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IMPROVED HARDY INEQUALITIES WITH EXACT REMAINDER TERMS

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Abstract. We set up several identities that imply some versions of the Hardy type inequalities. These equalities give a straightforward understanding of several Hardy type inequalities as well as the nonexistence of nontrivial optimizers. These identities also provide the "virtual" extremizers for many Hardy type inequalities.

1. Introduction

The main subject of this article is the following celebrated Hardy inequality that plays extremely important roles in many areas such as analysis, probability and partial differential equations:

$$\int_{\mathbb{R}^N} |\nabla u|^2\, dx \geqslant \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx. \tag{1.1}$$

As pointed out in [4, 8], we actually have the following identity that characterizes the form of the vanishing remainder terms and provides a simple and direct interpretation of the Hardy inequalities as well as the nonexistence of nontrivial optimizers:

$$\int_{\mathbb{R}^N} |\nabla u|^2\, dx = \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx + \int_{\mathbb{R}^N} \left|\nabla u + \frac{N-2}{2}\frac{u}{|x|}\frac{x}{|x|}\right|^2 dx. \tag{1.2}$$

We note that the second term on the RHS of (1.2) vanishes when $u(x) = c|x|^{-\frac{N-2}{2}}$. However, in the case $\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx$ is infinite unless $c = 0$. Hence we can say that (1.1) has "virtual" optimizer $|x|^{-\frac{N-2}{2}}$.

It is also worth mentioning that the following equality was proved in [28]:

$$\int_{\mathbb{R}^N} \left|\frac{x}{|x|}\cdot \nabla u\right|^2 dx = \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx + \int_{\mathbb{R}^N} \left|\frac{x}{|x|}\cdot\nabla u + \frac{N-2}{2}\frac{u}{|x|}\right|^2 dx. \tag{1.3}$$

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Since $\int\limits_{\mathbb{R}^N} |\nabla u|^2 dx \geqslant \int\limits_{\mathbb{R}^N} \left|\frac{x}{|x|}\cdot \nabla u\right|^2 dx$, (1.3) provides an improved version for the Hardy inequality . Actually, $\mathscr{R} = \frac{x}{|x|}\cdot\nabla$ is just the radial derivative since in the polar coordinate $(r,\sigma) = \left(|x|, \frac{x}{|x|}\right)$, we have $\frac{x}{|x|}\cdot\nabla u = \partial_r(u\sigma)$. We also note here that the operator $\mathscr{R}$ has appeared naturally in the literature. Indeed, a considerable effort has been devoted to investigate the functional and geometric inequalities on general homogeneous groups. However, as mentioned in [33], since these spaces do not have to be stratified or even graded, the concept of horizontal gradients does not make sense. Thus, it is logical to work with the full gradient. On the other hand, unless the homogeneous groups are abelian, the full gradient is not homogeneous. Nevertheless, on the homogeneous groups, the operator $\mathscr{R}$ is homogeneous of order -1 and thus, is reasonable to work with. Actually, the Hardy type inequalities with radial derivative have been studied extensively recently. See [9, 10, 11, 20, 21, 24, 25, 26, 29, 30, 32, 33, 34, 35, 36, 37], to name just a few.

The optimal constant $\left(\frac{N-2}{2}\right)^2$ in the Hardy inequality is never achieved. Hence, one may want to improve the Hardy inequalities by adding extra nonnegative terms to the RHS of (1.1). On the whole space $\mathbb{R}^N$, Ghoussoub and Moradifam proved in [18] that there is no strictly positive $V \in V^1((0,\infty))$ such that the inequality

$$\int\limits_{\mathbb{R}^N} |\nabla u|^2 dx - \left(\frac{N-2}{2}\right)^2 \int\limits_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx \geqslant \int\limits_{\mathbb{R}^N} V(|x|)\,|u|^2 dx$$

holds for all $u \in C_0^\infty(\mathbb{R}^N)$. However, it was showed that extra terms can be added to the Hardy inequality on bounded domains. For instance, let Ω be a bounded domain in $\mathbb{R}^N$, $N \geqslant 3$, with $0 \in \Omega$, then in order to investigate the stability of singular solutions of nonlinear elliptic equations, Brezis and Vázquez verified in [5] that for all $u \in W_0^{1,2}(\Omega)$:

$$\int\limits_{\Omega} |\nabla u|^2 dx - \left(\frac{N-2}{2}\right)^2 \int\limits_{\Omega} \frac{|u|^2}{|x|^2} dx \geqslant z_0^2 \omega_N^{\frac{2}{N}} |\Omega|^{-\frac{2}{N}} \int\limits_{\Omega} |u|^2 dx \tag{1.4}$$

where ω_N is the volume of the unit ball and $z_0 = 2.4048...$ is the first zero of the Bessel function $J_0(z)$. The constant $z_0^2 \omega_N^{\frac{2}{N}} |\Omega|^{-\frac{2}{N}}$ is optimal when Ω is a ball.

The first aim of this paper is to provide another look to (1.4) in the spirit of (1.2) and (1.3). More precisely, motivated by the results in [5] and [28], we will prove that

THEOREM 1. *For $u \in C_0^\infty(B_R)$, we have*

$$\begin{aligned} &\int\limits_{B_R} \left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx - \left(\frac{N-2}{2}\right)^2 \int\limits_{B_R} \frac{|u|^2}{|x|^2} dx \\ &= \frac{z_0^2}{R^2} \int\limits_{B_R} |u|^2 dx + \int\limits_{B_R} \left|\frac{x}{|x|}\cdot\nabla\left(\frac{|x|^{\frac{N-2}{2}}}{J_{0;R}(|x|)} u\right)\right|^2 \left|\frac{J_{0;R}(|x|)}{|x|^{\frac{N-2}{2}}}\right|^2 dx \end{aligned} \tag{1.5}$$

and

$$\int_{B_R} |\nabla u|^2 dx - \left(\frac{N-2}{2}\right)^2 \int_{B_R} \frac{|u|^2}{|x|^2} dx \tag{1.6}$$

$$= \frac{z_0^2}{R^2} \int_{B_R} |u|^2 dx + \int_{B_R} \left| \nabla \left(\frac{|x|^{\frac{N-2}{2}}}{J_{0;R}(|x|)} u \right) \right|^2 \left| \frac{J_{0;R}(|x|)}{|x|^{\frac{N-2}{2}}} \right|^2 dx.$$

Here $J_{0;R}(z) = J_0\left(\frac{z_0}{R} z\right)$. *The second term in the RHS of* (1.5) *vanishes if and only if* u *has the form*

$$u(x) = \frac{J_{0;R}(|x|)}{|x|^{\frac{N-2}{2}}} \phi\left(\frac{x}{|x|}\right)$$

for some functions $\phi : \mathbb{S}^{N-1} \to \mathbb{R}$. *The second term in the RHS of* (1.6) *vanishes if and only if* $u(x) = c \frac{J_{0;R}(|x|)}{|x|^{\frac{N-2}{2}}}$ *for some constant* c. *However, in these cases* $\int_{B_R} \frac{|u|^2}{|x|^2} dx$ *is infinite unless* $u = 0$.

As a consequence, we get

$$\int_{B_R} \left| \frac{x}{|x|} \cdot \nabla u \right|^2 dx \geqslant \left(\frac{N-2}{2}\right)^2 \int_{B_R} \frac{|u|^2}{|x|^2} dx + \frac{z_0^2}{R^2} \int_{B_R} |u|^2 dx \tag{1.7}$$

which implies

$$\int_{B_R} |\nabla u|^2 dx \geqslant \left(\frac{N-2}{2}\right)^2 \int_{B_R} \frac{|u|^2}{|x|^2} dx + \frac{z_0^2}{R^2} \int_{B_R} |u|^2 dx. \tag{1.8}$$

The constants $\left(\frac{N-2}{2}\right)^2$ and $\frac{z_0^2}{R^2}$ are sharp. The equalities in (1.7) and (1.8) are never achieved unless $u = 0$. However, we can say that they have "virtual" optimizers $\frac{J_{0;R}(|x|)}{|x|^{\frac{N-2}{2}}} \phi\left(\frac{x}{|x|}\right)$ and $\frac{J_{0;R}(|x|)}{|x|^{\frac{N-2}{2}}}$ respectively.

Now, since $z_0^2 \omega_N^{\frac{2}{N}} |\Omega|^{-\frac{2}{N}}$ is not attained in $W_0^{1,2}(\Omega)$, it is natural to conjecture that $z_0^2 \omega_N^{\frac{2}{N}} |\Omega|^{-\frac{2}{N}} \int_{\Omega} |u|^2 dx$ is just a first term of an infinite series of extra terms that can be added to the RHS of (1.4). This problem was investigated by many authors. We refer the interested reader to [1, 2, 3, 6, 7, 13, 15, 16, 17, 27], among others. See also the books [22, 23, 31] that are by now standard references on Hardy inequalities. In particular, in [14], the authors provided an infinite series expansion of Hardy's inequality that is in some sense optimal. It is also worth mentioning that in an attempt to improve, extend and unify several results in this direction, Ghoussoub and Moradifam [19] introduced the Hardy improving potentials-abbreviated as HI-potentials, and studied their connections to the Hardy inequalities. One of their results can be read as follows:

THEOREM A. *Let P be a decreasing nonnegative C^1-function on $(0,R)$. The following are equivalent:*

(1) P is a HI-potential on $(0,R)$, that is, the equation $y''(r)+\frac{1}{r}y'(r)+P(r)y(r)=0$ has a positive solution on $(0,R)$.

(2) For any $u \in W_0^{1,2}(B_R)$, there holds

$$\int_{B_R} |\nabla u|^2 dx - \left(\frac{N-2}{2}\right)^2 \int_{B_R} \frac{|u|^2}{|x|^2} dx \geqslant \int_{B_R} P(|x|)|u|^2 dx. \tag{1.9}$$

Our next purpose is to set up an improved version for the above result in the framework of equalities. More precisely, we would like to show that

THEOREM 2. *Assume that P is a HI-potential on $(0,R)$ and $\varphi_{P;R}$ is the positive solution of $y''(r)+\frac{1}{r}y'(r)+P(r)y(r)=0$ on $(0,R)$. For $u \in C_0^\infty(B_R)$, we have*

$$\begin{aligned} &\int_{B_R} \left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx - \left(\frac{N-2}{2}\right)^2 \int_{B_R} \frac{|u|^2}{|x|^2} dx \\ &= \int_{B_R} P(|x|)|u|^2 dx + \int_B \left|\frac{x}{|x|}\cdot\nabla\left(\frac{|x|^{\frac{N-2}{2}} u}{\varphi_{P;R}(|x|)}\right)\right|^2 \left|\frac{\varphi_{P;R}(|x|)}{|x|^{\frac{N-2}{2}}}\right|^2 dx \end{aligned} \tag{1.10}$$

and

$$\begin{aligned} &\int_{B_R} |\nabla u|^2 dx - \left(\frac{N-2}{2}\right)^2 \int_{B_R} \frac{|u|^2}{|x|^2} dx \\ &= \int_{B_R} P(|x|)|u|^2 dx + \int_B \left|\nabla\left(\frac{|x|^{\frac{N-2}{2}} u}{\varphi_{P;R}(|x|)}\right)\right|^2 \left|\frac{\varphi_{P;R}(|x|)}{|x|^{\frac{N-2}{2}}}\right|^2 dx. \end{aligned} \tag{1.11}$$

The second term in the RHS of (1.10) vanishes if and only if u has the form

$$u(x) = \frac{\varphi_{P;R}(|x|)}{|x|^{\frac{N-2}{2}}}\phi\left(\frac{x}{|x|}\right)$$

for some functions $\phi : \mathbb{S}^{N-1} \to \mathbb{R}$. The second term in the RHS of (1.11) vanishes if and only if u has the form

$$u(x) = c\frac{\varphi_{P;R}(|x|)}{|x|^{\frac{N-2}{2}}}.$$

However, in these cases $\int_{B_R} \frac{|u|^2}{|x|^2} dx$ is infinite unless $u=0$.

As a consequence, we have

$$\int_{B_R} \left| \frac{x}{|x|} \cdot \nabla u \right|^2 dx \geqslant \left(\frac{N-2}{2} \right)^2 \int_{B_R} \frac{|u|^2}{|x|^2} dx + \int_{B_R} P(|x|) |u|^2 dx. \tag{1.12}$$

From Theorem 2, we can again say that $\frac{\varphi_{P;R}(|x|)}{|x|^{\frac{N-2}{2}}}$ is the "virtual" optimizer for (1.9) and $\frac{\varphi_{P;R}(|x|)}{|x|^{\frac{N-2}{2}}} \phi \left(\frac{x}{|x|} \right)$ is the "virtual" optimizer for (1.12). Moreover, if there is no $c > 1$ such that cP is a HI-potential on $(0, R)$, then (1.9) and (1.12) are sharp in the sense that there is no $c > 1$ such that

$$\int_{B_R} |\nabla u|^2 dx \geqslant \int_{B_R} \left| \frac{x}{|x|} \cdot \nabla u \right|^2 dx \geqslant \left(\frac{N-2}{2} \right)^2 \int_{B_R} \frac{|u|^2}{|x|^2} dx + c \int_{B_R} P(|x|) |u|^2 dx.$$

The rest of this paper concerns the two-weight Hardy inequalities. There are many efforts to investigate the conditions of nonnegative weights V and W such that the following weighted Hardy type inequalities hold

$$\int V(x) |\nabla u|^2 dx \geqslant \int W(x) |u|^2 dx.$$

The interested reader is referred to, for examples, the books [19, 31]. We state here the result, that could be found in [18], on a necessary and sufficient condition of such a pair:

THEOREM B. *Let $0 < R \leqslant \infty$, V and W be positive C^1-functions on $(0, R)$ such that $\int_0^R \frac{1}{r^{N-1} V(r)} dr = \infty$ and $\int_0^R r^{N-1} V(r) dr < \infty$. Then the following are equivalent:*

(1) (V, cW) is a $N-$dimensional Bessel pair on $(0, R)$ for some $c > 0$.

(2) $\int_B V(|x|) |\nabla u|^2 dx \geqslant c \int_B W(|x|) |u|^2 dx$ for all $u \in C_0^\infty(B)$ for some $c > 0$.

Here we say that a couple of C^1-functions (V, W) is a $N-$dimensional Bessel pair on $(0, R)$ if the ordinary differential equation

$$y''(r) + \left(\frac{N-1}{r} + \frac{V_r(r)}{V(r)} \right) y'(r) + \frac{W(r)}{V(r)} y(r) = 0$$

has a positive solution on the interval $(0, R)$.

Our next goal is to set up the following result about the two-weight Hardy inequality in the spirit of [28]:

THEOREM 3. *Let W be a positive continuous function on $(0, R)$ such that there exists a C^1-function $\widetilde{W}$ on $(0, R)$ and $\frac{d\widetilde{W}}{dr}(r) = W(r) r^{N-1}$. Then for all $u \in C_0^\infty(B_R \setminus \{0\})$,* *we have*

$$\int_{B_R} \frac{4\widetilde{W}^2(|x|)}{W(|x|)|x|^{2N-2}} \left| \frac{x}{|x|} \cdot \nabla u(x) \right|^2 dx - \int_{B_R} W(|x|)|u|^2 dx$$
$$= \int_{B_R} \frac{4\left|\widetilde{W}(|x|)\right|}{W(|x|)|x|^{2N-2}} \left| \frac{x}{|x|} \cdot \nabla \left(u(x) \sqrt{\left|\widetilde{W}(|x|)\right|} \right) \right|^2 dx$$

and

$$\int_{B_R} \frac{4\widetilde{W}^2(|x|)}{W(|x|)|x|^{2N-2}} |\nabla u(x)|^2 dx - \int_{B_R} W(|x|)|u|^2 dx$$
$$= \int_{B_R} \frac{4\left|\widetilde{W}(|x|)\right|}{W(|x|)|x|^{2N-2}} \left| \nabla \left(u(x) \sqrt{\left|\widetilde{W}(|x|)\right|} \right) \right|^2 dx.$$

If $\widetilde{W}(0) = 0$, *then the above identities hold for any* $u \in C_0^\infty(B_R)$.

As a consequence, we get

$$\int_{\mathbb{R}^N} \frac{4\widetilde{W}^2(|x|)}{W(|x|)|x|^{2N-2}} \left| \frac{x}{|x|} \cdot \nabla u(x) \right|^2 dx \geqslant \int_{\mathbb{R}^N} W(|x|)|u|^2 dx$$

and

$$\int_{\mathbb{R}^N} \frac{4\widetilde{W}^2(|x|)}{W(|x|)|x|^{2N-2}} |\nabla u(x)|^2 dx \geqslant \int_{\mathbb{R}^N} W(|x|)|u|^2 dx.$$

We list here some direct applications of Theorem 3. If $W(r) = \frac{1}{r^{2a+2}}$, $a < \frac{N-2}{2}$, then $\widetilde{W}(r) = \int_0^r s^{N-3-2a} ds = \frac{1}{(N-2a-2)} r^{N-2a-2}$ and $\frac{4\widetilde{W}^2(|x|)}{W(|x|)|x|^{2N-2}} = \frac{4}{(N-2a-2)^2} \frac{1}{|x|^{2a}}$. Hence we get the weighted Hardy type inequalities

$$\int_{\mathbb{R}^N} \frac{1}{|x|^{2a}} |\nabla u(x)|^2 dx \geqslant \frac{(N-2a-2)^2}{4} \int_{\mathbb{R}^N} \frac{1}{|x|^{2a+2}} |u|^2 dx$$

with "virtual" optimizer $|x|^{a+1-\frac{N}{2}}$. In the critical case, $W(r) = \frac{1}{r^N}$, then $\widetilde{W}(r) = \ln r$ and $\frac{4\widetilde{W}^2(|x|)}{W(|x|)|x|^{2N-2}} = \frac{4|\ln|x||^2}{|x|^{N-2}}$. Hence we have for $u \in C_0^\infty\left(\mathbb{R}^N \setminus \{0\}\right)$

$$4\int_{\mathbb{R}^N} \frac{|\ln|x||^2}{|x|^{N-2}} \left| \frac{x}{|x|} \cdot \nabla u(x) \right|^2 dx \geqslant \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^N} dx$$

with "virtual" extremizer $\frac{1}{\sqrt{|\ln|x||}}\phi\left(\frac{x}{|x|}\right)$. Indeed, the equality happens in the above inequality if and only if $u(x)=\frac{1}{\sqrt{|\ln|x||}}\phi\left(\frac{x}{|x|}\right)$. However

$$\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^N}dx=\int_{\mathbb{S}^{N-1}}\phi^2(\sigma)\,d\sigma\int_0^\infty\frac{1}{r\ln|r|}dr$$

which is infinite unless $u=0$. Another example is when $W(r)=\frac{1}{r^N\left(\ln\frac{1}{r}\right)^2}$, $R=1$. In this case $\widetilde{W}(r)=\frac{1}{\ln\frac{1}{r}}$ and $\frac{4\widetilde{W}^2(|x|)}{W(|x|)|x|^{2N-2}}=\frac{4}{\left(\ln\frac{1}{|x|}\right)^2}\frac{|x|^N\left(\ln\frac{1}{|x|}\right)^2}{|x|^{2N-2}}=\frac{4}{|x|^{N-2}}$. Hence, we have the following critical Hardy inequality:

$$\int_{B_1}\frac{1}{|x|^{N-2}}\left|\frac{x}{|x|}\cdot\nabla u(x)\right|^2dx\geqslant\frac{1}{4}\int_{B_1}\frac{|u|^2}{|x|^N\left(\ln\frac{1}{|x|}\right)^2}dx$$

for $u\in C_0^\infty(B_1)$. The "virtual" optimizer of this critical Hardy inequality is $\sqrt{\ln\frac{1}{|x|}}\phi\left(\frac{x}{|x|}\right)$. When $N=2$, we get

$$\int_{B_1}\left|\frac{x}{|x|}\cdot\nabla u(x)\right|^2dx\geqslant\frac{1}{4}\int_{B_1}\frac{|u|^2}{|x|^2\left(\ln\frac{1}{|x|}\right)^2}dx$$

that implies

$$\int_{B_1}\left|\frac{x}{|x|}\cdot\nabla u(x)\right|^2dx\geqslant\frac{1}{4}\int_{B_1}\frac{|u|^2}{|x|^2\left(1+\ln\frac{1}{|x|}\right)^2}dx.$$

It was pointed out in [12] that the latter is equivalent to the critical case of the Sobolev-Lorentz inequality.

Our last aim in this paper is to set up an improved version of Theorem B in the setting of equalities. More precisely, we will show that

THEOREM 4. *Let $0<R\leqslant\infty$, V and W be positive radial C^1-functions on $B_R\setminus\{0\}$ such that $\int_0^R\frac{1}{r^{N-1}V(r)}dr=\infty$ and $\int_0^R r^{N-1}V(r)\,dr<\infty$. Assume that (V,W) is a $N-$dimensional Bessel pair on $(0,R)$. Then for all $u\in C_0^\infty(B_R)$:*

$$\int_{B_R}V(|x|)\left|\frac{x}{|x|}\cdot\nabla u\right|^2dx-\int_{B_R}W(|x|)|u|^2dx=\int_{B_R}V(|x|)\left|\frac{x}{|x|}\cdot\nabla\left(\frac{u}{\varphi_{V,W;R}}\right)\right|^2\varphi_{V,W;R}^2dx$$

and

$$\int_{B_R}V(|x|)|\nabla u|^2dx-\int_{B_R}W(|x|)|u|^2dx=\int_{B_R}V(|x|)\left|\nabla\left(\frac{u}{\varphi_{V,W;R}}\right)\right|^2\varphi_{V,W;R}^2dx$$

where $\varphi_{V,W;R}$ *is the positive solution of*

$$y''(r)+\left(\frac{N-1}{r}+\frac{V_r(r)}{V(r)}\right)y'(r)+\frac{W(r)}{V(r)}y(r)=0$$

on the interval $(0,R)$.

As a consequence

$$\int_{B_R} V(|x|)\left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx \geqslant \int_{B_R} W(|x|)|u|^2 dx$$

and

$$\int_{B_R} V(|x|)|\nabla u|^2 dx \geqslant \int_{B_R} W(|x|)|u|^2 dx.$$

Moreover, if there is no $c>1$ such that (V,cW) is a $N-$dimensional Bessel pair on $(0,R)$, then the above inequalities are optimal in the sense that there is no $c>1$ such that

$$\int_{B_R} V(|x|)|\nabla u|^2 dx \geqslant \int_{B_R} V(|x|)\left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx \geqslant c\int_{B_R} W(|x|)|u|^2 dx.$$

By applying Theorem 4 to some explicit Bessel pairs, we get the following Hardy type inequalities:

EXAMPLE 1. $(V,W)=\left(r^{-\lambda},\frac{(N-\lambda-2)^2}{4}r^{-\lambda-2}\right)$, $0\leqslant\lambda\leqslant N-2$, is a N-dimensional Bessel pair on $(0,\infty)$ with $\varphi_{V,W;\infty}(r)=r^{-\frac{N-\lambda-2}{2}}$. Hence, we have

$$\int_{\mathbb{R}^N}\frac{1}{|x|^{\lambda}}\left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx-\frac{(N-\lambda-2)^2}{4}\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{\lambda+2}}dx$$
$$=\int_{\mathbb{R}^N}\left|\frac{x}{|x|}\cdot\nabla\left(|x|^{\frac{N-\lambda-2}{2}}u\right)\right|^2\left|\frac{1}{|x|^{\frac{N-\lambda-2}{2}}}\right|^2 dx$$

and

$$\int_{\mathbb{R}^N}\frac{1}{|x|^{\lambda}}|\nabla u|^2 dx-\frac{(N-\lambda-2)^2}{4}\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{\lambda+2}}dx=\int_{\mathbb{R}^N}\left|\nabla\left(|x|^{\frac{N-\lambda-2}{2}}u\right)\right|^2\left|\frac{1}{|x|^{\frac{N-\lambda-2}{2}}}\right|^2 dx.$$

EXAMPLE 2. For any $R>0$, $(V,W)=\left(r^{-\lambda},\frac{(N-\lambda-2)^2}{4}r^{-\lambda-2}+\frac{z_0^2}{R^2}r^{-\lambda}\right)$, $0\leqslant\lambda\leqslant N-2$, is a N-dimensional Bessel pair on $(0,R)$ with $\varphi_{V,W;R}(r)=r^{-\frac{N-\lambda-2}{2}}J_0\left(\frac{rz_0}{R}\right)=$

$r^{-\frac{N-\lambda-2}{2}}J_{0;R}(r)$. Here $z_0 = 2.4048...$ is the first zero of the Bessel function $J_0(z)$. Then, for $u \in C_0^\infty(B_R)$, we have

$$\int_{B_R} \frac{1}{|x|^\lambda} \left| \frac{x}{|x|} \cdot \nabla u \right|^2 dx - \frac{(N-\lambda-2)^2}{4} \int_{B_R} \frac{|u|^2}{|x|^{\lambda+2}} dx$$
$$= \frac{z_0^2}{R^2} \int_{B_R} \frac{|u|^2}{|x|^\lambda} dx + \int_{B_R} \left| \frac{x}{|x|} \cdot \nabla \left(\frac{|x|^{\frac{N-\lambda-2}{2}}}{J_{0;R}(|x|)} u \right) \right|^2 \left| \frac{J_{0;R}(|x|)}{|x|^{\frac{N-\lambda-2}{2}}} \right|^2 dx$$

and

$$\int_{B_R} \frac{1}{|x|^\lambda} |\nabla u|^2 dx - \frac{(N-\lambda-2)^2}{4} \int_{B_R} \frac{|u|^2}{|x|^{\lambda+2}} dx$$
$$= \frac{z_0^2}{R^2} \int_{B_R} \frac{|u|^2}{|x|^\lambda} dx + \int_{B_R} \left| \nabla \left(\frac{|x|^{\frac{N-\lambda-2}{2}}}{J_{0;R}(|x|)} u \right) \right|^2 \left| \frac{J_{0;R}(|x|)}{|x|^{\frac{N-\lambda-2}{2}}} \right|^2 dx.$$

2. Some useful lemmata

We list here some important results that will be used to treat the integrals by parts in the following sections. The proofs of these results can be found in [19].

LEMMA 1. *Assume P is nonnegative on $(0,R)$, $a \geqslant 1$ and that the equation $y'' + \frac{a}{r}y' + P(r)y = 0$ has a positive solution φ on $(0,R)$. Then φ is decreasing on $(0,R)$ and has the following limiting behavior on the boundary*

$$\lim_{r\to 0} r\frac{\varphi'(r)}{\varphi(r)} = 0 \text{ and } \limsup_{r\to R} \frac{\varphi'(r)}{\varphi(r)} \leqslant 0.$$

LEMMA 2. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$ containing 0. Set $R = \sup_{x\in\partial\Omega}|x|$ and assume that $\varphi \in C^1(0,R)$ is a positive solution of the ODE*

$$y''(r) + \left(\frac{N-1}{r} + \frac{V_r(r)}{V(r)} \right) y'(r) + \frac{W(r)}{V(r)} y(r) = 0$$

on $(0,R)$ where V, $W \geqslant 0$ on $(0,R)$ such that $\int_0^R \frac{1}{r^{N-1}V(r)} dr = \infty$ and $\int_0^R r^{N-1}V(r)\,dr < \infty$. Setting $\psi(x) = \frac{u(x)}{\varphi(|x|)}$ for any $u \in C_0^\infty(\Omega)$, we then have the following properties:

(1) $\int_0^R V(r) \left(\frac{\varphi'(r)}{\varphi(r)} \right)^2 r^{N-1} dr < \infty$ *and* $\lim_{r\to 0} V(r) \frac{\varphi'(r)}{\varphi(r)} r^{N-1} = 0$.

(2) $\int_\Omega V(|x|)\,\varphi'(|x|)^2 \psi^2(x)\,dx < \infty$ *and* $\int_\Omega V(|x|)\,\varphi(|x|)^2 |\nabla\psi|^2(x)\,dx < \infty$.

(3) $\left|\int\limits_{\Omega} V(|x|)\varphi'(|x|)\varphi(|x|)\psi(x)\left(\frac{x}{|x|}\cdot\nabla\psi(x)\right)dx\right| < \infty.$

(4) $\lim_{r\to 0}\left|\int\limits_{\partial B_r} V(|x|)\varphi'(|x|)\varphi(|x|)\psi^2(x)\,ds\right| = 0.$

Now, let $u \in C_0^\infty(\Omega)$. We then extend u as zero outside Ω and may consider that $u \in C_0^\infty(\mathbb{R}^N)$. Hence, we can decompose u into spherical harmonics as follows:

$$u = \sum_{k=0}^{\infty} u_k = \sum_{k=0}^{\infty} f_k(r)\phi_k(\sigma),$$

where $\phi_k(\sigma)$ are the orthonormal eigenfunctions of the Laplace-Beltrami operator with corresponding eigenvalues $c_k = k(N+k-2)$, $k \geqslant 0$. We note that the corresponding components f_k are in $C_0^\infty(\Omega)$ and satisfy $f_k(r) = O\left(r^k\right)$, $f_k'(r) = O\left(r^{k-1}\right)$ as $r \downarrow 0$. In particular

$$\phi_0(\sigma) = 1,\ c_0 = 0 \text{ and } f_0(r) = \frac{1}{|\partial B_r|}\int\limits_{\partial B_r} u\,ds.$$

Also, for any $k \in \mathbb{N}$:

$$\Delta u_k = \left(\Delta f_k(r) - c_k\frac{f_k(r)}{r^2}\right)\phi_k(\sigma).$$

LEMMA 3. *Assume that the decomposition of u into the spherical harmonics is $u = \sum_{k=0}^{\infty} u_k = \sum_{k=0}^{\infty} f_k(r)\phi_k(\sigma)$ and assume that V is a positive radial C^1-function on $\mathbb{R}^N \setminus \{0\}$. Then we have:*

$$\int\limits_{\mathbb{R}^N} V(|x|)|u|^2\,dx = \sum_{k=0}^{\infty}\int\limits_{\mathbb{R}^N} V(|x|)|f_k(|x|)|^2\,dx.$$

$$\int\limits_{\mathbb{R}^N} V(|x|)\left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx = \sum_{k=0}^{\infty}\int\limits_{\mathbb{R}^N} V(|x|)|\nabla f_k(|x|)|^2\,dx.$$

$$\int\limits_{\mathbb{R}^N} V(|x|)|\nabla u|^2\,dx = \sum_{k=0}^{\infty}\int\limits_{\mathbb{R}^N} V(|x|)|\nabla f_k(|x|)|^2 + c_k V(|x|)\frac{|f_k(|x|)|^2}{|x|^2}dx.$$

Proof. By polar coordinate and direct computations, we have

$$\int\limits_{\mathbb{R}^N} V(|x|)|u|^2\,dx = \int\limits_0^{\infty}\int\limits_{\mathbb{S}^{N-1}} V(r)\left|\sum_{k=0}^{\infty} f_k(r)\phi_k(\sigma)\right|^2 r^{N-1}\,dr\,d\sigma = \sum_{k=0}^{\infty}\int\limits_{\mathbb{R}^N} V(|x|)|f_k(|x|)|^2\,dx.$$

Also

$$\int_{\mathbb{R}^N} V(|x|)\left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx = \int_0^\infty\int_{\mathbb{S}^{N-1}} V(r)\left|\sum_{k=0}^\infty \frac{\partial u_k}{\partial r}\right|^2 r^{N-1}drd\sigma$$

$$= \int_0^\infty\int_{\mathbb{S}^{N-1}} V(r)\left|\sum_{k=0}^\infty f_k'(r)\,\phi_k(\sigma)\right|^2 r^{N-1}drd\sigma$$

$$= \sum_{k=0}^\infty \left|\mathbb{S}^{N-1}\right|\int_0^\infty V(r)\left|f_k'(r)\right|^2 r^{N-1}dr$$

$$= \sum_{k=0}^\infty \int_{\mathbb{R}^N} V(|x|)\left|\nabla f_k(|x|)\right|^2.$$

Next, we note that

$$\int_{\mathbb{R}^N} uV(|x|)\,\Delta u dx$$

$$=\int_0^\infty\int_{\mathbb{S}^{N-1}} V(r)\sum_{k=0}^\infty f_k(r)\,\phi_k(\sigma)\sum_{k=0}^\infty\left(f_k''(r)+\frac{N-1}{r}f_k'(r)-c_k\frac{f_k(r)}{r^2}\right)\phi_k(\sigma)\,r^{N-1}drd\sigma$$

$$=\sum_{k=0}^\infty\left|\mathbb{S}^{N-1}\right|\int_0^\infty V(r)\,f_k(r)\left(f_k''(r)+\frac{N-1}{r}f_k'(r)-c_k\frac{f_k(r)}{r^2}\right)r^{N-1}drd\sigma$$

and

$$\int_{\mathbb{R}^N} u\nabla V(|x|)\cdot\nabla u dx$$

$$=\int_0^\infty\int_{\mathbb{S}^{N-1}}\left(\sum_{k=0}^\infty f_k(r)\,\phi_k(\sigma)\right)\frac{V'(r)}{r}\left[(r,\sigma)\cdot\sum_{k=0}^\infty\left(f_k'(r)\,\phi_k(\sigma)\,,\frac{f_k(r)}{r}\nabla_{\mathbb{S}^{N-1}}\phi_k(\sigma)\right)\right]r^{N-1}drd\sigma$$

$$=\sum_{k=0}^\infty\left|\mathbb{S}^{N-1}\right|\int_0^\infty V'(r)\,f_k(r)\,f_k'(r)\,r^{N-1}drd\sigma.$$

Hence

$$\int_{\mathbb{R}^N} V(|x|)\left|\nabla u\right|^2 dx=-\int_{\mathbb{R}^N} u\nabla\cdot(V(|x|)\,\nabla u)\,dx=-\int_{\mathbb{R}^N} uV(|x|)\,\Delta u dx-\int_{\mathbb{R}^N} u\nabla V(|x|)\cdot\nabla u dx$$

$$=-\sum_{k=0}^\infty\left|\mathbb{S}^{N-1}\right|\int_0^\infty\left[V(r)\,f_k(r)\left(f_k''(r)+\frac{N-1}{r}f_k'(r)-c_k\frac{f_k(r)}{r^2}\right)+V'(r)\,f_k(r)\,f_k'(r)\right]r^{N-1}dr$$

$$=\sum_{k=0}^\infty\int_{\mathbb{R}^N} V(|x|)\left|\nabla f_k(|x|)\right|^2+c_kV(|x|)\,\frac{\left|f_k(|x|)\right|^2}{|x|^2}dx.$$

3. Proof of Theorem 1

Proof of Theorem 1. By [28], we have

$$\int_{B_R} \left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx - \left(\frac{N-2}{2}\right)^2 \int_{B_R} \frac{|u|^2}{|x|^2} dx = \int_{B_R} |x|^{2-N} \left|\frac{x}{|x|}\cdot\nabla\left(|x|^{\frac{N-2}{2}} u\right)\right|^2 dx$$
$$= \int_{B_R} |x|^{2-N} \left|\frac{x}{|x|}\cdot\nabla v\right|^2 dx$$

with $v = |x|^{\frac{N-2}{2}} u$.

Let $\psi(x) = \frac{v(x)}{J_{0;R}(x)}$. We note that since J_0 is the Bessel function: $r^2 J_0''(r) + rJ_0'(r) + r^2 J_0 = 0$ on $(0, z_0)$, $J_{0;R}$ solves $r^2 J_{0;R}''(r) + rJ_{0;R}(r) + \frac{z_0^2}{R^2} r^2 J_{0;R} = 0$ on $(0, R)$. Then

$$\int_{B_R} |x|^{2-N} \left|\frac{x}{|x|}\cdot\nabla v\right|^2 dx = \int_{\mathbb{S}^{N-1}} \int_0^R |\partial_r v(r\sigma)|^2 r dr d\sigma$$
$$= \int_{\mathbb{S}^{N-1}} \int_0^R |\psi|^2 \left|J_{0;R}'\right|^2 r dr d\sigma + \int_{\mathbb{S}^{N-1}} \int_0^R |\partial_r \psi(r\sigma)|^2 |J_{0;R}|^2 r dr d\sigma + 2\int_{\mathbb{S}^{N-1}} \int_0^R \psi \partial_r \psi J_{0;R} J_{0;R}' r dr d\sigma.$$

Using integration by parts and Lemma 1, we get

$$2\int_{\mathbb{S}^{N-1}} \int_0^R \psi \partial_r \psi J_{0;R} J_{0;R}' r dr d\sigma = -\int_{\mathbb{S}^{N-1}} \int_0^R |\psi|^2 \left(J_{0;R} J_{0;R}' r\right)' dr d\sigma$$
$$= -\int_{\mathbb{S}^{N-1}} \int_0^R |\psi|^2 \left|J_{0;R}'\right|^2 r dr d\sigma - \int_{\mathbb{S}^{N-1}} \int_0^R |\psi|^2 \left(J_{0;R} J_{0;R}' + J_{0;R} J_{0;R}'' r\right) dr d\sigma.$$

Hence

$$\int_{B_R} |x|^{2-N} \left|\frac{x}{|x|}\cdot\nabla v\right|^2 dx$$
$$= \int_{\mathbb{S}^{N-1}} \int_0^R |\partial_r \psi|^2 |J_{0;R}|^2 r dr d\sigma - \int_{\mathbb{S}^{N-1}} \int_0^R |v|^2 \left(\frac{J_{0;R}' + J_{0;R}'' r}{J_{0;R}}\right) dr d\sigma$$
$$= \int_{\mathbb{S}^{N-1}} \int_0^R |\partial_r \psi|^2 |J_{0;R}|^2 r dr d\sigma + \frac{z_0^2}{R^2} \int_{\mathbb{S}^{N-1}} \int_0^R |v|^2 r dr d\sigma.$$

Noting that

$$\int_{\mathbb{S}^{N-1}} \int_0^R |v|^2 r dr d\sigma = \int_{B_R} |u|^2 dx,$$

we obtain

$$\begin{aligned}
&\int_{B_R}\left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx-\left(\frac{N-2}{2}\right)^2\int_{B_R}\frac{|u|^2}{|x|^2}dx\\
=&\frac{z_0^2}{R^2}\int_{B_R}|u|^2\,dx+\int_{\mathbb{S}^{N-1}}\int_0^R|\partial_r\psi|^2\,|J_{0;R}|^2\,\frac{1}{r^{N-2}}r^{N-1}drd\sigma\\
=&\frac{z_0^2}{R^2}\int_{B_R}|u|^2\,dx+\int_{B_R}\left|\frac{x}{|x|}\cdot\nabla\left(\frac{|x|^{\frac{N-2}{2}}u}{J_{0;R}(|x|)}\right)\right|^2\left|\frac{J_{0;R}(|x|)}{|x|^{\frac{N-2}{2}}}\right|^2 dx.
\end{aligned}$$

We now decompose u into spherical harmonics: $u=\sum_{k=0}^{\infty}u_k=\sum_{k=0}^{\infty}f_k(r)\,\phi_k(\sigma)$. By Lemma 3, we have that

$$\begin{aligned}
&\int_{B_R}|\nabla u|^2\,dx-\left(\frac{N-2}{2}\right)^2\int_{B_R}\frac{|u|^2}{|x|^2}dx\\
=&\int_{B_R}\left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx-\left(\frac{N-2}{2}\right)^2\int_{B_R}\frac{|u|^2}{|x|^2}dx+\sum_{k=0}^{\infty}c_k\int_{B_R}\frac{|f_k(|x|)|^2}{|x|^2}dx\\
=&\frac{z_0^2}{R^2}\int_{B_R}|u|^2\,dx+\int_{B_R}\left|\frac{x}{|x|}\cdot\nabla\left(\frac{|x|^{\frac{N-2}{2}}u}{J_{0;R}(|x|)}\right)\right|^2\left|\frac{J_{0;R}(|x|)}{|x|^{\frac{N-2}{2}}}\right|^2 dx+\sum_{k=0}^{\infty}c_k\int_{B_R}\frac{|f_k(|x|)|^2}{|x|^2}dx.
\end{aligned}$$

Noting that $\frac{|x|^{\frac{N-2}{2}}u}{J_{0;R}(|x|)}=\sum_{k=0}^{\infty}\frac{r^{\frac{N-2}{2}}}{J_{0;R}(r)}f_k(r)\,\phi_k(\sigma)$ and using Lemma 3, we have

$$\begin{aligned}
&\int_{B_R}\left|\nabla\left(\frac{|x|^{\frac{N-2}{2}}u}{J_{0;R}(|x|)}\right)\right|^2\left|\frac{J_{0;R}(|x|)}{|x|^{\frac{N-2}{2}}}\right|^2 dx\\
=&\int_{B_R}\left|\frac{x}{|x|}\cdot\nabla\left(\frac{|x|^{\frac{N-2}{2}}u}{J_{0;R}(|x|)}\right)\right|^2\left|\frac{J_{0;R}(|x|)}{|x|^{\frac{N-2}{2}}}\right|^2 dx+\sum_{k=0}^{\infty}c_k\int_{B_R}\left|\frac{J_{0;R}(|x|)}{|x|^{\frac{N-2}{2}}}\right|^2\frac{\left|\frac{|x|^{\frac{N-2}{2}}f_k(|x|)}{J_{0;R}(|x|)}\right|^2}{|x|^2}dx\\
=&\int_{B_R}\left|\frac{x}{|x|}\cdot\nabla\left(\frac{|x|^{\frac{N-2}{2}}u}{J_{0;R}(|x|)}\right)\right|^2\left|\frac{J_{0;R}(|x|)}{|x|^{\frac{N-2}{2}}}\right|^2 dx+\sum_{k=0}^{\infty}c_k\int_{B_R}\frac{|f_k(|x|)|^2}{|x|^2}dx.
\end{aligned}$$

Hence

$$\int_{B_R}|\nabla u|^2\,dx-\left(\frac{N-2}{2}\right)^2\int_{B_R}\frac{|u|^2}{|x|^2}dx$$

$$= \frac{z_0^2}{R^2} \int_{B_R} |u|^2\, dx + \int_{B_R} \left| \nabla \left(\frac{|x|^{\frac{N-2}{2}} u}{J_{0;R}(|x|)} \right) \right|^2 \left| \frac{J_{0;R}(|x|)}{|x|^{\frac{N-2}{2}}} \right|^2 dx.$$

Now, if $u = \frac{J_{0;R}(|x|)}{|x|^{\frac{N-2}{2}}}$, then

$$\int_{B_R} \frac{|u|^2}{|x|^2} dx = \left|\mathbb{S}^{N-1}\right| \int_0^R \frac{|J_{0;R}(r)|^2}{r} dr.$$

Noting that $J_{0;R}(r)$ is a positive decreasing function on $(0,R)$, we have for some $\varepsilon \in (0,R)$ that

$$\int_0^R \frac{|J_{0;R}(r)|^2}{r} dr \geqslant \int_0^{\varepsilon} \frac{|J_{0;R}(\varepsilon)|^2}{r} dr = \infty.$$

4. Proof of Theorem 2

Proof of Theorem 2. Let $\varphi_{P;R}$ be the positive solution of $y''(r) + \frac{1}{r} y'(r) + P(r) y(r) = 0$ on $(0,R)$. We recall that

$$\int_{B_R} \left| \frac{x}{|x|} \cdot \nabla u \right|^2 dx - \left(\frac{N-2}{2} \right)^2 \int_{B_R} \frac{|u|^2}{|x|^2} dx = \int_B |x|^{2-N} \left| \frac{x}{|x|} \cdot \nabla v \right|^2 dx$$

where $v = |x|^{\frac{N-2}{2}} u$. Letting $\psi(x) = \frac{v(x)}{\varphi_{P;R}(|x|)}$ and using the polar coordinate, we have

$$\int_B |x|^{2-N} \left| \frac{x}{|x|} \cdot \nabla v \right|^2 dx = \int_{\mathbb{S}^{N-1}} \int_0^1 |\partial_r v(r\sigma)|^2\, r dr d\sigma$$

$$= \int_{\mathbb{S}^{N-1}} \int_0^1 |\psi|^2 \left|\varphi'_{P;R}\right|^2 r dr d\sigma + \int_{\mathbb{S}^{N-1}} \int_0^1 |\partial_r \psi|^2 |\varphi_{P;R}|^2 r dr d\sigma + 2 \int_{\mathbb{S}^{N-1}} \int_0^1 \psi \psi' \varphi_{P;R} \varphi'_{P;R} r dr d\sigma$$

$$= \int_{\mathbb{S}^{N-1}} \int_0^1 |\psi|^2 \left|\varphi'_{P;R}\right|^2 r dr d\sigma + \int_{\mathbb{S}^{N-1}} \int_0^1 |\partial_r \psi|^2 |\varphi_{P;R}|^2 r dr d\sigma - \int_{\mathbb{S}^{N-1}} \int_0^1 |\psi|^2 \left(\varphi_{P;R} \varphi'_{P;R} r \right)' dr d\sigma.$$

Using Lemma 1 to treat the integration by parts, we obtain

$$2 \int_{\mathbb{S}^{N-1}} \int_0^1 \psi \psi' \varphi_{P;R} \varphi'_{P;R} r dr d\sigma = - \int_{\mathbb{S}^{N-1}} \int_0^1 |\psi|^2 \left(\varphi_{P;R} \varphi'_{P;R} r \right)' dr d\sigma.$$

Hence, we can deduce that

$$
\begin{aligned}
&\int_B |x|^{2-N}\left|\frac{x}{|x|}\cdot\nabla v\right|^2 dx\\
&=\int_{\mathbb{S}^{N-1}}\int_0^1 |\partial_r\psi|^2\,|\varphi_{P;R}|^2\,r dr d\sigma-\int_{\mathbb{S}^{N-1}}\int_0^1 |\psi|^2\left(\varphi_{P;R}\varphi'_{P;R}+\varphi_{P;R}\varphi''_{P;R}r\right)dr d\sigma\\
&=\int_{\mathbb{S}^{N-1}}\int_0^1 |\partial_r\psi|^2\,|\varphi_{P;R}|^2\,r dr d\sigma-\int_{\mathbb{S}^{N-1}}\int_0^1 |v|^2\left(\frac{\varphi'_{P;R}+\varphi''_{P;R}r}{\varphi_{P;R}}\right)dr d\sigma\\
&=\int_{\mathbb{S}^{N-1}}\int_0^1 |\partial_r\psi|^2\,|\varphi_{P;R}|^2\,r dr d\sigma+\int_{\mathbb{S}^{N-1}}\int_0^1 r\,|v|^2\,P dr d\sigma\\
&=\int_{B_R} P(|x|)\,|u|^2 dx+\int_B\left|\frac{x}{|x|}\cdot\nabla\left(\frac{|x|^{\frac{N-2}{2}}u}{\varphi_{P;R}(|x|)}\right)\right|^2\left|\frac{\varphi_{P;R}(|x|)}{|x|^{\frac{N-2}{2}}}\right|^2 dx.
\end{aligned}
$$

We now decompose u into spherical harmonics: $u=\sum_{k=0}^{\infty}u_k=\sum_{k=0}^{\infty}f_k(r)\,\phi_k(\sigma)$. By Lemma 3, we have that

$$
\begin{aligned}
&\int_{B_R}|\nabla u|^2 dx-\left(\frac{N-2}{2}\right)^2\int_{B_R}\frac{|u|^2}{|x|^2}dx\\
&=\int_{B_R}\left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx-\left(\frac{N-2}{2}\right)^2\int_{B_R}\frac{|u|^2}{|x|^2}dx+\sum_{k=0}^{\infty}c_k\int_{B_R}\frac{|f_k(|x|)|^2}{|x|^2}dx\\
&=\int_{B_R}P(|x|)\,|u|^2 dx+\int_{B_R}\left|\frac{x}{|x|}\cdot\nabla\left(\frac{|x|^{\frac{N-2}{2}}u}{\varphi_{P;R}(|x|)}\right)\right|^2\left|\frac{\varphi_{P;R}(|x|)}{|x|^{\frac{N-2}{2}}}\right|^2 dx+\sum_{k=0}^{\infty}c_k\int_{B_R}\frac{|f_k(|x|)|^2}{|x|^2}dx.
\end{aligned}
$$

Noting that $\frac{|x|^{\frac{N-2}{2}}u}{\varphi_{P;R}(|x|)}=\sum_{k=0}^{\infty}\frac{r^{\frac{N-2}{2}}}{\varphi_{P;R}(r)}f_k(r)\,\phi_k(\sigma)$ and using Lemma 3, we have

$$
\begin{aligned}
&\int_{B_R}\left|\nabla\left(\frac{|x|^{\frac{N-2}{2}}u}{\varphi_{P;R}(|x|)}\right)\right|^2\left|\frac{\varphi_{P;R}(|x|)}{|x|^{\frac{N-2}{2}}}\right|^2 dx\\
&=\int_{B_R}\left|\frac{x}{|x|}\cdot\nabla\left(\frac{|x|^{\frac{N-2}{2}}u}{\varphi_{P;R}(|x|)}\right)\right|^2\left|\frac{\varphi_{P;R}(|x|)}{|x|^{\frac{N-2}{2}}}\right|^2 dx+\sum_{k=0}^{\infty}c_k\int_{B_R}\left|\frac{\varphi_{P;R}(|x|)}{|x|^{\frac{N-2}{2}}}\right|^2\frac{\left|\frac{|x|^{\frac{N-2}{2}}f_k(|x|)}{\varphi_{P;R}(|x|)}\right|^2}{|x|^2}dx\\
&=\int_{B_R}\left|\frac{x}{|x|}\cdot\nabla\left(\frac{|x|^{\frac{N-2}{2}}u}{\varphi_{P;R}(|x|)}\right)\right|^2\left|\frac{\varphi_{P;R}(|x|)}{|x|^{\frac{N-2}{2}}}\right|^2 dx+\sum_{k=0}^{\infty}c_k\int_{B_R}\frac{|f_k(|x|)|^2}{|x|^2}dx.
\end{aligned}
$$

Hence

$$\begin{aligned}
&\int_{B_R} |\nabla u|^2\,dx - \left(\frac{N-2}{2}\right)^2 \int_{B_R} \frac{|u|^2}{|x|^2}dx \\
&= \int_{B_R} P(|x|)\,|u|^2\,dx + \int_{B_R} \left|\nabla\left(\frac{|x|^{\frac{N-2}{2}}u}{\varphi_{P;R}(|x|)}\right)\right|^2 \left|\frac{\varphi_{P;R}(|x|)}{|x|^{\frac{N-2}{2}}}\right|^2 dx.
\end{aligned}$$

Now, if $u = \frac{\varphi_{P;R}(|x|)}{|x|^{\frac{N-2}{2}}}$, then we have

$$\int_{B_R} \frac{|u|^2}{|x|^2}dx = \left|\mathbb{S}^{N-1}\right| \int_0^R \frac{|\varphi_{P;R}(r)|^2}{r}dr.$$

By Lemma 1, $\varphi_{P;R}$ is a positive decreasing function on $(0,R)$. Hence we have for some $\varepsilon \in (0,R)$ that

$$\int_0^R \frac{|\varphi_{P;R}(r)|^2}{r}dr \geqslant \int_0^{\varepsilon} \frac{|\varphi_{P;R}(\varepsilon)|^2}{r}dr = \infty.$$

5. Proof of Theorem 3

Proof of Theorem 3. By using the polar coordinates $(r,\sigma) = \left(|x|, \frac{x}{|x|}\right) \in (0,R) \times \mathbb{S}^{N-1}$ and integrations by parts, we have

$$\begin{aligned}
\int_{B_R} W(|x|)\,|u|^2\,dx &= \int_0^R W(r)\,r^{N-1} \int_{\mathbb{S}^{N-1}} |u(r\sigma)|^2\,d\sigma dr \\
&= -\int_0^R \widetilde{W}(r)\,2 \int_{\mathbb{S}^{N-1}} u(r\sigma)\,\sigma\cdot\nabla u(r\sigma)\,d\sigma dr \\
&= -2\int_0^R \frac{\widetilde{W}(r)}{\sqrt{W(r)}r^{N-1}} \int_{\mathbb{S}^{N-1}} \sqrt{W(r)}u(r\sigma)\,\sigma\cdot\nabla u(r\sigma)\,r^{N-1}d\sigma dr \\
&= -2\int_{B_R} \sqrt{W(|x|)}u(x)\,\frac{\widetilde{W}(|x|)}{\sqrt{W(|x|)}\,|x|^{N-1}}\left(\frac{x}{|x|}\cdot\nabla u(x)\right)dx.
\end{aligned}$$

We note here that when treating the integration by parts, the boundary term will vanish as long as $u \in C_0^\infty(B_R\setminus\{0\})$ or $\lim_{r\downarrow 0}\widetilde{W}(r) = 0$.

Then, we get

$$2\int_{B_R} W(|x|)\,|u|^2\,dx = -4\int_{B_R} \sqrt{W(|x|)}u(x)\,\frac{\widetilde{W}(|x|)}{\sqrt{W(|x|)}\,|x|^{N-1}}\left(\frac{x}{|x|}\cdot\nabla u(x)\right)dx$$

and so

$$\begin{aligned}
&\int_{B_R} W(|x|)\,|u|^2\,dx \\
&= -\int_{B_R} W(|x|)\,|u|^2\,dx - 2\int_{B_R} \sqrt{W(|x|)}\,u(x)\,2\frac{\widetilde{W}(|x|)}{\sqrt{W(|x|)}\,|x|^{N-1}}\left(\frac{x}{|x|}\cdot\nabla u(x)\right)dx \\
&= -\int_{B_R} \left|\sqrt{W(|x|)}\,u(x) + 2\frac{\widetilde{W}(|x|)}{\sqrt{W(|x|)}\,|x|^{N-1}}\left(\frac{x}{|x|}\cdot\nabla u(x)\right)\right|^2 dx \\
&\quad + 4\int_{B_R} \frac{\widetilde{W}^2(x)}{W(|x|)\,|x|^{2N-2}}\left|\frac{x}{|x|}\cdot\nabla u(x)\right|^2 dx.
\end{aligned}$$

We note that

$$\begin{aligned}
&\int_{B_R} \left|\sqrt{W(|x|)}\,u(x) + 2\frac{\widetilde{W}(|x|)}{\sqrt{W(|x|)}\,|x|^{N-1}}\left(\frac{x}{|x|}\cdot\nabla u(x)\right)\right|^2 dx \\
&= \int_{B_R} \left|\frac{2\sqrt{\left|\widetilde{W}(|x|)\right|}}{\sqrt{W(|x|)}\,|x|^{N-1}}\,\frac{\widetilde{W}(|x|)}{\left|\widetilde{W}(|x|)\right|}\left(\left(\frac{x}{|x|}\cdot\nabla u(x)\right)\sqrt{\left|\widetilde{W}(|x|)\right|} + \frac{W(|x|)\,|x|^{N-1}}{2\sqrt{\left|\widetilde{W}(|x|)\right|}}\,\frac{\widetilde{W}(|x|)}{\left|\widetilde{W}(|x|)\right|}\right)\right|^2 dx \\
&= \int_{B_R} \frac{4\left|\widetilde{W}(|x|)\right|}{W(|x|)\,|x|^{2N-2}}\left|\frac{x}{|x|}\cdot\nabla\left(u(x)\sqrt{\left|\widetilde{W}(|x|)\right|}\right)\right|^2 dx.
\end{aligned}$$

We now decompose u into spherical harmonics: $u = \sum_{k=0}^{\infty} u_k = \sum_{k=0}^{\infty} f_k(r)\,\phi_k(\sigma)$. By Lemma 3, we have that

$$\begin{aligned}
&4\int_{B_R} \frac{\widetilde{W}^2(x)}{W(|x|)\,|x|^{2N-2}}\,|\nabla u|^2\,dx - \int_{B_R} W(|x|)\,|u|^2\,dx \\
&= 4\int_{B_R} \frac{\widetilde{W}^2(x)}{W(|x|)\,|x|^{2N-2}}\left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx - \int_{B_R} W(|x|)\,|u|^2\,dx + \sum_{k=0}^{\infty} c_k 4\int_{B_R} \frac{\widetilde{W}^2(x)}{W(|x|)\,|x|^{2N-2}}\,\frac{|f_k(|x|)|^2}{|x|^2}\,dx \\
&= \int_{B_R} \frac{4\left|\widetilde{W}(|x|)\right|}{W(|x|)\,|x|^{2N-2}}\left|\frac{x}{|x|}\cdot\nabla\left(u(x)\sqrt{\left|\widetilde{W}(|x|)\right|}\right)\right|^2 dx + \sum_{k=0}^{\infty} c_k 4\int_{B_R} \frac{\widetilde{W}^2(x)}{W(|x|)\,|x|^{2N-2}}\,\frac{|f_k(|x|)|^2}{|x|^2}\,dx.
\end{aligned}$$

Noting that $u(x)\sqrt{\left|\widetilde{W}(|x|)\right|} = \sum_{k=0}^{\infty}\sqrt{\left|\widetilde{W}(r)\right|}f_k(r)\,\phi_k(\sigma)$ and using Lemma 3, we have

$$\begin{aligned}
&\int_{B_R}\frac{4\left|\widetilde{W}(|x|)\right|}{W(|x|)|x|^{2N-2}}\left|\nabla\left(u(x)\sqrt{\left|\widetilde{W}(|x|)\right|}\right)\right|^2\varphi_{V,W;R}^2dx \\
&=\int_{B_R}\frac{4\left|\widetilde{W}(|x|)\right|}{W(|x|)|x|^{2N-2}}\left|\frac{x}{|x|}\cdot\nabla\left(u(x)\sqrt{\left|\widetilde{W}(|x|)\right|}\right)\right|^2dx \\
&\quad+\sum_{k=0}^{\infty}c_k\int_{B_R}\frac{4\left|\widetilde{W}(|x|)\right|}{W(|x|)|x|^{2N-2}}\varphi_{V,W;R}^2\frac{\left|\sqrt{\left|\widetilde{W}(|x|)\right|}f_k(|x|)\right|^2}{|x|^2}dx \\
&=\int_{B_R}\frac{4\left|\widetilde{W}(|x|)\right|}{W(|x|)|x|^{2N-2}}\left|\frac{x}{|x|}\cdot\nabla\left(u(x)\sqrt{\left|\widetilde{W}(|x|)\right|}\right)\right|^2dx \\
&\quad+\sum_{k=0}^{\infty}c_k4\int_{B_R}\frac{\widetilde{W}^2(x)}{W(|x|)|x|^{2N-2}}\frac{|f_k(|x|)|^2}{|x|^2}dx.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_{B_R}\frac{4\widetilde{W}^2(|x|)}{W(|x|)|x|^{2N-2}}|\nabla u(x)|^2dx-\int_{B_R}W(|x|)|u|^2dx \\
&=\int_{B_R}\frac{4\left|\widetilde{W}(|x|)\right|}{W(|x|)|x|^{2N-2}}\left|\nabla\left(u(x)\sqrt{\left|\widetilde{W}(|x|)\right|}\right)\right|^2dx.
\end{aligned}$$

6. Proof of Theorem 4

Proof of Theorem 4. By polar coordinate

$$\begin{aligned}
&\int_{B_R}V(|x|)\left|\frac{x}{|x|}\cdot\nabla u\right|^2dx-\int_{B_R}W(|x|)|u|^2dx \\
&=\int_{\mathbb{S}^{N-1}}\int_0^R V(r)|\partial_r u(r\sigma)|^2r^{N-1}drd\sigma-\int_{\mathbb{S}^{N-1}}\int_0^R W(r)|u(r\sigma)|^2r^{N-1}drd\sigma
\end{aligned}$$

Let $u(x)=\varphi_{V,W;R}(x)\,\psi(x)$, then

$$\int_0^R V(r)|\partial_r u(r\sigma)|^2r^{N-1}dr$$

$$
\begin{aligned}
&= \int_0^R V(r) \left| \varphi'_{V,W;R}(r)\psi + \varphi_{V,W;R} \partial_r \psi(r\sigma) \right|^2 r^{N-1} dr \\
&= \int_0^R V(r) \varphi'_{V,W;R}(r)^2 |\psi|^2 r^{N-1} dr + \int_0^R V(r) \varphi^2_{V,W;R} |\partial_r \psi(r\sigma)|^2 r^{N-1} dr \\
&\quad + 2\int_0^R V(r) \varphi'_{V,W;R} \varphi_{V,W;R} \psi \partial_r \psi r^{N-1} dr.
\end{aligned}
$$

Using Lemma 2 to treat the integrations by parts, we obtain

$$
\begin{aligned}
&2\int_0^R V(r) \varphi'_{V,W;R} \varphi_{V,W;R} \psi \partial_r \psi r^{N-1} dr \\
&= -\int_0^R |\psi|^2 \partial_r \left[V(r) \varphi'_{V,W;R} \varphi_{V,W;R} r^{N-1} \right] dr \\
&= -\int_0^R V(r) \varphi'_{V,W;R}(r)^2 |\psi|^2 r^{N-1} dr - \int_0^R |\psi|^2 V'(r) \varphi'_{V,W;R} \varphi_{V,W;R} r^{N-1} dr \\
&\quad - \int_0^R |\psi|^2 V(r) \varphi_{V,W;R} \varphi''_{V,W;R} r^{N-1} dr - (N-1) \int_0^R |\psi|^2 V(r) \varphi'_{V,W;R} \varphi_{V,W;R} r^{N-2} dr
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^R V(r) |\partial_r u(r\sigma)|^2 r^{N-1} dr \\
&= \int_0^R V(r) \varphi^2_{V,W;R} |\partial_r \psi(r\sigma)|^2 r^{N-1} dr \\
&\quad - \int_0^R \left[V \varphi''_{V,W;R} + \frac{(N-1)}{r} V \varphi'_{V,W;R} + V'(r) \varphi'_{V,W;R} \right] |\psi|^2 \varphi_{V,W;R} r^{N-1} dr \\
&= \int_0^R V(r) \varphi^2_{V,W;R} |\partial_r \psi(r\sigma)|^2 r^{N-1} dr + \int_0^R W(r) |u|^2 r^{N-1} dr.
\end{aligned}
$$

Hence, we have

$$
\int_{\mathbb{S}^{N-1}} \int_0^R V(r) |\partial_r u(r\sigma)|^2 r^{N-1} dr d\sigma - \int_{\mathbb{S}^{N-1}} \int_0^R W(r) |u(r\sigma)|^2 r^{N-1} dr d\sigma
$$

$$= \int_{\mathbb{S}^{N-1}} \int_0^R V(r)\, \varphi_{V,W;R}^2 \left|\partial_r \psi(r\sigma)\right|^2 r^{N-1} dr d\sigma = \int_{B_R} V(|x|) \left| \frac{x}{|x|} \cdot \nabla \left(\frac{u}{\varphi_{V,W;R}} \right) \right|^2 \varphi_{V,W;R}^2 dx.$$

We now decompose u into spherical harmonics: $u = \sum_{k=0}^{\infty} u_k = \sum_{k=0}^{\infty} f_k(r)\, \phi_k(\sigma)$. By Lemma 3, we have that

$$\begin{aligned}
&\int_{B_R} V(|x|) |\nabla u|^2 dx - \int_{B_R} W(|x|) |u|^2 dx \\
&= \int_{B_R} V(|x|) \left| \frac{x}{|x|} \cdot \nabla u \right|^2 dx - \int_{B_R} W(|x|) |u|^2 dx + \sum_{k=0}^{\infty} c_k \int_{B_R} V(|x|) \frac{|f_k(|x|)|^2}{|x|^2} dx \\
&= \int_{B_R} V(|x|) \left| \frac{x}{|x|} \cdot \nabla \left(\frac{u}{\varphi_{V,W;R}} \right) \right|^2 \varphi_{V,W;R}^2 dx + \sum_{k=0}^{\infty} c_k \int_{B_R} V(|x|) \frac{|f_k(|x|)|^2}{|x|^2} dx.
\end{aligned}$$

Noting that $\frac{u}{\varphi_{V,W;R}} = \sum_{k=0}^{\infty} \frac{f_k(r)}{\varphi_{V,W;R}(r)} \phi_k(\sigma)$ and using Lemma 3, we have

$$\begin{aligned}
&\int_{B_R} V(|x|) \left| \nabla \left(\frac{u}{\varphi_{V,W;R}} \right) \right|^2 \varphi_{V,W;R}^2 dx \\
&= \int_{B_R} V(|x|) \left| \frac{x}{|x|} \cdot \nabla \left(\frac{u}{\varphi_{V,W;R}} \right) \right|^2 \varphi_{V,W;R}^2 dx + \sum_{k=0}^{\infty} c_k \int_{B_R} V(|x|)\, \varphi_{V,W;R}^2 \frac{\left| \frac{f_k(|x|)}{\varphi_{V,W;R}(|x|)} \right|^2}{|x|^2} dx \\
&= \int_{B_R} V(|x|) \left| \frac{x}{|x|} \cdot \nabla \left(\frac{u}{\varphi_{V,W;R}} \right) \right|^2 \varphi_{V,W;R}^2 dx + \sum_{k=0}^{\infty} c_k \int_{B_R} V(|x|) \frac{|f_k(|x|)|^2}{|x|^2} dx.
\end{aligned}$$

Hence

$$\int_{B_R} V(|x|) |\nabla u|^2 dx - \int_{B_R} W(|x|) |u|^2 dx = \int_{B_R} V(|x|) \left| \nabla \left(\frac{u}{\varphi_{V,W;R}} \right) \right|^2 \varphi_{V,W;R}^2 dx.$$

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