2003), Art. 53. `http://www.emis.de/journals/JIPAM/images/135_02_JIPAM/135_02_www.pdf`.

[16] S. KAIJSER, L. E. PERSSON AND A. ÖBERG, *On Carleman and Knopp's inequalities*, J. Approx. Theory **117** (2002), 140–151.

[17] J. A. KNOX AND H. J. BROTHERS, *Novel series-based approximations to e*, College Math. J. **30** (1999), 269–275.

[18] A. KUFNER AND L. E. PERSSON, *Weighted Inequalities of Hardy Type*, World Scientific, New Jersey/London, Singapore/Hong Kong, 2003.

[19] J.-L. LI, *Notes on an inequality involving the constant e*, J. Math. Anal. Appl. **250** (2000), 722–725.

[20] H.-P. LIU AND L. ZHU, *New strengthened Carleman's inequality and Hardy's inequality*, J. Inequal. Appl. J. Inequal. Appl. **2007**, Art. ID 84104, 7 pp. `http://link.springer.com/article/10.1155/2007/84104/fulltext.html`.

[21] Z. LÜ, Y. GAO AND Y. WEI, *Note on the Carleman's inequality and Hardy's inequality*, Comput. Math. Appl. **59** (2010), 94–97.
[22] C. MORTICI AND Y. HU, *On some convergences to the constant e and improvements of Carleman's inequality*, Carpathian J. Math. **31** (2015), 249–254.
[23] C. MORTICI AND X.-J. JANG, *Estimates of $(1+x)^{1/x}$ Involved in Carleman's Inequality and Keller's Limit*, Filomat **29**, 7 (2015), 1535–1539.
[24] J. PEČARIĆ AND K. B. STOLARSKY, *Carleman's inequality: history and new generalizations*, Aequationes Math. **61** (2001), 49–62.
[25] G. PÓLYA, *Proof of an inequality*, Proc. London Math. Soc. **24** (1926), 57.
[26] Z. XIE AND Y. ZHONG, *A best approximation for constant e and an improvement to Hardy's inequality*, J. Math. Anal. Appl. **252** (2000), 994–998.
[27] P. YAN AND G.-Z. SUN, *A strengthened Carleman's inequality*, J. Math. Anal. Appl. **240** (1999), 290–293.
[28] B.-C. YANG AND L. DEBNATH, *Some inequalities involving the constant e, and an application to Carleman's inequality*, J. Math. Anal. Appl. **223** (1998), 347–353.
[29] B.-C. YANG, *On Hardy's inequality*, J. Math. Anal. Appl. **234** (1999), 717–722.
[30] X.-J. YANG, *On Carleman's inequality*, J. Math. Anal. Appl. **253** (2001), 691–694.
[31] X.-J. YANG, *Approximations for constant e and their applications*, J. Math. Anal. Appl. **262** (2001), 651–659.

(Received November 12, 2016)

Chao-Ping Chen
School of Mathematics and Informatics
Henan Polytechnic University
Jiaozuo City 454000, Henan Province, China
e-mail: chenchaoping@sohu.com

Richard B. Paris
Division of Computing and Mathematics
University of Abertay
Dundee, DD1 1HG, UK
e-mail: R.Paris@abertay.ac.uk