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EXACT LOWER AND UPPER BOUNDS ON THE INCOMPLETE GAMMA FUNCTION

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Abstract. Lower and upper bounds $B_a(x)$ on the incomplete gamma function $\Gamma(a,x)$ are given for all real a and all real $x>0$. These bounds $B_a(x)$ are exact in the sense that $B_a(x) \underset{x\downarrow 0}{\sim} \Gamma(a,x)$ and $B_a(x) \underset{x\to\infty}{\sim} \Gamma(a,x)$. Moreover, the relative errors of these bounds are rather small for other values of x, away from 0 and ∞.

1. Statements of main results

Take any real a and any real $x > 0$. The corresponding value of the incomplete gamma function is given by the formula

$$\Gamma(a,x) = \int_x^\infty t^{a-1} e^{-t}\, dt. \tag{1.1}$$

Let

$$b_a := \begin{cases} \Gamma(a+1)^{1/(a-1)} & \text{if } a \in (-1,\infty)\setminus\{1\}, \\ e^{1-\gamma} & \text{if } a = 1, \end{cases} \tag{1.2}$$

where $\gamma = 0.577\dots$ is the Euler constant.

One may note here that $b_a > 0$ for all $a > -1$. The value of b_a at $a = 1$ is defined in (1.2) by continuity (see Lemma 3.5 and its proof for details).

Consider next

$$G_a(x) := \begin{cases} x^{-2} e^{-x} & \text{if } a = -1, \\ \dfrac{(x+b_a)^a - x^a}{a b_a} e^{-x} & \text{if } a \in (-1,\infty)\setminus\{0\}, \\ e^{-x} \ln \dfrac{x+1}{x} & \text{if } a = 0. \end{cases} \tag{1.3}$$

One may note here that $G_a(x)$ is continuous in $a \geqslant -1$ for each $x > 0$.

Further, introduce

$$g_a(x) := \left(\frac{(x+2)^a - x^a - 2^a}{2a} + \Gamma(a) \right) e^{-x} \tag{1.4}$$

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for $a > 0$.

Let us say that a bound $B_a(x)$ on $\Gamma(a,x)$ is exact at $x = 0$ if $B_a(x) \underset{x \downarrow 0}{\sim} \Gamma(a,x)$; similarly defined is the exactness at $x = \infty$. As usual, we write $u \sim v$ for $u/v \to 1$.

THEOREM 1.1. *Take any real $a \geqslant -1$. Then (for all real $x > 0$)*

$$
\begin{aligned}
&\Gamma(a,x) < G_a(x) && \text{if} \quad -1 \leqslant a < 1, && (1.5)\\
&g_a(x) = G_a(x) = \Gamma(a,x) = e^{-x} && \text{if} \quad a = 1, && (1.6)\\
&g_a(x) < G_a(x) < \Gamma(a,x) && \text{if} \quad 1 < a < 2, && (1.7)\\
&g_a(x) = G_a(x) = \Gamma(a,x) = e^{-x}(1+x) && \text{if} \quad a = 2, &&\\
&\Gamma(a,x) < g_a(x) < G_a(x) && \text{if} \quad 2 < a < 3, &&\\
&\Gamma(a,x) = g_a(x) = e^{-x}(2+2x+x^2) < G_a(x) && \text{if} \quad a = 3, &&\\
&g_a(x) < \Gamma(a,x) < G_a(x) && \text{if} \quad a > 3. && (1.8)
\end{aligned}
$$

Also, for each real $a \geqslant 0$ the bound $G_a(x)$ on $\Gamma(a,x)$ is exact both at $x = 0$ and at $x = \infty$. Further, for each real $a \geqslant 1$ the bound $g_a(x)$ on $\Gamma(a,x)$ is exact both at $x = 0$ and at $x = \infty$. Moreover, the bound $G_a(x)$ is exact at $x = \infty$ for each real $a \geqslant -1$.

Thus, for $a > 3$ the bounds $G_a(x)$ and $g_a(x)$ on $\Gamma(a,x)$ bracket $\Gamma(a,x)$ from above and below, respectively.

In the simple cases $a = 1$ and $a = 2$, the bounds $G_a(x)$ and $g_a(x)$ on $\Gamma(a,x)$ are exact in sense that they coincide with $\Gamma(a,x)$. In the same sense, the bound $g_a(x)$ on $\Gamma(a,x)$ is exact if $a = 3$, in contrast with the bound $G_a(x)$.

If $1 < a < 2$, then the lower bound $G_a(x)$ on $\Gamma(a,x)$ is better (that is, closer to $\Gamma(a,x)$) than the lower bound $g_a(x)$. If $2 < a \leqslant 3$, then, vice versa, the upper bound $g_a(x)$ on $\Gamma(a,x)$ is better than the upper bound $G_a(x)$.

Theorem 1.1, which concerns the case $a \geqslant -1$, is complemented by the following result.

THEOREM 1.2. *Take any real $a < -1$. Then (for all real $x > 0$)*

$$
g_a^{\mathsf{lo}}(x) < \Gamma(a,x) < g_a^{\mathsf{up}}(x), \tag{1.9}
$$

where

$$
g_a^{\mathsf{lo}}(x) := \frac{x^a e^{-x}(x-a-1)}{(x-a)^2 + a} \quad \text{and} \quad g_a^{\mathsf{up}}(x) := \frac{x^a e^{-x}}{x-a}.
$$

Also, each of the bounds $g_a^{\mathsf{lo}}(x)$ and $g_a^{\mathsf{up}}(x)$ on $\Gamma(a,x)$ is exact both at $x = 0$ and at $x = \infty$.

Actually, the statements in this theorem concerning $g_a^{\mathsf{up}}(x)$ hold for all $a < 0$.

The bounds on $\Gamma(a,x)$ presented in Theorems 1.1 and 1.2 are rather simple and appear natural. In particular, we shall see in Section 3 that the different pieces in the proofs of these bounds fit together tightly.

2. Discussion

Another nice asymptotic exactness property of the bracketing bounds $g_a^{\mathsf{lo}}(x)$ and $g_a^{\mathsf{up}}(x)$ on $\Gamma(a,x)$ is as follows.

PROPOSITION 2.1.

$$g_a^{\mathsf{lo}}(x) \underset{a\to-\infty}{\sim} \Gamma(a,x) \underset{a\to-\infty}{\sim} g_a^{\mathsf{up}}(x) \quad \textit{uniformly in } x>0. \tag{2.1}$$

Equivalently,

$$\max_{x>0} \frac{g_a^{\mathsf{up}}(x)}{g_a^{\mathsf{lo}}(x)} \underset{a\to-\infty}{\longrightarrow} 1. \tag{2.2}$$

In contrast with (2.1)–(2.2), bounds $g_a(x)$ and $G_a(x)$ on $\Gamma(a,x)$ in (1.8) – which bracket $\Gamma(a,x)$ for $a \geqslant 3$ – exhibit the following explosion phenomenon:

PROPOSITION 2.2.

$$\max_{x>0} \frac{G_a(x)}{\Gamma(a,x)} \underset{a\to\infty}{\longrightarrow} \infty \quad \textit{and} \quad \max_{x>0} \frac{\Gamma(a,x)}{g_a(x)} \underset{a\to\infty}{\longrightarrow} \infty. \tag{2.3}$$

One can find quite a few bounds on the incomplete gamma function in the literature, including papers [6, 17, 2, 16, 10, 12, 15, 9, 5, 3, 11, 18, 8].

A distinctive feature of our bounds on $\Gamma(a,x)$ is their exactness both at $x=0$ and at $x=\infty$. It appears that this feature can be found only in few other papers.

Apparently the first of them was the paper by Gautschi [6], containing the inequalities

$$H(p,1/2,v) < e^{v^p} \int_v^\infty e^{-u^p}\,du \leqslant H(p,c_p,v) \tag{2.4}$$

for real $p>1$ and real $v>0$, where $H(p,c,v) := c\big((v^p+1/c)^{1/p}-v\big)$ and $c_p := \Gamma(1+1/p)^{p/(p-1)}$.

As noted in [6], it is easy to rewrite inequalities (2.4) in terms of the incomplete gamma function. Indeed, using the substitutions $p=1/a$, $v=x^{1/p}=x^a$, and $u=t^{1/p}=t^a$, we see that the second inequality in (2.4) (for $p>1$) becomes the (non-strict version of the) case of inequality (1.5) for $a\in(0,1)$. The limit case $p=\infty$ of the second inequality in (2.4) similarly corresponds to the case $a=0$ of inequality (1.5).

Thus, the second inequality in (2.4) can be considered a special case of (1.5), and it is therefore exact at $x=0$ and at $x=\infty$ – or, in terms of (2.4), at $v=0$ and at $v=\infty$.

However, it is easy to see that the lower bound $H(p,1/2,v)$ on $e^{v^p}\int_v^\infty e^{-u^p}\,du$ in (2.4) is exact only at $v=\infty$, but not at $v=0$. The bound $g_a(x)$, defined in (1.4), can then be viewed as a "corrected" version of $H(p,1/2,v)$ that is exact, for appropriate values of a, both at $x=0$ and at $x=\infty$.

It was pointed out in the review of the paper [6] in Mathematical Reviews [1] that "As it stands, the proof is only valid if p is an integer, but, in a correction, the author has indicated a modification which validates it for all $p>1$." Apparently [7], no proof of (2.4) for the values $p \in (1,\infty)\setminus\mathbb{Z}$ – which correspond to $a \in (0,1)\setminus\{\frac{1}{2},\frac{1}{3},\frac{1}{4},\dots\}$ – has so far been published.

Gautschi's result was complemented in [18], where it was shown that $\Gamma(a,x) > G_a(x)$ for $a \in (1,2)$ and $\Gamma(a,x) < G_a(x)$ for $a > 2$ (again, with $x > 0$); cf. Theorem 1.1 of the present paper. The method used in [18] was based on results in [14], restated, however, in terms of the function $H_{f,g} := \frac{f'}{g'}g - f$, which differs only by the sign factor $\operatorname{sign}(g')$ from the function $\tilde{\rho}$ introduced and used in [14].

REMARK 2.3. For "most" real values of a, Theorems 1.1 and 1.2 taken together provide both lower and upper bounds on $\Gamma(a,x)$, each of which is exact both at $x=0$ and at $x=\infty$. However, there are a few gaps in the coverage by Theorems 1.1 and 1.2:

Gap 1: the absence of a lower bound on $\Gamma(a,x)$ for $a \in [-1,1)$;

Gap 2: the absence of an upper bound on $\Gamma(a,x)$ for $a \in (1,2)$;

Gap 3: the absence of a lower bound on $\Gamma(a,x)$ for $a \in (2,3)$.

Moreover, we shall address

Gap 4: for $a \in [-1,0)$, the upper bound $G_a(x)$ on $\Gamma(a,x)$ is exact only at $x=\infty$, but not at $x=0$. $\square$

To fill these gaps, and to address the explosion phenomenon presented in Proposition 2.2, we can use the following shift technique.

Integrating by parts, we have

$$\Gamma(a,x) = x^{a-1}e^{-x} + (a-1)\Gamma(a-1,x) \tag{2.5}$$

for all real a and all $x > 0$. Iterating this recursion in a, we see that

$$\Gamma(a,x) = x^{a-1}e^{-x}\sum_{j=0}^{k-1}(a-1)_j x^{-j} + (a-1)_k\Gamma(a-k,x) \tag{2.6}$$

for all natural k and all real $x > 0$, where $(u)_j := \prod_{i=0}^{j-1}(u-i)$, the jth falling factorial of u.

Replacing now $\Gamma(a-k,x)$ on the right-hand side of (2.6) by a bound $B_{a-k}(x)$ on $\Gamma(a-k,x)$, we obtain the new, modified bound

$$B_{a;k}(x) := x^{a-1}e^{-x}\sum_{j=0}^{k-1}(a-1)_j x^{-j} + (a-1)_k B_{a-k}(x) \tag{2.7}$$

on $\Gamma(a,x)$, which may be thought of as the (forward) k-shift of the bound $B_{a-k}(x)$ on $\Gamma(a-k,x)$. In particular, if $(a-1)_k \geqslant 0$, then this forward k-shift will transform a lower (respectively, upper) bound $B_{a-k}(x)$ on $\Gamma(a-k,x)$ into a lower (respectively, upper) bound $B_{a;k}(x)$ on $\Gamma(a,x)$. Similarly, if $(a-1)_k \leqslant 0$, then the forward k-shift will transform a lower (respectively, upper) bound into an upper (respectively, lower) one.

For any bound $B_a(x)$ on $\Gamma(a,x)$, consider the corresponding (signed) error and (signed) relative error of the approximation of $\Gamma(a,x)$ by the bound $B_a(x)$:

$$\Delta B_a(x) := B_a(x) - \Gamma(a,x) \quad \text{and} \quad \delta B_a(x) := \frac{\Delta B_a(x)}{\Gamma(a,x)}.$$

It is then obvious from (2.6) and (2.7) that

$$\Delta B_{a;k}(x) = (a-1)_k \Delta B_{a-k}(x). \tag{2.8}$$

Now we obtain the following simple "relative-error-taming".

PROPOSITION 2.4. *Take any natural k, any real $a \geqslant k$, and any real $x > 0$.*

(i) If $B_{a-k}(x)$ is a lower bound on $\Gamma(a-k,x)$, then $B_{a;k}(x)$ is a lower bound on $\Gamma(a,x)$, and $\delta B_{a-k}(x) \leqslant \delta B_{a;k}(x) \leqslant 0$.

(ii) If $B_{a-k}(x)$ is an upper bound on $\Gamma(a-k,x)$, then $B_{a;k}(x)$ is also an upper bound on $\Gamma(a,x)$, and $0 \leqslant \delta B_{a;k}(x) \leqslant \delta B_{a-k}(x)$.

This follows immediately from (2.8) and the inequality

$$\Gamma(a,x) \geqslant (a-1)_k \Gamma(a-k,x);$$

in turn, the latter inequality follows immediately, in the conditions of Proposition 2.4, from identity (2.6).

So, if $a \geqslant k \in \mathbb{N}$, then the forward k-shift can only reduce the absolute value of the relative error of a bound $B_a(x)$ on $\Gamma(a,x)$. Immediately from Theorem 1.1 and Proposition 2.4, we obtain

COROLLARY 2.5. *Take any real a and any real $x > 0$.*

(i) If $a > 2$ and $k = \lceil a \rceil - 2$, then $1 < a-k \leqslant 2$ and $\delta G_{a-k}(x) \leqslant \delta G_{a;k}(x) \leqslant 0$.

(ii) If $a > 3$ and $k = \lceil a \rceil - 3$, then $2 < a-k \leqslant 3$ and $0 \leqslant \delta g_{a;k}(x) \leqslant \delta g_{a-k}(x)$.

(iii) If $a > 4$ and $k = \lceil a \rceil - 4$, then $3 < a-k \leqslant 4$ and $\delta g_{a-k}(x) \leqslant \delta g_{a;k}(x) \leqslant 0 \leqslant \delta G_{a;k}(x) \leqslant \delta G_{a-k}(x)$.

Before stating the following proposition, let us note that $\delta B_a(x) > -1$ whenever $B_a(x) > 0$.

PROPOSITION 2.6. *Take any real a_*. Then*

(i) $\delta G_a(x)$ is bounded away from -1 and ∞ over all $(a,x) \in [1,a_*] \times (0,\infty)$;

(ii) $\delta g_a(x)$ is bounded away from -1 and ∞ over all $(a,x) \in [2,a_*] \times (0,\infty)$.

We see that, in particular, Corollary 2.5 and Proposition 2.6, taken together, provide a lower bound and an upper bound on $\Gamma(a,x)$ with relative errors bounded away from -1 and ∞ uniformly over all $(a,x)\in[2,\infty)\times(0,\infty)$. This fully addresses the explosion phenomenon described in Proposition 2.2. Of course, the trade-off when using the shifted, better bounds $G_{a;k}(x)$ and $g_{a;k}(x)$ for large k is that they are more complicated than the "original" bounds $G_a(x)$ and $g_a(x)$.

The following proposition provides simple, if not very precise, bounds on $\Gamma(a,x)$, to be used in the proof of Proposition 2.6, which is clearly of a qualitative nature.

PROPOSITION 2.7. *Take any real $a\geqslant 1$. Then $\Gamma(a,x)\geqslant x^{a-1}e^{-x}$ for all real $x>0$ and $\Gamma(a,x)\leqslant x^{a-1}e^{-x}/\big(1-(a-1)/x\big)$ for all real $x>a-1$.*

The shift technique also allows us to fill the gaps described in Remark 2.3.

Along with the forward shift described above, here we can use the corresponding backward shift. To obtain such a shift, let us begin by rewriting the forward-shift identity (2.6) in a "backward" manner:

$$\Gamma(a,x)=\frac{1}{(a-1+k)_k}\Big(\Gamma(a+k,x)-x^{a-1+k}e^{-x}\sum_{j=0}^{k-1}(a-1+k)_j x^{-j}\Big) \tag{2.9}$$

for all real a, all natural k such that $(a-1+k)_k\neq 0$, and all real $x>0$. Replacing here $\Gamma(a+k,x)$ by a bound $B_{a+k}(x)$ on $\Gamma(a+k,x)$, we obtain the "backward-shifted" version of the bound $B_{a+k}(x)$:

$$B_{a;-k}(x):=\frac{1}{(a-1+k)_k}\Big(B_{a+k}(x)-x^{a-1+k}e^{-x}\sum_{j=0}^{k-1}(a-1+k)_j x^{-j}\Big). \tag{2.10}$$

In particular, if $(a-1+k)_k>0$, then this backward k-shift will transform a lower (respectively, upper) bound $B_{a+k}(x)$ on $\Gamma(a+k,x)$ into a lower (respectively, upper) bound $B_{a;-k}(x)$ on $\Gamma(a,x)$. Similarly, if $(a-1+k)_k<0$, then the backward k-shift will transform a lower (respectively, upper) bound into an upper (respectively, lower) one.

Now we are ready to state the following propositions.

PROPOSITION 2.8. *Take any real $a<1$ and recall (2.7) and (1.9). Then for the forward 2-shift*

$$\begin{align} g^{\mathsf{lo}}_{a;2}(x) &= x^{a-1}e^{-x}\big(1+(a-1)/x\big)+(a-1)(a-2)\,g^{\mathsf{lo}}_{a-2}(x) \tag{2.11}\\ &= \frac{e^{-x}x^a(x+3-a)}{x^2+(4-2a)x+(a-1)(a-2)} \tag{2.12}\end{align}$$

of the lower bound $g^{\mathsf{lo}}_{a-2}(x)$ on $\Gamma(a-2,x)$ for all real $x>0$ we have

$$\Gamma(a,x)>g^{\mathsf{lo}}_{a;2}(x), \tag{2.13}$$

so that $g^{\mathsf{lo}}_{a;2}(x)$ is a lower bound on $\Gamma(a,x)$. Moreover, for each real $a<1$ the lower bound $g^{\mathsf{lo}}_{a;2}(x)$ on $\Gamma(a,x)$ is exact at $x=\infty$.

PROPOSITION 2.9. *Take any* $a \in (-2,1)$, *and recall* (2.10), (1.3), *and* (1.5). *If* $a \neq 0$, *consider the backward* 1*-shift*

$$G_{a;-1}(x) = \tfrac{1}{a}\left(G_{a+1}(x) - x^a e^{-x}\right) \tag{2.14}$$

of the bound $G_{a+1}(x)$ *on* $\Gamma(a+1,x)$. *Define* $G_{0;-1}(x)$ *by continuity:*

$$G_{0;-1}(x) := \lim_{a\to 0} G_{a;-1}(x) = e^{-x}\Big[\Big(1+\frac{x}{b_1}\Big)\ln\Big(1+\frac{b_1}{x}\Big) - 1\Big]. \tag{2.15}$$

Then for all real $x > 0$

$$\Gamma(a,x) > G_{a;-1}(x), \tag{2.16}$$

so that $G_{a;-1}(x)$ *is a lower bound on* $\Gamma(a,x)$. *Moreover, for each* $a \in (-2,1)$ *the lower bound* $G_{a;-1}(x)$ *on* $\Gamma(a,x)$ *is exact at* $x = 0$.

PROPOSITION 2.10. *Take any* $a \in (1,3)$, *and recall* (2.7), (1.3), *and Theorem* 1.1. *Then for the forward* 1*-shift*

$$G_{a;1}(x) = x^{a-1}e^{-x} + (a-1)\,G_{a-1}(x)$$

of the bound $G_{a-1}(x)$ *on* $\Gamma(a-1,x)$ *for all real* $x > 0$ *we have*

$$\Gamma(a,x) < G_{a;1}(x) \quad \text{if } 1 < a < 2, \tag{2.17}$$
$$\Gamma(a,x) = G_{a;1}(x) \quad \text{if } a = 2, \tag{2.18}$$
$$\Gamma(a,x) > G_{a;1}(x) \quad \text{if } 2 < a < 3. \tag{2.19}$$

Moreover, for each $a \in (1,3)$ *the bound* $G_{a;1}(x)$ *on* $\Gamma(a,x)$ *is exact both at* $x = 0$ *and at* $x = \infty$.

PROPOSITION 2.11. *Take any real* $a < 0$. *Then for the forward* 1*-shift*

$$g^{\mathsf{lo}}_{a;1}(x) = x^{a-1}e^{-x} + (a-1)\,g^{\mathsf{lo}}_{a-1}(x) = \frac{e^{-x}x^a(1-a+x)}{(x-a)^2 - a + 2x} \tag{2.20}$$

of the lower bound $g^{\mathsf{lo}}_{a-1}(x)$ *on* $\Gamma(a-1,x)$ *for all real* $x > 0$ *we have*

$$\Gamma(a,x) < g^{\mathsf{lo}}_{a;1}(x) < g^{\mathsf{up}}_{a}(x), \tag{2.21}$$

so that $g^{\mathsf{lo}}_{a;1}(x)$ *is an upper bound on* $\Gamma(a,x)$, *which is an improvement of the upper bound* $g^{\mathsf{up}}_{a}(x)$ *on* $\Gamma(a,x)$. *Moreover, for each real* $a < 0$ *the upper bound* $g^{\mathsf{lo}}_{a;1}(x)$ *on* $\Gamma(a,x)$ *is exact both at* $x = 0$ *and at* $x = \infty$.

Propositions 2.8–2.11 fill the four gaps listed in Remark 2.3. In particular, inequalities (2.17) and (2.19) in Proposition 2.10 cover Gaps 2 and 3, respectively, whereas Proposition 2.11 covers Gap 4. Finally, Gap 1 is covered by the following immediate corollary of Propositions 2.8 and 2.9:

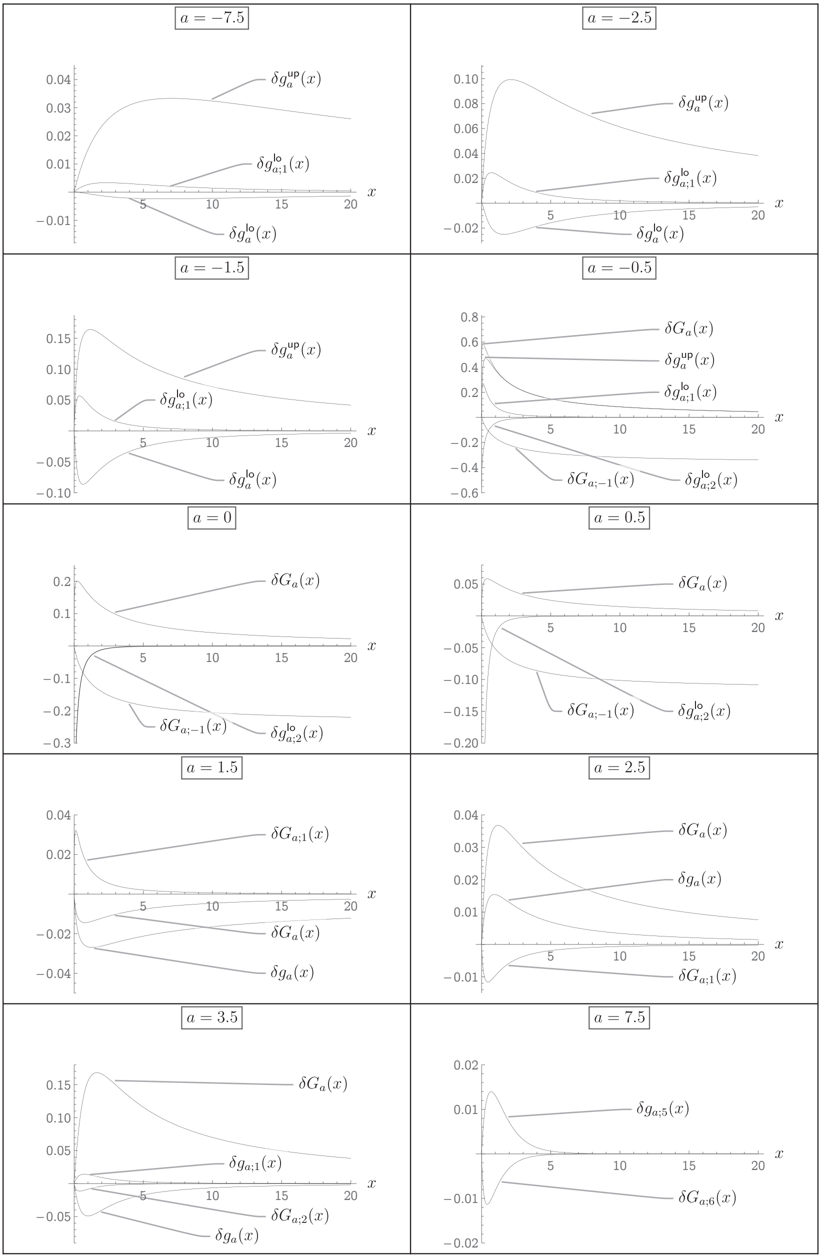

Figure 1: Graphs of signed relative errors of bounds on $\Gamma(a,x)$.

COROLLARY 2.12. *Take any* $a \in (-2,1)$. *Then for all real* $x > 0$

$$\Gamma(a,x) > h_a(x) := G_{a;-1}(x) \vee g^{\mathsf{lo}}_{a;2}(x),$$

so that $h_a(x)$ *is a lower bound on* $\Gamma(a,x)$. *Moreover, for each* $a \in (-2,1)$ *the lower bound* $h_a(x)$ *on* $\Gamma(a,x)$ *is exact both at* $x = 0$ *and at* $x = \infty$.

The drawback of the bound $h_a(x)$ on $\Gamma(a,x)$ in Corollary 2.12 is that, in contrast with all the other bounds on $\Gamma(a,x)$ given in this paper, the bound h_a is not a real-analytic function, but rather the maximum of two real-analytic functions, $G_{a;-1}$ and $g^{\mathsf{lo}}_{a;2}$.

Figure 1 shows graphs of the signed relative errors of various bounds on $\Gamma(a,x)$ presented above for selected values of a, namely, for $a \in \{-7.5, -2.5, -1.5, -0.5, 0, 0.5, 1.5, 2.5, 3.5, 7.5\}$.

3. Proofs

The proofs are based mainly on the following "special-case l'Hospital-type rules for monotonicity" given in [14, Propositions 4.1 and 4.3]:

PROPOSITION A. *Let* $-\infty \leqslant A < B \leqslant \infty$. *Let* f *and* g *be differentiable functions defined on the interval* (A,B) *such that the functions* f *and* f' *do not take on the zero value and do not change their respective signs on* (A,B). *Suppose also that* $f(A+) = g(A+) = 0$ *or* $f(B-) = g(B-) = 0$. *Consider the ratio* $r := g/f$ *and the "derivative" ratio* $\rho := g'/f'$. *Then we have the following:*

(i) If ρ *is increasing or decreasing on* (A,B), *then* r *is so as well, respectively.*

(ii) If ρ *is increasing-decreasing or decreasing-increasing on* (A,B), *then* r *is so as well, respectively.*

Here we say that a function r on (A,B) is increasing-decreasing if there is some point $C \in [A,B]$ such that r is increasing on (A,C) and decreasing on (C,B). The term "decreasing-increasing" is defined similarly, so that r is decreasing-increasing if and only if the function $-r$ is increasing-decreasing. In particular, if r is increasing or decreasing on the entire interval (A,B), then r is both increasing-decreasing and decreasing-increasing on (A,B).

In this paper the terms "increasing" and "decreasing" are understood in the strict sense: namely, as "strictly increasing" and "strictly decreasing", respectively.

General versions of this special l'Hospital-type rule for monotonicity, without the assumption that $f(A+) = g(A+) = 0$ or $f(B-) = g(B-) = 0$ are also known; see again [14] and references therein, especially [13].

Next, let us say that a function $h\colon (0,\infty) \to \mathbb{R}$ is strictly concave-convex if, for some $c \in (0,\infty)$, the function h is strictly concave on $(0,c]$ and strictly convex on $[c,\infty)$. Let us say that h is strictly convex-concave if $-h$ is strictly concave-convex.

LEMMA 3.1. *Let a function* $h\colon (0,\infty)\to\mathbb{R}$ *be such that* $h(\infty-)\in\mathbb{R}$. *Then, if* h *is strictly concave-convex, then* h *is increasing-decreasing; if* h *is strictly convex-concave, then* h *is decreasing-increasing.*

Proof. Without loss of generality, h is strictly convex-concave. Hence, h is decreasing-increasing on $(0,c]$ and increasing-decreasing on $[c,\infty)$, for some $c\in(0,\infty)$. Moreover, if h were decreasing on $[d,\infty)$ for some real $d\geqslant c$, then, because h is strictly concave on $[d,\infty)$, we would have $h(\infty-)=-\infty$, which would contradict the condition $h(\infty-)\in\mathbb{R}$. Thus, h is increasing on $[c,\infty)$ and decreasing-increasing on $(0,c]$, which implies that h is decreasing-increasing on $(0,\infty)$.

This completes the proof of Lemma 3.1.

Proof of Theorem 1.1. This follows immediately from Propositions 3.2, 3.3, and 3.4 below.

PROPOSITION 3.2. *Take any real* $a\geqslant -1$. *Then (for all real* $x>0$*)*

$$\Gamma(a,x)\begin{cases} <G_a(x) & \text{if } a\in[-1,1)\cup(2,\infty),\\ >G_a(x) & \text{if } a\in(1,2),\\ =G_a(x) & \text{if } a\in\{1,2\}.\end{cases} \tag{3.1}$$

Also, for each real $a\geqslant 0$ *the bound* $G_a(x)$ *on* $\Gamma(a,x)$ *is exact both at* $x=0$ *and at* $x=\infty$. *In fact, this bound is exact at* $x=\infty$ *for each real* $a\geqslant -1$.

PROPOSITION 3.3. *Take any real* $a\geqslant 1$. *Then (for all real* $x>0$*)*

$$\Gamma(a,x)\begin{cases} >g_a(x) & \text{if } a\in(1,2)\cup(3,\infty),\\ <g_a(x) & \text{if } a\in(2,3),\\ =g_a(x) & \text{if } a\in\{1,2,3\}.\end{cases} \tag{3.2}$$

Also, the bound $g_a(x)$ *on* $\Gamma(a,x)$ *is exact both at* $x=0$ *and at* $x=\infty$. *In fact, this bound is exact at* $x=\infty$ *for each real* $a\geqslant -1$.

PROPOSITION 3.4. *For all real* $x>0$

$$g_a(x)<G_a(x) \quad \text{if} \quad a\in(1,2)\cup(2,\infty). \tag{3.3}$$

To prove Proposition 3.2, we shall need the following two lemmas.

LEMMA 3.5. b_a *is continuously increasing in real* $a>-1$ *from* $b_{(-1)+}=0$ *to* $b_0=1$ *to* $b_1=e^{1-\gamma}$ *to* $b_2=2$ *to* $b_{\infty-}=\infty$.

Proof. The most essential ingredient of this proof is Proposition A, stated in the beginning of this section. Indeed, for $a\in(-1,\infty)\setminus\{1\}$ we have $\ln b_a=\frac{\ln\Gamma(a+1)}{a-1}$, and the "derivative" ratio for the ratio $\frac{\ln\Gamma(a+1)}{a-1}$ is $\frac{d}{da}\ln\Gamma(a+1)$, which is increasing in

$a \in (-1,\infty)$, since the function Γ is strictly log convex. So, by part (i) of Proposition A, b_a is increasing in $a \in (-1,1)$ and in $a \in (1,\infty)$.

Moreover, using the well known fact (see e.g. [4, formula (1.2.12)]) that

$$\Gamma'(1) = -\gamma, \tag{3.4}$$

the identity $\Gamma(a+1) = a\Gamma(a)$, and l'Hospital's rule, we see that b_a is continuous in a at $a = 1$ and hence in all real $a > -1$. Therefore, b_a is increasing in all real $a > -1$.

The equality $b_{(-1)+} = 0$ follows immediately from the identity $\Gamma(a+1) = \frac{\Gamma(a+2)}{a+1}$ for $a \neq 1$. The equalities $b_0 = 1$ and $b_2 = 2$ are trivial. Finally, the equality $b_{\infty-} = \infty$ follows easily from Stirling's formula.

LEMMA 3.6. *We have*

$$\begin{aligned} a < b_a < 2 \quad & \textit{if } a \in (-1,2), \\ a > b_a > 2 \quad & \textit{if } a \in (2,\infty). \end{aligned}$$

Proof. If $a \in (-1,0]$, then the inequality $a < b_a$ is obvious and the inequality $b_a < 2$ follows by Lemma 3.5.

Take now any real $a > 0$. Let

$$h(a) := (a-1)\ln(b_a/a) = \ln\Gamma(a+1) + (1-a)\ln a.$$

Then $h''(a) = \psi'(a+1) - \frac{1}{a} - \frac{1}{a^2} < \psi'(a+1) - \frac{1}{a}$, where, as usual, $\psi := (\ln\Gamma)' = \Gamma'/\Gamma$, and, by [4, formula (1.2.14)],

$$\psi'(a+1) = \sum_{k=0}^{\infty} \frac{1}{(a+1+k)^2} < \int_a^{\infty} \frac{dx}{x^2} = \frac{1}{a}.$$

So, $h'' < 0$ and hence h is strictly concave on $(0,\infty)$. Also, $h(1) = h(2) = 0$. Hence, $h < 0$ on $(0,1) \cup (2,\infty)$ and $h > 0$ on $(1,2)$. Now the inequalities $a < b_a$ for $a \in (0,1) \cup (1,2)$ and $a > b_a$ for $a \in (2,\infty)$ follow immediately from the definition of h. The inequalities $b_a < 2$ for $a \in (0,1) \cup (1,2)$ and $b_a > 2$ for $a \in (2,\infty)$, as well as the inequalities $a < b_a < 2$ for $a = 1$, follow immediately from Lemma 3.5.

This completes the proof of Lemma 3.6.

The following two very simple lemmas will be used repeatedly.

LEMMA 3.7. *For each real a we have $\Gamma(a,x) \sim x^{a-1}e^{-x}$ as $x \to \infty$.*

This follows immediately by the l'Hospital rule.

LEMMA 3.8. *Take any real a. Then*

$$\Gamma(a,x) \underset{x\downarrow 0}{\sim} \begin{cases} -x^a/a & \textit{if } a < 0, \\ -\ln x & \textit{if } a = 0, \\ \Gamma(a) & \textit{if } a > 0. \end{cases} \tag{3.5}$$

Moreover,

$$\Gamma(0,x) = -\ln x - \gamma + O(x) \tag{3.6}$$

as $x \downarrow 0$, *where, again,* γ *is the Euler constant.*

Proof. The first two asymptotic relations in (3.5) follow immediately by the l'Hospital rule; the third asymptotic relation in (3.5) follows immediately by, say, the dominated convergence theorem.

To prove (3.6), use the identities $\int_0^\infty e^{-t}\ln t\,dt = \Gamma'(1)$ and (3.4) to write

$$\int_0^\infty e^{-t}\ln t\,dt = -\gamma.$$

So, integration by parts yields

$$\begin{aligned}\Gamma(0,x) = \int_x^\infty \frac{1}{t}e^{-t}\,dt &= -e^{-x}\ln x - \gamma - \int_0^x e^{-t}\ln t\,dt \\ &= -(1-x)\ln x + O(x^2|\ln x|) - \gamma - \int_0^x \ln t\,dt + O(x^2|\ln x|) \\ &= -\ln x - \gamma + x + O(x^2|\ln x|) = -\ln x - \gamma + O(x)\end{aligned}$$

as $x \downarrow 0$, which proves (3.6) as well.

Proof of Proposition 3.2. The cases with $a \in \{1,2\}$ in (3.1) are straightforward.

Take now any $a \in [-1,\infty)\setminus\{1,2\}$. By the mean value theorem, $G_a(x) \sim x^{a-1}e^{-x}$ as $x \to \infty$, and now the exactness of the bound $G_a(x)$ on $\Gamma(a,x)$ at $x = \infty$ follows by Lemma 3.7.

The exactness of the bound $G_a(x)$ on $\Gamma(a,x)$ at $x = 0$ for each real $a \geqslant 0$ follows immediately from (3.5) and (1.3).

It remains to prove the inequalities in (3.1). This proof relies on Lemma 3.6 and the "special-case l'Hospital-type rules for monotonicity" cited in Proposition A.

We are going to apply Proposition A to the functions $f = f_a$ and $g = G_a$, where

$$f_a(x) := \Gamma(a,x) \tag{3.7}$$

and G_a defined by (1.3). Then for $a \in (-1,\infty)\setminus\{0,1,2\}$

$$\rho(x) = \rho_a(x) := \frac{G_a'(x)}{f_a'(x)} = x\,\frac{(1+b_a/x)^a - 1}{ab_a} - \frac{(1+b_a/x)^{a-1} - 1}{b_a} \underset{x\to\infty}{\longrightarrow} 1 \tag{3.8}$$

and

$$\rho''(x)x^{a+1}(b_a+x)^{3-a} = (a-1)\big(b_a(b_a-a) - (2-b_a)x\big).$$

Consider now the case $a \in (1,2)$.

Then, by Lemma 3.6, for $c := c_a := b_a(b_a-a)/(2-b_a) \in (0,\infty)$ we have $\rho'' > 0$ on the interval $(0,c)$, and $\rho'' < 0$ on the interval (c,∞). So, by Lemma 3.1, ρ is decreasing-increasing on $(0,\infty)$. Also, $f(\infty-) = g(\infty-) = 0$. Therefore, by part (ii) of Proposition A, $r = g/f$ is decreasing-increasing on $(0,\infty)$. Also, by the exactness of the bound $G_a(x)$ on $\Gamma(a,x)$, we have $r(0+) = r(\infty-) = 1$. It follows that $r < 1$ on $(0,\infty)$, which means that the second inequality in (3.1) holds.

The first inequality in (3.1) is proved quite similarly for $a \in (-1,0) \cup (0,1) \cup (2,\infty)$ – except that for $a \in (-1,0)$ by Lemma 3.5 we have $r(0+) = 1/b_a > 1$, rather than $r(0+) = 1$.

In the remaining cases $a = -1$ and $a = 0$, the proof of the first inequality in (3.1) is similar and even easier, especially in the case $a = -1$, where $\rho(x) = 1 + 2/x$ is obviously decreasing in $x > 0$; in the case $a = 0$, we have $\rho''(x)x(1+x)^3 = x - 1$.

This completes the proof of Proposition 3.2.

Proof of Proposition 3.3. This proof is very similar to, and even a bit simpler than, the proof of Proposition 3.2.

Indeed, the cases with $a \in \{1,2,3\}$ in (3.2) are straightforward.

Take now any $a \in [1,\infty) \setminus \{1,2,3\}$. By the mean value theorem, $g_a(x) \sim x^{a-1}e^{-x}$ as $x \to \infty$, and now the exactness of the bound $g_a(x)$ at $x = \infty$ follows by Lemma 3.7.

For any $a > 0$ (and hence for any $a > 1$), we have the trivial equalities $\Gamma(a,0+) = \Gamma(a) = g_a(0+) \in (0,\infty)$, so that the bound $g_a(x)$ is exact at $x = 0$.

It remains to prove the inequalities in (3.2).

We are going to apply Proposition A to the functions $f = f_a$ and $g = g_a$, with f_a defined by (3.7) and g_a defined by (1.4). Then

$$\rho(x) = \frac{g'(x)}{f'(x)} = x\,\frac{(1+2/x)^a - 1}{2a} - \frac{(1+2/x)^{a-1} - 1}{2} + \frac{C_a}{a}\,x^{1-a} \underset{x\to\infty}{\longrightarrow} 1 \tag{3.9}$$

and

$$\rho''(x)\frac{x^{a+1}}{2(a-1)} = (2-a)(2+x)^{a-3} + C_a/2, \tag{3.10}$$

where

$$C_a := \Gamma(a+1) - 2^{a-1}. \tag{3.11}$$

Note that the ratio $\Gamma(a+1)/2^{a-1}$ is strictly log convex in a and takes value 1 when $a \in \{1,2\}$. So,

$$\text{for } a > 1 \text{ we have } C_a < 0 \text{ iff } a < 2\text{, and } C_a = 0 \text{ iff } a = 2. \tag{3.12}$$

Consider now the case $a \in (2,3)$.

Then, by (3.10), ρ is strictly concave-convex and hence, by (3.9) and Lemma 3.1, ρ is increasing-decreasing, on $(0,\infty)$. Also, $f(\infty-) = g(\infty-) = 0$. Therefore, by part (ii) of Proposition A, $r = g/f$ is increasing-decreasing on $(0,\infty)$. Also, by the exactness of the bound $g_a(x)$ at $x = 0$ and $x = \infty$, we have $r(0+) = r(\infty-) = 1$. It follows that $r > 1$ on $(0,\infty)$, which means that the second inequality in (3.2) holds.

The first inequality in (3.2) is proved quite similarly.

This completes the proof of Proposition 3.3.

Proof of Proposition 3.4. Take indeed any $a \in (1,2)\cup(2,\infty)$ and any real $x>0$. Recall the definition of C_a in (3.11). Consider

$$d(x) := \Big(\frac{g_a(x)}{G_a(x)}-1\Big)2((x+b_a)^a-x^a) = (2-b_a)x^a+b_a(x+2)^a-2(x+b_a)^a+2b_aC_a,$$

which equals $g_a(x)-G_a(x)$ in sign, and then

$$d_1(u) := d'(1/u)u^{a-1}/a = 2-2(1+b_au)^{a-1}+b_a((1+2u)^{a-1}-1)$$

and

$$\frac{d_1'(u)}{2(a-1)b_a} = (1+2u)^{a-2}-(1+b_au)^{a-2}<0$$

for $u>0$, since, in view of Lemma 3.5, b_a-2 equals $a-2$ in sign. So, d_1 is decreasing on $(0,\infty)$, from $d_1(0+)=0$. It follows that $d_1<0$ and hence d is decreasing on $(0,\infty)$, from $d(0+)=0$. Thus, $d<0$ on $(0,\infty)$, which completes the proof of Proposition 3.4.

Proof of Theorem 1.2. Take indeed any real $a<-1$ and $x>0$.

Consider first the lower bound $g_a^{\mathsf{lo}}(x)$ on $\Gamma(a,x)$ and, within this consideration, let $g := g_a^{\mathsf{lo}}$, for the simplicity of writing. Let then $f := f_a$, with f_a as in (3.7), and let $r = g/f$ and $\rho = g'/f'$, as in Proposition A. Then

$$\rho(x) = \frac{u^4+2(a-1)u^2-4au-a^2}{(u^2+a)^2}, \quad \text{with} \quad u := x-a,$$

so that $u>-a>1$ and hence $u^2>u>-a$ and $u^2+a>0$. Next,

$$\rho'(x)(u^2+a)^3/(4x) = x^2-a(a+1)$$

and $a(a+1)>0$. So, ρ is decreasing-increasing on $(0,\infty)$. Also, $f(\infty-)=g(\infty-)=0$. Therefore, by part (ii) of Proposition A, $r=g/f$ is decreasing-increasing on $(0,\infty)$. Also, $\rho(0+)=1=\rho(\infty-)$, $f(0+)=\infty=g(0+)$, and $f(\infty-)=0=g(\infty-)$, whence, by the l'Hospital rule for limits, $r(0+)=1=r(\infty-)$. Thus, $r<1$ on $(0,\infty)$. We conclude that the first inequality in (1.9) and the exactness properties concerning the lower bound $g_a^{\mathsf{lo}}(x)$ on $\Gamma(a,x)$ do hold.

The corresponding proof for the upper bound $g_a^{\mathsf{up}}(x)$ is similar and even simpler. Indeed, letting here $g := g_a^{\mathsf{up}}$ and, as before, $f := f_a$, $r=g/f$, and $\rho=g'/f'$, we have

$$\rho(x) = \frac{(x-a)^2+x}{(x-a)^2} \quad \text{and} \quad \rho'(x) = -\frac{x+a}{(x-a)^3},$$

so that ρ is increasing-decreasing on $(0,\infty)$. Also, $f(\infty-)=g(\infty-)=0$. Therefore, by part (ii) of Proposition A, $r=g/f$ is increasing-decreasing on $(0,\infty)$. Also, $\rho(0+)=1=\rho(\infty-)$, $f(0+)=\infty=g(0+)$, and $f(\infty-)=0=g(\infty-)$, whence, by the l'Hospital rule for limits, $r(0+)=1=r(\infty-)$. Thus, $r>1$ on $(0,\infty)$. We conclude that the second inequality in (1.9) and the exactness properties concerning the upper bound $g_a^{\mathsf{up}}(x)$ on $\Gamma(a,x)$ do hold.

Theorem 1.2 is now proved.

Proof of Proposition 2.1. The ratio in (2.2) equals $\frac{(x-a)^2+a}{(x-a-1)(x-a)}$, and its partial derivative in x equals $a(a+1)-x^2$ in sign. So, for each $a<-1$ this ratio attains its maximum in $x>0$ at $x=\sqrt{a(a+1)}$, and the value of this maximum is $\frac{2}{1+\sqrt{1+1/a}}\to 1$ as $a\to-\infty$. Thus, asymptotic relations (2.2) and (2.1) are verified.

Proof of Proposition 2.2. Let indeed $a\to\infty$. By Stirling's formula, $b_a\sim a/e$ and hence $G_a(a)=(a/e)^a(1+1/e)^a e^{o(a)}$, whereas $\Gamma(a,a)\leqslant\Gamma(a)=(a/e)^a e^{o(a)}$, so that

$$\max_{x>0}\frac{G_a(x)}{\Gamma(a,x)}\geqslant\frac{G_a(a)}{\Gamma(a,a)}\geqslant(1+1/e)^a e^{o(a)}\to\infty,$$

and the first asymptotic relation in (2.3) follows.

Letting now $w(t):=w_a(t):=(a-1)\ln t-t$, we have $w(a-1)=(a-1)\ln\frac{a-1}{e}$, $w'(a-1)=0$, and $w''(t)=-\frac{a-1}{t^2}>-\frac{1}{a-1}$ for $t>a-1$, so that

$$\begin{aligned}\Gamma(a,a-1)&=\int_{a-1}^{\infty}e^{w(t)}dt>\int_{a-1}^{\infty}\exp\Big\{w(a-1)-\frac{(t-(a-1))^2}{2(a-1)}\Big\}dt\\&=\Big(\frac{a-1}{e}\Big)^{a-1}\sqrt{\frac{\pi(a-1)}{2}}.\end{aligned}$$

On the other hand, it is easy to see that

$$g_a(a-1)\sim\frac{e^2-1}{2}\Big(\frac{a-1}{e}\Big)^{a-1}.$$

So,

$$\max_{x>0}\frac{\Gamma(a,x)}{g_a(x)}\geqslant\frac{\Gamma(a,a-1)}{g_a(a-1)}=\frac{2+o(1)}{e^2-1}\sqrt{\frac{\pi a}{2}}\to\infty,$$

and the second asymptotic relation in (2.3) follows as well.

Proof of Proposition 2.6. Since $\Gamma(a,x)$, $G_a(x)$, and $g_a(x)$ are (strictly) positive and continuous in (a,x), it follows that $\delta G_a(x)$ and $\delta g_a(x)$ are bounded away from -1 and ∞ over all $a\in[1,a_*]$ and $x\in[0,a]$.

On the other hand, by the mean value theorem and Lemma 3.5,

$$x^{a-1}e^{-x}\leqslant G_a(x)\leqslant(x+b_a)^{a-1}e^{-x}\leqslant x^{a-1}e^{-x}(1+b_a/a)^{a-1}\leqslant x^{a-1}e^{-x}(1+b_{a_*})^{a_*-1}$$

for all $a\in[1,a_*]$ and $x\in[a,\infty)$.

Again by the mean value theorem and in view of (3.12),

$$g_a(x)\geqslant(x^{a-1}+C_a/a)e^{-x}\geqslant x^{a-1}e^{-x}$$

for all real $a \geqslant 2$ and $x > 0$. Also, once again by the mean value theorem,

$$g_a(x) \leqslant ((x+2)^{a-1} + \Gamma(a))e^{-x} \leqslant x^{a-1}e^{-x}((1+2/a)^{a-1} + \Gamma(a)/a^{a-1}) \leqslant x^{a-1}e^{-x}K$$

for some universal positive real constant K and all $a \in [2, a_*]$ and $x \in [a, \infty)$.

Finally, by Proposition 2.7 (to be proved next),

$$x^{a-1}e^{-x} \leqslant \Gamma(a,x) \leqslant ax^{a-1}e^{-x}$$

for all $a \in [1, \infty)$ and $x \in [a, \infty)$.

Collecting all the pieces together, we complete the proof of Proposition 2.6.

Proof of Proposition 2.7. For all real $x > 0$

$$\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}\,dt \geqslant x^{a-1}\int_x^\infty e^{-t}\,dt = x^{a-1}e^{-x},$$

which proves the first inequality in Proposition 2.7.

Next, for $t > x > a-1$, let $h(t) := (a-1)\ln t - t$. Since the function h is strictly concave, for $t > x$ we have

$$h(t) < h_x(t) := h(x) + h'(x)(t-x) = (a-1)\ln x - x + ((a-1)/x - 1)(t-x)$$

So,

$$\Gamma(a,x) = \int_x^\infty e^{h(t)}\,dt < \int_x^\infty e^{h_x(t)}\,dt = \frac{x^{a-1}e^{-x}}{1-(a-1)/x},$$

which proves the second inequality in Proposition 2.7 as well.

Proof of Proposition 2.8. Inequality (2.13) follows immediately from the discussion of forward-shift bounds in the paragraph containing formula (2.7) and the first inequality in (1.9). The exactness of the bound $g^{\mathsf{lo}}_{a;2}(x)$ on $\Gamma(a,x)$ at $x = \infty$ follows immediately from (2.12) and Lemma 3.7.

Proof of Proposition 2.9. Consider first the case $a \neq 0$, so that $a \in (-2,1) \setminus \{0\}$. Then inequality (2.16) follows immediately from equality (2.14) and inequalities (1.5) and (1.7), and the exactness of the bound $G_{a;-1}(x)$ on $\Gamma(a,x)$ at $x = 0$ follows by (2.14), (1.3), and Lemma 3.8.

To complete the proof of Proposition 2.9, consider now the case $a = 0$. The second equality in (2.15) can be obtained as follows. In view of (2.14) and (1.3), write $G_{a;-1}(x)$ as the ratio with denominator $a(a+1)b_{a+1}$; replace $(a+1)b_{a+1}$ in the denominator by $\lim_{a\to 0}(a+1)b_{a+1} = b_1$; finally, use l'Hospital's rule.

The exactness of the bound $G_{0;-1}(x)$ on $\Gamma(0,x)$ at $x = 0$ follows by (2.15) and Lemma 3.8. The non-strict version of inequality (2.16) for $a = 0$ follows by continuity from inequality (2.16) for $a \neq 0$.

However, the strict inequality (2.16) for $a = 0$ requires proof, which is somewhat similar to the proofs of inequalities in Theorems 1.1 and 1.2. Again, we are going to apply Proposition A, now to the functions $f=\Gamma(0,\cdot)$ and $g=G_{0;-1}$, where $G_{0;-1}$ is as in (2.15). Then for real $x>0$

$$\begin{aligned}
\rho(x) &:= \frac{g'(x)}{f'(x)} = x\Big(1+\frac{x-1}{b_1}\Big)\ln\frac{b_1+x}{x}-x+1,\\
\rho'(x) &= \frac{1}{b_1+x}-2+\frac{b_1+2x-1}{b_1}\ln\frac{b_1+x}{x},\\
\rho''(x) &= \frac{d_2(x)}{b_1x(b_1+x)^2},\\
d_2(x) &:= 2x(b_1+x)^2\ln\frac{b_1+x}{x}-b_1(b_1(3x-1)+b_1^2+2x^2),\\
\rho'''(x) &= b_1\,\frac{(b_1-1)b_1-x(3-b_1)}{x^2(b_1+x)^2}.
\end{aligned}$$

By Lemma 3.5, $1<b_1<2$. So, ρ''' is $+-$ on $(0,\infty)$ – that is, there is some $c\in[0,\infty]$ such that $\rho'''>0$ on $(0,c)$ and $\rho'''<0$ on (c,∞) (in this case, we actually have $c\in(0,\infty)$). So, ρ'' is increasing-decreasing on $(0,\infty)$.

Also, $x^3\rho''(x)\to(1-b_1/3)b_1>0$ as $x\to\infty$. So, ρ'' is $-+$ on $(0,\infty)$. So, ρ is strictly concave-convex on $(0,\infty)$. Also, $\rho(\infty-)=b_1/2\in\mathbb{R}$. So, by Lemma 3.1, ρ is increasing-decreasing on $(0,\infty)$. Also, $f(\infty-)=g(\infty-)=0$. Therefore, by part (ii) of Proposition A, $r=g/f$ is increasing-decreasing on $(0,\infty)$.

Also, for real $x>0$

$$\begin{aligned}
r'(x) &= \frac{d(x)}{b_1e^{2x}x\Gamma(0,x)^2},\\
d(x) &:= (b_1+x)(\ln(b_1+x)-\ln x)-b_1\\
&\quad +e^x\Gamma(0,x)(b_1(x-1)+x(1-b_1-x)(\ln(b_1+x)-\ln x)).
\end{aligned}$$

Making here the substitutions $\ln(b_1+x)=\ln b_1+c_1x$, $e^x=1+c_2x$, and, in accordance with (3.6) and (1.2), $\Gamma(0,x)=-\ln x-\gamma+c_3x=-\ln x+\ln b_1-1+c_3x$, where $c_j=c_j(x)=O(1)$ as $x\downarrow0$ for each $j\in\{1,2,3\}$, we see that $r'(0+)=1/b_1-1<0$. So, the increasing-decreasing function r is actually decreasing everywhere on $(0,\infty)$. Also, the already established exactness of the bound $G_{0;-1}(x)$ on $\Gamma(0,x)$ at $x=0$ means that $r(0+)=1$. Thus, $r<1$ on $(0,\infty)$; that is, inequality (2.16) holds for $a=0$.

This completes the proof of Proposition 2.9.

Proof of Proposition 2.10. This follows immediately from Proposition 2.4 (with $k=1$) and Theorem 1.1. More specifically, the case $1<a<2$ follows from part (ii) of Proposition 2.4 and (1.5); the case $a=2$ follows from parts (i) and (ii) of Proposition 2.4 and (1.6); and the case $2<a<3$ follows from part (i) of Proposition 2.4 and (1.7).

Proof of Proposition 2.11. The first inequality in (2.21) follows immediately from the first equality in (2.20), the first inequality in (1.9), and identity (2.5). The second inequality in (2.21) follows because

$$g_a^{\mathsf{up}}(x) - g_{a;1}^{\mathsf{lo}}(x) = \frac{x^{1+a}e^{-x}}{(x-a)\big((x-a)^2 - a + 2x\big)} > 0$$

for $a < 0$ and $x > 0$.

The exactness of the upper bound $g_{a;1}^{\mathsf{lo}}(x)$ on $\Gamma(a,x)$ at $x = 0$ and at $x = \infty$ follows immediately from inequalities (2.21) and the exactness of $g_a^{\mathsf{up}}(x)$ at $x = 0$ and at $x = \infty$.

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