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BILINEAR WEIGHTED HARDY–TYPE INEQUALITIES IN DISCRETE AND q–CALCULUS FRAMEWORKS

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Abstract. We characterize Hardy inequality in weighted Lebesgue sequence spaces involving discrete bilinear Hardy operator $\left(\sum_{i=-\infty}^{n} a_i\right)\left(\sum_{i=-\infty}^{n} b_i\right)$ and then we apply this information to characterize the inequality with q-bilinear Hardy operator

$$\mathscr{H}_q(f,g)(x) := \left(\int_0^\infty \chi_{(0,x]}(t)f(t)\,d_qt\right)\left(\int_0^\infty \chi_{(0,x]}(t)g(t)\,d_qt\right)$$

for all possible indices of summation.

1. Introduction

The weighted Hardy inequality

$$\left(\int_0^\infty (Hf(x))^s u(x)\,dx\right)^{1/s} \leqslant C\left(\int_0^\infty f^p(x)v(x)\,dx\right)^{1/p}, \quad f \geqslant 0, \tag{1}$$

where $Hf(x) = \int_0^x f(t)dt$ is the Hardy operator has been well settled now for all choices of indices p and s. A complete and comprehensive description of the development of such Hardy inequalities can be found in the books [12], [16], [17] and references therein.

The discrete version of the inequality (1) has the form

$$\left[\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n} a_i\right)^s u_n\right]^{\frac{1}{s}} \leqslant C\left(\sum_{n=1}^{\infty} a_n^p v_n\right)^{\frac{1}{p}}, \tag{2}$$

where $u = \{u_n\}, v = \{v_n\}, a = \{a_n\}, n \in \mathbb{N}$ are sequences of non-negative numbers and the constant C is the best possible. The characterization of the inequality (2) for various

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combinations of the indices p and s can be found in [2], [3], [4], [5], [11], [20, § 1.4] with classical results in [12] and historical review in [16]. For our purpose, we mention these results in Section 2.

Recently, Cañestro et. al. [6] and Krepela [15] considered the bilinear Hardy operator

$$H_2(f,g)(x) = Hf(x)\cdot Hg(x)$$

and characterized the corresponding inequality

$$\left(\int_0^\infty (H_2(f,g)(x))^s u(x)\,dx\right)^{1/s} \leqslant C\left(\int_0^\infty f^{p_1}(x)v_1(x)\,dx\right)^{1/p_1} \times \left(\int_0^\infty g^{p_2}(x)v_2(x)\,dx\right)^{1/p_2}, \quad f,g \geqslant 0 \tag{3}$$

for different combinations of indices p_1, p_2, s. Let us mention that in very recent papers [10], [14] several scales of equivalent conditions for the inequality (3) have been obtained as well as [22] and a short survey [23] devoted to the subject.

One of the aims of this paper is to make a systematic study of the discrete version of (3). To this end we study first the inequality

$$\left[\sum_{n=-\infty}^{\infty}\left(\sum_{i=-\infty}^{n} a_i\right)^s\left(\sum_{i=-\infty}^{n} b_i\right)^s u_n\right]^{\frac{1}{s}} \leqslant C\left(\sum_{n=-\infty}^{\infty} a_n^{p_1} v_n\right)^{\frac{1}{p_1}}\left(\sum_{n=-\infty}^{\infty} b_n^{p_2} w_n\right)^{\frac{1}{p_2}} \tag{4}$$

for all possible cases of parameters $p_1, p_2, s \in (0,\infty)$.

During the last decade, a lot of interest has been developed by many authors to investigate q-calculus. Tremendous amount of papers have been published, many books have been written and the heat is still on. The notion of q-calculus, sometimes also regarded as calculus without limits, was initiated by F.H. Jackson [13] (see also [7]) who defined derivative and integral in the framework of q-calculus. This notion has variety of applications, not only in mathematical sciences, but also in other sciences and engineering. Here, we refer to the books [7], [8], [9], [21] for development and applications in q-calculus. In recent papers [1], [18], those authors characterized the Hardy inequality (1) in the framework of q-calculus. Our next aim, in this paper, is to study the q-analogue of the bilinear Hardy inequality (3).

The paper is organized as follows: Section 2 contains preliminary information required for rest of the paper. Here, we collect various characterizations of the inequality (2) in all possible available cases. In Section 3, we provide results concerning the discrete bilinear inequality (4) for all the possible cases. In Section 4, we provide brief description about basics of q-calculus and prove a couple of lemmas that will be used in Section 5 where we prove results for the q-analogue of the inequality (3).

Throughout the article, products of the form $0\cdot\infty$ are assumed to be equal to 0. The sign $A \lesssim B$ means $A \leqslant cB$ with an insignificant constant c; $A \approx B$ means that $A \lesssim B \lesssim A$ and $A \cong B$ stands for $A = cB$. Also $\mathbb{Z}$ denotes the set of all integers, and χ_E denotes the characteristic function (indicator) of a set $E \subset (0,\infty)$. We use the symbols $:=$ and $=:$ for definition of new quantities. If $1 \leqslant p \leqslant \infty$, then $p' := \frac{p}{p-1}$ for $1 < p < \infty$, $p' := \infty$ for $p = 1$ and $p' := 1$ for $p = \infty$ and we use *iff* as *if and only if*.

2. Preliminaries

Let $u=\{u_n\}, v=\{v_n\}, a=\{a_n\}, n\in\mathbb{N}$ be sequences of non-negative numbers.

THEOREM A. *Let* $0<p,s<\infty$. *Then the inequality* (2) *holds if and only if*
(i) *If* $1<p\leqslant s<\infty$, $A_1<\infty$, *where*

$$A_1 := \sup_{n\in\mathbb{N}} \left(\sum_{i=n}^{\infty} u_i\right)^{\frac{1}{s}} \left(\sum_{i=1}^{n} v_i^{1-p'}\right)^{\frac{1}{p'}}. \tag{5}$$

(ii) *If* $0<p\leqslant 1, p\leqslant s<\infty$, $A_2<\infty$, *where*

$$A_2 := \sup_{n\in\mathbb{N}} \left(\sum_{i=n}^{\infty} u_i\right)^{\frac{1}{s}} v_n^{-\frac{1}{p}}. \tag{6}$$

(iii) *If* $0<s<p$, $p>1$, $\frac{1}{r}:=\frac{1}{s}-\frac{1}{p}$, $A_3<\infty$, *where*

$$A_3 := \left\{\sum_{n=1}^{\infty} \left(\sum_{i=n}^{\infty} u_i\right)^{\frac{r}{s}} \left(\sum_{i=1}^{n} v_i^{1-p'}\right)^{\frac{r}{s'}} v_n^{1-p'}\right\}^{\frac{1}{r}}, \tag{7}$$

$$A_3 \approx \tilde{A}_3 := \left\{\sum_{n=1}^{\infty} \left(\sum_{i=n}^{\infty} u_i\right)^{\frac{r}{p}} \left(\sum_{i=1}^{n} v_i^{1-p'}\right)^{\frac{r}{p'}} u_n\right\}^{\frac{1}{r}} \tag{8}$$

with suitable modification for $s=1$.
(iv) *If* $0<s<p\leqslant 1$, $\frac{1}{r}:=\frac{1}{s}-\frac{1}{p}$, $A_4<\infty$, *where*

$$A_4 := \left\{\sum_{n=1}^{\infty} u_n \left(\sum_{i=n}^{\infty} u_i\right)^{\frac{r}{p}} \max_{1\leqslant i\leqslant n} v_i^{-\frac{r}{p}}\right\}^{\frac{1}{r}}. \tag{9}$$

Moreover, $C\approx A_i, i=1,2,3,4$.

For the dual discrete Hardy inequality of the form

$$\left[\sum_{n=1}^{\infty} \left(\sum_{i=n}^{\infty} a_i\right)^{s} u_n\right]^{\frac{1}{s}} \leqslant C \left(\sum_{n=1}^{\infty} a_n^p v_n\right)^{\frac{1}{p}} \tag{10}$$

we have the following.

THEOREM B. *Let* $0<p,s<\infty$. *Then the inequality* (10) *holds iff*
(i) *If* $1<p\leqslant s<\infty$, $B_1<\infty$, *where*

$$B_1 := \sup_{n\in\mathbb{N}} \left(\sum_{i=1}^{n} u_i\right)^{\frac{1}{s}} \left(\sum_{i=n}^{\infty} v_i^{1-p'}\right)^{\frac{1}{p'}}. \tag{11}$$

(ii) *If* $0 < p \leqslant 1, p \leqslant s < \infty,\ B_2 < \infty$, *where*

$$B_2 := \sup_{n\in\mathbb{N}} \left(\sum_{i=1}^{n} u_i\right)^{\frac{1}{s}} v_n^{-\frac{1}{p}}.$$

(iii) *If* $0 < s < p, p > 1,\ \frac{1}{r} := \frac{1}{s} - \frac{1}{p},\ B_3 < \infty$, *where*

$$B_3 := \left\{\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n} u_i\right)^{\frac{r}{s}} \left(\sum_{i=n}^{\infty} v_i^{1-p'}\right)^{\frac{r}{s'}} v_n^{1-p'}\right\}^{\frac{1}{r}},$$

$$B_3 \approx \tilde{B}_3 := \left\{\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n} u_i\right)^{\frac{r}{p}} \left(\sum_{i=n}^{\infty} v_i^{1-p'}\right)^{\frac{r}{p'}} u_n\right\}^{\frac{1}{r}}. \tag{12}$$

(iv) *If* $0 < s < p \leqslant 1,\ \frac{1}{r} := \frac{1}{s} - \frac{1}{p}$, *then* $B_4 < \infty$, *where*

$$B_4 := \left\{\sum_{n=1}^{\infty} u_n \left(\sum_{i=1}^{n} u_i\right)^{\frac{r}{p}} \max_{i\geqslant n} v_i^{-\frac{r}{p}}\right\}^{\frac{1}{r}}.$$

Moreover, $C \approx B_i, i = 1,2,3,4.$

Theorem A(i) is proved in ([2], Theorem 2) and ([4], Theorem 1(v)(b)). Theorem A(ii) is proved in ([4], Theorem 1(iv)). Theorem A(iii) is proved in ([4], Theorem 1(viii)) and [5]. And Theorem A(iv) is proved in ([4], Theorem 1(vii)) for $p = 1$, ([11], Theorem 9.2) and ([20], Theorem 1.9). The proof of Theorem B is analogous.

We also need the reverse Hölder inequalities for the weighted sequence ℓ_p-spaces, $1 < p < \infty$:

$$\left(\sum_{n=1}^{\infty} d_n^p z_n\right)^{\frac{1}{p}} = \sup_h \left(\sum_{n=1}^{\infty} d_n h_n\right) \left(\sum_{n=1}^{\infty} h_n^{p'} z_n^{1-p'}\right)^{-\frac{1}{p'}}, \tag{13}$$

$$\left(\sum_{n=1}^{\infty} d_n^p z_n\right)^{\frac{1}{p}} = \sup_h \left(\sum_{n=1}^{\infty} d_n h_n z_n\right) \left(\sum_{n=1}^{\infty} h_n^{p'} z_n\right)^{-\frac{1}{p'}}. \tag{14}$$

We also make use of the following (see [3], Lemma 2 and Lemma 3):

LEMMA C. *Let* $r > 0,\ 1 \leqslant n < N \leqslant \infty$. *Then*

$$\sum_{k=n}^{N} a_k \left(\sum_{j=k}^{N} a_j\right)^{r-1} \approx \left(\sum_{i=n}^{N} a_i\right)^{r} \approx \sum_{k=n}^{N} a_k \left(\sum_{j=n}^{k} a_j\right)^{r-1}.$$

REMARK 1. Without any loss of generality, in Theorems A and B, the summations $\sum_{n=1}^{\infty}$ and $\sum_{i=1}^{n}$ can be replaced by $\sum_{n=-\infty}^{\infty}$ and $\sum_{i=-\infty}^{n}$, respectively. The same is true for (13) and (14).

3. Discrete bilinear Hardy inequality

Let $0 < p_1, p_2, s < \infty$, $u = \{u_n\}, v = \{v_n\}, w = \{w_n\}, n \in \mathbb{Z}$ be sequences of non-negative numbers. We study the inequality

$$\left[\sum_{n=-\infty}^{\infty}\left(\sum_{i=-\infty}^{n} a_i\right)^s \left(\sum_{i=-\infty}^{n} b_i\right)^s u_n\right]^{\frac{1}{s}} \leqslant C\left(\sum_{n=-\infty}^{\infty} a_n^{p_1} v_n\right)^{\frac{1}{p_1}}\left(\sum_{n=-\infty}^{\infty} b_n^{p_2} w_n\right)^{\frac{1}{p_2}} \tag{15}$$

with arbitrary non-negative sequences $a = \{a_n\}$ and $b = \{b_n\}, n \in \mathbb{Z}$ and the best possible constant C.

The problem is divided for the next zones of parameters p_1, p_2 and s.

I_1. $1 < \min(p_1,p_2) \leqslant \max(p_1,p_2) \leqslant s < \infty$,
I_2. $0 < \min(p_1,p_2) \leqslant 1 < \max(p_1,p_2) \leqslant s < \infty$,
I_3. $0 < \max(p_1,p_2) \leqslant \min(1,s) < \infty$,
II_1. $1 < \min(p_1,p_2) \leqslant s < \max(p_1,p_2) < \infty$,
II_2. $0 < \min(p_1,p_2) \leqslant \min(1,s) \leqslant 1 < \max(p_1,p_2) < \infty$,
II_3. $0 < \min(p_1,p_2) \leqslant s < \max(p_1,p_2) \leqslant 1$,
III_1. $0 < s < \min(p_1,p_2)$, $\min(p_1,p_2) > 1$,
III_2. $0 < s < \min(p_1,p_2) \leqslant 1 < \max(p_1,p_2) < \infty$,
IV. $0 < s < \min(p_1,p_2) \leqslant \max(p_1,p_2) \leqslant 1$.

Each of the following theorems characterizes the related case of the inequality (15).

THEOREM 1. *(Case I_1.) Let $1 < p_1, p_2 \leqslant s$. Then $C \approx \mathscr{A}_1$, where*

$$\mathscr{A}_1 := \sup_{n\in\mathbb{Z}}\left(\sum_{i=n}^{\infty} u_i\right)^{\frac{1}{s}}\left(\sum_{i=-\infty}^{n} v_i^{1-p_1'}\right)^{\frac{1}{p_1'}}\left(\sum_{i=-\infty}^{n} w_i^{1-p_2'}\right)^{\frac{1}{p_2'}}.$$

Proof. Denote $U_n := \left(\sum_{i=-\infty}^{n} b_i\right)^s u_n$, $\|a\|_{p_1} := \left(\sum_{n=-\infty}^{\infty} a_n^{p_1} v_n\right)^{\frac{1}{p_1}}$, $\|b\|_{p_2} := \left(\sum_{n=-\infty}^{\infty} b_n^{p_2} w_n\right)^{\frac{1}{p_2}}$. We have, according to Theorem A(i),

$$\begin{aligned}
C &= \sup_b \sup_a \left[\sum_{n=-\infty}^{\infty}\left(\sum_{i=-\infty}^{n} a_i\right)^s\left(\sum_{i=-\infty}^{n} b_i\right)^s u_n\right]^{\frac{1}{s}} \|a\|_{p_1}^{-1}\|b\|_{p_2}^{-1} \\
&= \sup_b \|b\|_{p_2}^{-1}\left\{\sup_a \left[\sum_{n=-\infty}^{\infty}\left(\sum_{i=-\infty}^{n} a_i\right)^s U_n\right]^{\frac{1}{s}} \|a\|_{p_1}^{-1}\right\} \\
&\overset{(5)}{\approx} \sup_b \|b\|_{p_2}^{-1} \sup_{n\in\mathbb{Z}}\left(\sum_{i=n}^{\infty} U_i\right)^{\frac{1}{s}}\left(\sum_{i=-\infty}^{n} v_i^{1-p_1'}\right)^{\frac{1}{p_1'}}
\end{aligned}$$

$$= \sup_{n\in\mathbb{Z}} \left(\sum_{i=-\infty}^{n} v_i^{1-p_1'}\right)^{\frac{1}{p_1'}} \sup_b \left[\sum_{i=n}^{\infty}\left(\sum_{j=-\infty}^{i} b_j\right)^s u_i\right]^{\frac{1}{s}} \|b\|_{p_2}^{-1}$$

$$\overset{(5)}{\approx} \sup_{n\in\mathbb{Z}} \left(\sum_{i=-\infty}^{n} v_i^{1-p_1'}\right)^{\frac{1}{p_1'}} \sup_{m\geqslant n} \left(\sum_{i=m}^{\infty} u_i\right)^{\frac{1}{s}} \left(\sum_{i=-\infty}^{m} w_i^{1-p_2'}\right)^{\frac{1}{p_2'}}$$

$$= \sup_{n\in\mathbb{Z}} \left(\sum_{i=n}^{\infty} u_i\right)^{\frac{1}{s}} \left(\sum_{i=-\infty}^{n} v_i^{1-p_1'}\right)^{\frac{1}{p_1'}} \left(\sum_{i=-\infty}^{n} w_i^{1-p_2'}\right)^{\frac{1}{p_2'}} = \mathscr{A}_1.$$

THEOREM 2. *(Case I_2.) Let* $0 < \min(p_1, p_2) \leqslant 1 < \max(p_1, p_2) \leqslant s < \infty$. *Then* $C \approx \mathscr{A}_2$, *where*

(a) *If* $0 < p_1 \leqslant 1 < p_2 \leqslant s$, *then*

$$\mathscr{A}_2 := \sup_{n\in\mathbb{Z}} \left(\sum_{i=n}^{\infty} u_i\right)^{\frac{1}{s}} \left(\sum_{i=-\infty}^{n} w_i^{1-p_2'}\right)^{\frac{1}{p_2'}} \sup_{k\leqslant n} v_k^{-\frac{1}{p_1}}.$$

(b) *If* $0 < p_2 \leqslant 1 < p_1 \leqslant s$, *then*

$$\mathscr{A}_2 := \sup_{n\in\mathbb{Z}} \left(\sum_{i=n}^{\infty} u_i\right)^{\frac{1}{s}} \left(\sum_{i=-\infty}^{n} v_i^{1-p_1'}\right)^{\frac{1}{p_1'}} \sup_{k\leqslant n} w_k^{-\frac{1}{p_2}}.$$

Proof. (a) Using the similar arguments as in the previous Theorem 1, we have

$$C = \sup_b \|b\|_{p_2}^{-1} \left\{ \sup_a \left[\sum_{n=-\infty}^{\infty}\left(\sum_{i=-\infty}^{n} a_i\right)^s U_n\right]^{\frac{1}{s}} \|a\|_{p_1}^{-1} \right\}$$

$$\overset{(6)}{\approx} \sup_{n\in\mathbb{Z}} v_n^{-\frac{1}{p_1}} \sup_b \left[\sum_{i=n}^{\infty}\left(\sum_{j=-\infty}^{i} b_j\right)^s u_i\right]^{\frac{1}{s}} \|b\|_{p_2}^{-1}$$

$$\overset{(5)}{\approx} \sup_{n\in\mathbb{Z}} v_n^{-\frac{1}{p_1}} \sup_{m\geqslant n} \left(\sum_{i=m}^{\infty} u_i\right)^{\frac{1}{s}} \left(\sum_{i=-\infty}^{m} w_i^{1-p_2'}\right)^{\frac{1}{p_2'}} = \mathscr{A}_2.$$

Similar for the case (b).

Analogously, we obtain the following.

THEOREM 3. *(Case I_3.) Let* $0 < \max(p_1, p_2) \leqslant \min(1, s) < \infty$. *Then* $C \approx \mathscr{A}_3$, *where*

$$\mathscr{A}_3 := \sup_{n\in\mathbb{Z}} v_n^{-\frac{1}{p_1}} \left(\sum_{j=n}^{\infty} u_j\right)^{\frac{1}{s}} \sup_{i\leqslant n} w_i^{-\frac{1}{p_2}} + \sup_{n\in\mathbb{Z}} w_n^{-\frac{1}{p_2}} \left(\sum_{j=n}^{\infty} u_j\right)^{\frac{1}{s}} \sup_{i\leqslant n} v_i^{-\frac{1}{p_1}}.$$

THEOREM 4. *(Case II_1.) Let* $1 < \min(p_1, p_2) \leqslant s < \max(p_1, p_2) < \infty$, $\frac{1}{r_i} = \frac{1}{s} - \frac{1}{p_i}, i = 1, 2$. *Then* $C \approx \mathscr{A}_4$, *where*

(a) *If* $1 < p_1 \leqslant s < p_2$, *then*

$$\mathscr{A}_4 := \sup_{n\in\mathbb{Z}} \left(\sum_{l=-\infty}^{n} v_l^{1-p_1'}\right)^{\frac{1}{p_1'}} \left(\sum_{i=n}^{\infty} u_i \left(\sum_{k=i}^{\infty} u_k\right)^{\frac{r_2}{p_2}} \left(\sum_{j=-\infty}^{i} w_j^{1-p_2'}\right)^{\frac{r_2}{p_2'}}\right)^{\frac{1}{r_2}}.$$

(b) *If* $1 < p_2 \leqslant s < p_1$, *then*

$$\mathscr{A}_4 := \sup_{n\in\mathbb{Z}} \left(\sum_{l=-\infty}^{n} w_l^{1-p_2'}\right)^{\frac{1}{p_2'}} \left(\sum_{i=n}^{\infty} u_i \left(\sum_{k=i}^{\infty} u_k\right)^{\frac{r_1}{p_1}} \left(\sum_{j=-\infty}^{i} v_j^{1-p_1'}\right)^{\frac{r_1}{p_1'}}\right)^{\frac{1}{r_1}}.$$

Proof. (a) We have

$$\begin{aligned} C &= \sup_b \|b\|_{p_2}^{-1} \left\{ \sup_a \left[\sum_{n=-\infty}^{\infty} \left(\sum_{i=-\infty}^{n} a_i\right)^s U_n\right]^{\frac{1}{s}} \|a\|_{p_1}^{-1} \right\} \\ &\overset{(5)}{\approx} \sup_{n\in\mathbb{Z}} \left(\sum_{i=-\infty}^{n} v_i^{1-p_1'}\right)^{\frac{1}{p_1'}} \sup_b \left[\sum_{i=n}^{\infty} \left(\sum_{j=-\infty}^{i} b_j\right)^s u_i\right]^{\frac{1}{s}} \|b\|_{p_2}^{-1} \\ &\overset{(8)}{\approx} \sup_{n\in\mathbb{Z}} \left(\sum_{l=-\infty}^{n} v_l^{1-p_1'}\right)^{\frac{1}{p_1'}} \left(\sum_{i=n}^{\infty} u_i \left(\sum_{k=i}^{\infty} u_k\right)^{\frac{r_2}{p_2}} \left(\sum_{j=-\infty}^{i} w_j^{1-p_2'}\right)^{\frac{r_2}{p_2'}}\right)^{\frac{1}{r_2}} = \mathscr{A}_4. \end{aligned}$$

Similar for the case (b).

Analogously, we prove the following.

THEOREM 5. *(Case II_2.) Let* $0 < \min(p_1, p_2) \leqslant \min(1, s) \leqslant 1 < \max(p_1, p_2) < \infty$, $\frac{1}{r_i} := \frac{1}{s} - \frac{1}{p_i}, i = 1, 2$. *Then* $C \approx \mathscr{A}_5$, *where*

(a) *If* $0 < p_1 \leqslant s \leqslant 1 < p_2$, *then*

$$\mathscr{A}_5 := \sup_{n\in\mathbb{Z}} v_n^{-\frac{1}{p_1}} \left(\sum_{i=n}^{\infty} u_i \left(\sum_{k=i}^{\infty} u_k\right)^{\frac{r_2}{p_2}} \left(\sum_{j=-\infty}^{i} w_j^{1-p_2'}\right)^{\frac{r_2}{p_2'}}\right)^{\frac{1}{r_2}}.$$

(b) *If* $0 < p_2 \leqslant s \leqslant 1 < p_1$, *then*

$$\mathscr{A}_5 := \sup_{n\in\mathbb{Z}} w_n^{-\frac{1}{p_2}} \left(\sum_{i=n}^{\infty} u_i \left(\sum_{k=i}^{\infty} u_k\right)^{\frac{r_1}{p_1}} \left(\sum_{j=-\infty}^{i} v_j^{1-p_1'}\right)^{\frac{r_1}{p_1'}}\right)^{\frac{1}{r_1}}.$$

THEOREM 6. *(Case II_3.) Let* $0 < \min(p_1, p_2) \leqslant s < \max(p_1, p_2) \leqslant 1$, $\frac{1}{r_i} := \frac{1}{s} - \frac{1}{p_i}, i = 1,2$. *Then* $C \approx \mathscr{A}_6$, *where*

(a) *If* $0 < p_1 \leqslant s < p_2 \leqslant 1$, *then*

$$\mathscr{A}_6 := \sup_{n\in\mathbb{Z}} v_n^{-\frac{1}{p_1}} \left(\sum_{i=n}^{\infty} u_i \left(\sum_{k=i}^{\infty} u_k \right)^{\frac{r_2}{p_2}} \max_{j\leqslant i} w_j^{-\frac{r_2}{p_2}} \right)^{\frac{1}{r_2}}.$$

(b) *If* $0 < p_2 \leqslant s < p_1 \leqslant 1$, *then*

$$\mathscr{A}_6 := \sup_{n\in\mathbb{Z}} w_n^{-\frac{1}{p_2}} \left(\sum_{i=n}^{\infty} u_i \left(\sum_{k=i}^{\infty} u_k \right)^{\frac{r_1}{p_1}} \max_{j\leqslant i} v_j^{-\frac{r_1}{p_1}} \right)^{\frac{1}{r_1}}.$$

Proof. (a) We have

$$\begin{aligned} C &= \sup_b \|b\|_{p_2}^{-1} \left\{ \sup_a \left[\sum_{n=-\infty}^{\infty} \left(\sum_{i=-\infty}^{n} a_i \right)^s U_n \right]^{\frac{1}{s}} \|a\|_{p_1}^{-1} \right\} \\ &\overset{(6)}{\approx} \sup_{n\in\mathbb{Z}} v_n^{-\frac{1}{p_1}} \sup_b \left[\sum_{i=n}^{\infty} \left(\sum_{j=-\infty}^{i} b_j \right)^s u_i \right]^{\frac{1}{s}} \|b\|_{p_2}^{-1} \\ &\overset{(9)}{\approx} \sup_{n\in\mathbb{Z}} v_n^{-\frac{1}{p_1}} \left(\sum_{i=n}^{\infty} u_i \left(\sum_{k=i}^{\infty} u_k \right)^{\frac{r_2}{p_2}} \max_{j\leqslant i} w_j^{-\frac{r_2}{p_2}} \right)^{\frac{1}{r_2}} = \mathscr{A}_6. \end{aligned}$$

Similar for the case (b).

Let

$$\mathscr{V}_n := \max_{j\leqslant n} v_j^{-\frac{r_1}{p_1}},\ \mathscr{V}_0 := 0,\ \mathscr{W}_n := \max_{j\leqslant n} w_j^{-\frac{r_2}{p_2}},\ \mathscr{W}_0 := 0,$$

then

$$\mathscr{V}_n = \sum_{j=-\infty}^{n} (\mathscr{V}_j - \mathscr{V}_{j-1}) =: \sum_{j=-\infty}^{n} \tilde{v}_j,\ \mathscr{W}_n = \sum_{j=-\infty}^{n} (\mathscr{W}_j - \mathscr{W}_{j-1}) =: \sum_{j=-\infty}^{n} \tilde{w}_j.$$

Also, let

$$V_n := v_n^{1-p_1'} \left(\sum_{k=-\infty}^{n} v_k^{1-p_1'} \right)^{\frac{r_1}{s'}},\ W_n := w_n^{1-p_2'} \left(\sum_{k=-\infty}^{n} w_k^{1-p_2'} \right)^{\frac{r_2}{s'}},\ \tilde{u}_n := \sum_{k=n}^{\infty} u_k.$$

THEOREM 7. *(Case III_1.) Let* $0 < s < \min(p_1, p_2)$, $\min(p_1, p_2) > 1$, $\frac{1}{r_i} := \frac{1}{s} - \frac{1}{p_i}, i = 1,2$. *Then* $C \approx \mathscr{A}_{7.1} + \mathscr{A}_{7.2}$ *in case* (a) *and* $C \approx \mathscr{B}_{7.1} + \mathscr{B}_{7.2}$ *in case* (b)*, where*

(a) *If* $0 < s < p_1 < p_2 < \infty,\ p_1 > 1,$ *then for* $\frac{1}{s} \leqslant \frac{1}{p_1} + \frac{1}{p_2}$,

$$\mathscr{A}_{7.1} := \sup_{n\in\mathbb{Z}} \left(\sum_{k=n}^{\infty} W_k \tilde{u}_k^{\frac{r_2}{s}}\right)^{\frac{1}{r_2}} \left(\sum_{j=-\infty}^{n} V_j\right)^{\frac{1}{r_1}}, \mathscr{A}_{7.2} := \sup_{n\in\mathbb{Z}} \left(\sum_{k=-\infty}^{n} W_k\right)^{\frac{1}{r_2}} \left(\sum_{j=n}^{\infty} \tilde{u}_j^{\frac{r_1}{s}} V_j\right)^{\frac{1}{r_1}}.$$

For $\frac{1}{s} > \frac{1}{p_1} + \frac{1}{p_2}$,

$$\mathscr{A}_{7.1} := \left(\sum_{n=-\infty}^{\infty} W_n \tilde{u}_n^{\frac{r_2}{s}} \left(\sum_{k=n}^{\infty} W_k \tilde{u}_k^{\frac{r_2}{s}}\right)^{\frac{r_2}{p_1-r_2}} \left(\sum_{j=-\infty}^{n} V_j\right)^{\frac{p_1 r_2}{r_1(p_1-r_2)}}\right)^{\frac{1}{r_2}-\frac{1}{p_1}},$$

$$\mathscr{A}_{7.2} := \left(\sum_{n=-\infty}^{\infty} W_n \left(\sum_{k=-\infty}^{n} W_k\right)^{\frac{r_2}{p_1-r_2}} \left(\sum_{j=n}^{\infty} \tilde{u}_j^{\frac{r_1}{p_1}} V_j\right)^{\frac{p_1 r_2}{r_1(p_1-r_2)}}\right)^{\frac{1}{r_2}-\frac{1}{p_1}}.$$

(b) *If* $0 < s < p_2 < p_1 < \infty,\ p_2 > 1,$ *then for* $\frac{1}{s} \leqslant \frac{1}{p_1} + \frac{1}{p_2}$,

$$\mathscr{B}_{7.1} := \sup_{n\in\mathbb{Z}} \left(\sum_{k=n}^{\infty} V_k \tilde{u}_k^{\frac{r_1}{s}}\right)^{\frac{1}{r_1}} \left(\sum_{j=-\infty}^{n} W_j\right)^{\frac{1}{r_2}}, \mathscr{B}_{7.2} := \sup_{n\in\mathbb{Z}} \left(\sum_{k=-\infty}^{n} V_k\right)^{\frac{1}{r_1}} \left(\sum_{j=n}^{\infty} \tilde{u}_j^{\frac{r_2}{s}} W_j\right)^{\frac{1}{r_2}}.$$

For $\frac{1}{s} > \frac{1}{p_1} + \frac{1}{p_2}$,

$$\mathscr{B}_{7.1} := \left(\sum_{n=-\infty}^{\infty} V_n \tilde{u}_n^{\frac{r_1}{s}} \left(\sum_{k=n}^{\infty} V_k \tilde{u}_k^{\frac{r_1}{s}}\right)^{\frac{r_1}{p_2-r_1}} \left(\sum_{j=-\infty}^{n} W_j\right)^{\frac{p_2 r_1}{r_2(p_2-r_1)}}\right)^{\frac{1}{r_1}-\frac{1}{p_2}},$$

$$\mathscr{B}_{7.2} := \left(\sum_{n=-\infty}^{\infty} V_n \left(\sum_{k=-\infty}^{n} V_k\right)^{\frac{r_1}{p_2-r_1}} \left(\sum_{j=n}^{\infty} \tilde{u}_j^{\frac{r_2}{p_2}} W_j\right)^{\frac{p_2 r_1}{r_2(p_2-r_1)}}\right)^{\frac{1}{r_1}-\frac{1}{p_2}}.$$

Proof. (a) We have

$$C = \sup_b \|b\|_{p_2}^{-1} \left\{ \sup_a \left[\sum_{n=-\infty}^{\infty} \left(\sum_{i=-\infty}^{n} a_i\right)^s U_n\right]^{\frac{1}{s}} \|a\|_{p_1}^{-1} \right\}$$

$$\overset{(7)}{\approx} \sup_b \left(\sum_{n=-\infty}^{\infty} \left(\sum_{k=n}^{\infty} U_k\right)^{\frac{r_1}{s}} v_n^{1-p_1'} \left(\sum_{j=-\infty}^{n} v_j^{1-p_1'}\right)^{\frac{r_1}{s'}}\right)^{\frac{1}{r_1}} \|b\|_{p_2}^{-1}$$

$$= \sup_b \left(\sum_{n=-\infty}^{\infty} \left(\sum_{k=n}^{\infty} \left(Hb(k)\right)^s u_k\right)^{\frac{r_1}{s}} V_n\right)^{\frac{1}{r_1}} \|b\|_{p_2}^{-1},$$

where $Hb(n) := \sum_{k=-\infty}^{n} b_k$. By duality, for $\frac{r_1}{s} > 1$,

$$C \overset{(7)}{\approx} \sup_b \left[\sup_h \sum_{n=-\infty}^{\infty} \sum_{k=n}^{\infty} \Big(Hb(k)\Big)^s u_k h_n V_n \left(\sum_{j=-\infty}^{\infty} h_j^{\frac{r_1}{r_1-s}} V_j \right)^{\frac{s-r_1}{r_1}} \right]^{\frac{1}{s}} \|b\|_{p_2}^{-1}$$

$$= \sup_b \left[\sup_h \sum_{k=-\infty}^{\infty} \Big(Hb(k)\Big)^s u_k \left(\sum_{n=-\infty}^{k} h_n V_n \right) \left(\sum_{j=-\infty}^{\infty} h_j^{\frac{r_1}{r_1-s}} V_j \right)^{\frac{s-r_1}{r_1}} \right]^{\frac{1}{s}}$$

$$\|b\|_{p_2}^{-1} = \sup_h \left(\sum_{j=-\infty}^{\infty} h_j^{\frac{p_1}{s}} V_j \right)^{-\frac{1}{p_1}} \sup_b \left(\sum_{n=-\infty}^{\infty} \Big(Hb(n)\Big)^s u_n \tilde{H}h(n) \right)^{\frac{1}{s}} \|b\|_{p_2}^{-1},$$

where $\tilde{H}h(n) = \sum_{j=-\infty}^{n} h_j V_j$ and because $\frac{r_1}{r_1-s} = \frac{p_1}{s}$. Then

$$C \approx \sup_h \left(\sum_{j=-\infty}^{\infty} h_j^{\frac{p_1}{s}} V_j \right)^{-\frac{1}{p_1}} \left(\sum_{n=-\infty}^{\infty} \left(\sum_{k=n}^{\infty} u_k \tilde{H}h(k) \right)^{\frac{r_2}{s}} W_n \right)^{\frac{1}{r_2}}.$$

Since $\sum_{k=n}^{\infty} u_k \tilde{H}h(k) \approx \tilde{u}_n \tilde{H}h(n) + \sum_{j=n}^{\infty} h_j V_j \tilde{u}_j$, then $C \approx J_1 + J_2$, where

$$J_1^s := \sup_h \left(\sum_{n=-\infty}^{\infty} \Big(\tilde{H}h(n)\Big)^{\frac{r_2}{s}} \tilde{u}_n^{\frac{r_2}{s}} W_n \right)^{\frac{s}{r_2}} \left(\sum_{j=-\infty}^{\infty} h_j^{\frac{p_1}{s}} V_j \right)^{-\frac{s}{p_1}},$$

$$J_2^s := \sup_h \left(\sum_{n=-\infty}^{\infty} \left(\sum_{j=n}^{\infty} h_j V_j \tilde{u}_j \right)^{\frac{r_2}{s}} W_n \right)^{\frac{s}{r_2}} \left(\sum_{j=-\infty}^{\infty} h_j^{\frac{p_1}{s}} V_j \right)^{-\frac{s}{p_1}}.$$

We have two cases

$$\text{(i)} \quad \frac{1}{s} \leqslant \frac{1}{p_1} + \frac{1}{p_2} \Leftrightarrow 1 < \frac{p_1}{s} \leqslant \frac{r_2}{s},$$

$$\text{(ii)} \quad \frac{1}{s} > \frac{1}{p_1} + \frac{1}{p_2} \Leftrightarrow \frac{p_1}{s} > \frac{r_2}{s} > 1.$$

Since $\frac{p_1}{s} > 1$, in case (i) we apply (5) and (11), then

$$J_1 = \sup_{n\in\mathbb{Z}} \left(\sum_{k=n}^{\infty} W_k \tilde{u}_k^{\frac{r_2}{s}} \right)^{\frac{1}{r_2}} \left(\sum_{j=-\infty}^{n} V_j \right)^{\frac{1}{r_1}} = \mathscr{A}_{7.1},$$

$$J_2 = \sup_{n\in\mathbb{Z}} \left(\sum_{k=-\infty}^{n} W_k \right)^{\frac{1}{r_2}} \left(\sum_{j=n}^{\infty} \tilde{u}_j^{\frac{r_1}{s}} V_j \right)^{\frac{1}{r_1}} = \mathscr{A}_{7.2}.$$

In the case (ii) we apply (8) and (12):

$$J_1 = \left(\sum_{n=-\infty}^{\infty} W_n \tilde{u}_n^{\frac{r_2}{s}} \left(\sum_{k=n}^{\infty} W_k \tilde{u}_k^{\frac{r_2}{s}} \right)^{\frac{r_2}{p_1-r_2}} \left(\sum_{j=-\infty}^{n} V_j \right)^{\frac{p_1 r_2}{r_1(p_1-r_2)}} \right)^{\frac{1}{r_2}-\frac{1}{p_1}} = \mathscr{A}_{7.1},$$

$$J_2 = \left(\sum_{n=-\infty}^{\infty} W_n \left(\sum_{k=-\infty}^{n} W_k \right)^{\frac{r_2}{p_1-r_2}} \left(\sum_{j=n}^{\infty} \tilde{u}_j^{\frac{r_1}{p_1}} V_j \right)^{\frac{p_1 r_2}{r_1(p_1-r_2)}} \right)^{\frac{1}{r_2}-\frac{1}{p_1}} = \mathscr{A}_{7.2}.$$

Similar for the case (b).

THEOREM 8. *(Case III_2.) Let* $0 < s < \min(p_1, p_2) \leqslant 1 < \max(p_1, p_2)$, $\frac{1}{r_i} = \frac{1}{s} - \frac{1}{p_i}, i = 1,2$. *Then* $C \approx \mathscr{A}_{8.1} + \mathscr{A}_{8.2}$ *in case* (a) *and* $C \approx \mathscr{B}_{8.1} + \mathscr{B}_{8.2}$ *in case* (b)*, where*

(a) *If* $0 < s < p_1 \leqslant 1 < p_2 < \infty$, *then for* $\frac{1}{s} \leqslant \frac{1}{p_1} + \frac{1}{p_2}$,

$$\mathscr{A}_{8.1} := \sup_{n\in\mathbb{Z}} \left(\sum_{k=n}^{\infty} W_k \tilde{u}_k^{\frac{r_2}{s}} \right)^{\frac{1}{r_2}} \left(\sum_{j=-\infty}^{n} \tilde{v}_j \right)^{\frac{1}{r_1}}, \mathscr{A}_{8.2} := \sup_{n\in\mathbb{Z}} \left(\sum_{k=-\infty}^{n} W_k \right)^{\frac{1}{r_2}} \left(\sum_{j=n}^{\infty} \tilde{u}_k^{\frac{r_1}{s}} \tilde{v}_j \right)^{\frac{1}{r_1}}.$$

For $\frac{1}{s} > \frac{1}{p_1} + \frac{1}{p_2}$,

$$\mathscr{A}_{8.1} := \left(\sum_{n=-\infty}^{\infty} W_n \tilde{u}_n^{\frac{r_2}{s}} \left(\sum_{k=n}^{\infty} W_k \tilde{u}_k^{\frac{r_2}{s}} \right)^{\frac{r_2}{p_1-r_2}} \left(\sum_{j=-\infty}^{n} \tilde{v}_j \right)^{\frac{p_1 r_2}{r_1(p_1-r_2)}} \right)^{\frac{1}{r_2}-\frac{1}{p_1}},$$

$$\mathscr{A}_{8.2} := \left(\sum_{n=-\infty}^{\infty} W_n \left(\sum_{k=-\infty}^{n} W_k \right)^{\frac{r_2}{p_1-r_2}} \left(\sum_{j=n}^{\infty} \tilde{u}_j^{\frac{r_1}{p_1}} \tilde{v}_j \right)^{\frac{p_1 r_2}{r_1(p_1-r_2)}} \right)^{\frac{1}{r_2}-\frac{1}{p_1}}.$$

(b) *If* $0 < s < p_2 \leqslant 1 < p_1 < \infty$, *then for* $\frac{1}{s} \leqslant \frac{1}{p_1} + \frac{1}{p_2}$,

$$\mathscr{B}_{8.1} := \sup_{n\in\mathbb{Z}} \left(\sum_{k=n}^{\infty} V_k \tilde{u}_k^{\frac{r_1}{s}} \right)^{\frac{1}{r_1}} \left(\sum_{j=-\infty}^{n} \tilde{w}_j \right)^{\frac{1}{r_2}}, \mathscr{B}_{8.2} := \sup_{n\in\mathbb{Z}} \left(\sum_{k=-\infty}^{n} V_k \right)^{\frac{1}{r_1}} \left(\sum_{j=n}^{\infty} \tilde{u}_j^{\frac{r_2}{s}} \tilde{w}_j \right)^{\frac{1}{r_2}}.$$

For $\frac{1}{s} > \frac{1}{p_1} + \frac{1}{p_2}$,

$$\mathscr{B}_{8.1} := \left(\sum_{n=-\infty}^{\infty} V_n \tilde{u}_n^{\frac{r_1}{s}} \left(\sum_{k=n}^{\infty} V_k \tilde{u}_k^{\frac{r_1}{s}} \right)^{\frac{r_1}{p_2-r_1}} \left(\sum_{j=-\infty}^{n} \tilde{w}_j \right)^{\frac{p_2 r_1}{r_2(p_2-r_1)}} \right)^{\frac{1}{r_1}-\frac{1}{p_2}},$$

$$\mathscr{B}_{8.2} := \left(\sum_{n=-\infty}^{\infty} V_n \left(\sum_{k=-\infty}^{n} V_k \right)^{\frac{r_1}{p_2-r_1}} \left(\sum_{j=n}^{\infty} \tilde{u}_j^{\frac{r_2}{p_2}} \tilde{w}_j \right)^{\frac{p_2 r_1}{r_2(p_2-r_1)}} \right)^{\frac{1}{r_1}-\frac{1}{p_2}}.$$

Proof. (a) We have

$$C = \sup_b \|b\|_{p_2}^{-1} \left\{ \sup_a \left[\sum_{n=-\infty}^{\infty} \left(\sum_{i=-\infty}^{n} a_i \right)^s U_n \right]^{\frac{1}{s}} \|a\|_{p_1}^{-1} \right\}$$
$$\overset{(9)}{\approx} \sup_b \left(\sum_{n=-\infty}^{\infty} U_n \left(\sum_{k=n}^{\infty} U_k \right)^{\frac{r_1}{p_1}} \max_{j \leqslant n} v_j^{-\frac{r_1}{p_1}} \right)^{\frac{1}{r_1}} \|b\|_{p_2}^{-1}.$$

Using the above notations, we obtain

$$C \approx \sup_b \left(\sum_{n=-\infty}^{\infty} \tilde{v}_n \sum_{j=n}^{\infty} U_j \left(\sum_{k=j}^{\infty} U_k \right)^{\frac{r_1}{p_1}} \right)^{\frac{1}{r_1}} \|b\|_{p_2}^{-1}.$$

By Lemma C

$$C \approx \sup_b \left(\sum_{j=-\infty}^{\infty} \tilde{v}_j \left(\sum_{n=j}^{\infty} U_n \right)^{\frac{r_1}{s}} \right)^{\frac{1}{r_1}} \|b\|_{p_2}^{-1}$$
$$\approx \sup_b \sup_h \left(\sum_{j=-\infty}^{\infty} \left(\sum_{n=j}^{\infty} U_n \right) h_j \tilde{v}_j \right)^{\frac{1}{s}} \left(\sum_{j=-\infty}^{\infty} h_j^{\frac{r_1}{r_1-s}} \tilde{v}_j \right)^{\frac{s-r_1}{r_1}} \|b\|_{p_2}^{-1}$$
$$= \sup_b \sup_h \left(\sum_{n=-\infty}^{\infty} U_n \sum_{j=-\infty}^{n} \tilde{v}_j h_j \right)^{\frac{1}{s}} \left(\sum_{j=-\infty}^{\infty} h_j^{\frac{r_1}{r_1-s}} \tilde{v}_j \right)^{\frac{s-r_1}{r_1}} \|b\|_{p_2}^{-1}$$
$$= \sup_b \sup_h \left(\sum_{n=-\infty}^{\infty} U_n \sum_{j=-\infty}^{n} \tilde{v}_j h_j \right)^{\frac{1}{s}} \left(\sum_{j=-\infty}^{\infty} h_j^{\frac{r_1}{r_1-s}} \tilde{v}_j \right)^{\frac{s-r_1}{r_1}} \|b\|_{p_2}^{-1}.$$

Now, $\frac{r_1}{r_1-s} = \frac{p_1}{s}$, so

$$C \approx \sup_h \left(\sum_{j=-\infty}^{\infty} h_j^{\frac{p_1}{s}} \tilde{v}_j \right)^{-\frac{1}{p_1}} \sup_b \left(\sum_{n=-\infty}^{\infty} U_n \tilde{H}_1 h(n) \right)^{\frac{1}{s}} \|b\|_{p_2}^{-1},$$

where $\tilde{H}_1 h(n) := \sum_{j=-\infty}^{n} \tilde{v}_j h_j$. Then

$$C \overset{(7)}{\approx} \sup_h \left(\sum_{j=-\infty}^{\infty} h_j^{\frac{p_1}{s}} \tilde{v}_j \right)^{-\frac{1}{p_1}} \left(\sum_{n=-\infty}^{\infty} \left(\sum_{k=n}^{\infty} u_k \tilde{H}_1 h(k) \right)^{\frac{r_2}{s}} W_n \right)^{\frac{1}{r_2}}.$$

Since $\tilde{u}_n = \sum_{k=n}^{\infty} u_k$, then

$$\sum_{k=n}^{\infty} u_k \tilde{H}_1 h(k) = \sum_{k=n}^{\infty} u_k \left(\sum_{j=-\infty}^{k} h_j \tilde{v}_j \right) \approx \tilde{u}_n \tilde{H}_1 h(n) + \sum_{j=n}^{\infty} h_j \tilde{v}_j \tilde{u}_j.$$

Thus, $C \approx J_3 + J_4$, where

$$J_3^s := \sup_h \left(\sum_{n=-\infty}^{\infty} \left(\tilde{H}_1 h(n)\right)^{\frac{r_2}{s}} \tilde{u}_n^{\frac{r_2}{s}} W_n \right)^{\frac{s}{r_2}} \left(\sum_{j=-\infty}^{\infty} h_j^{\frac{p_1}{s}} \tilde{v}_j \right)^{-\frac{s}{p_1}},$$

$$J_4^s := \sup_h \left(\sum_{n=-\infty}^{\infty} \left(\sum_{j=n}^{\infty} h_j \tilde{v}_j \tilde{u}_j \right)^{\frac{r_2}{s}} W_n \right)^{\frac{s}{r_2}} \left(\sum_{j=-\infty}^{\infty} h_j^{\frac{p_1}{s}} \tilde{v}_j \right)^{-\frac{s}{p_1}}.$$

We have 2 cases

$$\text{(i)} \quad \frac{1}{s} \leqslant \frac{1}{p_1} + \frac{1}{p_2} \Leftrightarrow \frac{p_1}{s} \leqslant \frac{r_2}{s},$$
$$\text{(ii)} \quad \frac{1}{s} > \frac{1}{p_1} + \frac{1}{p_2} \Leftrightarrow \frac{p_1}{s} > \frac{r_2}{s}.$$

Since $\frac{p_1}{s} > 1$, in (i) we apply (5) and (11), then

$$J_3 = \sup_n \left(\sum_{k=n}^{\infty} W_k \tilde{u}_k^{\frac{r_2}{s}} \right)^{\frac{1}{r_2}} \left(\sum_{j=-\infty}^{n} \tilde{v}_j \right)^{\frac{1}{r_1}} = \mathscr{A}_{8.1},\ J_4 = \sup_n \left(\sum_{k=-\infty}^{n} W_k \right)^{\frac{1}{r_2}} \left(\sum_{j=n}^{\infty} \tilde{u}_j^{\frac{r_1}{s}} \tilde{v}_j \right)^{\frac{1}{r_1}} = \mathscr{A}_{8.2}.$$

In the (ii) we apply (8) and (12), then

$$J_3 = \left(\sum_{n=-\infty}^{\infty} W_n \tilde{u}_n^{\frac{r_2}{s}} \left(\sum_{k=n}^{\infty} W_k \tilde{u}_k^{\frac{r_2}{s}} \right)^{\frac{r_2}{p_1 - r_2}} \left(\sum_{j=-\infty}^{n} \tilde{v}_j \right)^{\frac{p_1 r_2}{r_1 (p_1 - r_2)}} \right)^{\frac{1}{r_2} - \frac{1}{p_1}} = \mathscr{A}_{8.1},$$

$$J_4 = \left(\sum_{n=-\infty}^{\infty} W_n \left(\sum_{k=-\infty}^{n} W_k \right)^{\frac{r_2}{p_1 - r_2}} \left(\sum_{j=n}^{\infty} \tilde{u}_j^{\frac{r_1}{p_1}} \tilde{v}_j \right)^{\frac{p_1 r_2}{r_1 (p_1 - r_2)}} \right)^{\frac{1}{r_2} - \frac{1}{p_1}} = \mathscr{A}_{8.2}.$$

Similar for the case (b).

THEOREM 9. *(Case IV.) Let* $0 < s < \min(p_1, p_2) \leqslant \max(p_1, p_2) \leqslant 1$, $\frac{1}{r_i} = \frac{1}{s} - \frac{1}{p_i}$, $i = 1, 2$. *Then* $C \approx \mathscr{A}_{9.1} + \mathscr{A}_{9.2}$ *in case (a) and* $C \approx \mathscr{B}_{9.1} + \mathscr{B}_{9.2}$ *in case (b), where*

(a) *If* $0 < s < p_1 \leqslant p_2 \leqslant 1$, *then for* $\frac{1}{s} \leqslant \frac{1}{p_1} + \frac{1}{p_2}$,

$$\mathscr{A}_{9.1} := \sup_{n \in \mathbb{Z}} \left(\sum_{k=n}^{\infty} \tilde{u}_k^{\frac{r_2}{s}} \tilde{w}_k \right)^{\frac{1}{r_2}} \left(\sum_{j=-\infty}^{n} \tilde{v}_j \right)^{\frac{1}{r_1}},\ \mathscr{A}_{9.2} := \sup_{n \in \mathbb{Z}} \left(\sum_{k=-\infty}^{n} \tilde{w}_k \right)^{\frac{1}{r_2}} \left(\sum_{j=n}^{\infty} \tilde{u}_j^{\frac{r_1}{s}} \tilde{v}_j \right)^{\frac{1}{r_1}}.$$

For $\frac{1}{s} > \frac{1}{p_1} + \frac{1}{p_2}$,

$$\mathscr{A}_{9.1} := \left(\sum_{n=-\infty}^{\infty} \tilde{u}_n^{\frac{r_2}{s}} \tilde{w}_n \left(\sum_{k=n}^{\infty} \tilde{u}_k^{\frac{r_2}{s}} \tilde{w}_k \right)^{\frac{r_2}{p_1 - r_2}} \left(\sum_{j=-\infty}^{n} \tilde{v}_j \right)^{\frac{p_1 r_2}{r_1 (p_1 - r_2)}} \right)^{\frac{1}{r_2} - \frac{1}{p_1}},$$

$$\mathscr{A}_{9.2} := \left(\sum_{n=-\infty}^{\infty} \tilde{w}_n \left(\sum_{k=-\infty}^{n} \tilde{w}_k \right)^{\frac{r_2}{p_1 - r_2}} \left(\sum_{j=n}^{\infty} \tilde{u}_j^{\frac{r_1}{p_1}} \tilde{v}_j \right)^{\frac{p_1 r_2}{r_1 (p_1 - r_2)}} \right)^{\frac{1}{r_2} - \frac{1}{p_1}}.$$

(b) *If* $0 < s < p_2 < p_1 \leqslant 1$, *then for* $\frac{1}{s} \leqslant \frac{1}{p_1} + \frac{1}{p_2}$,

$$\mathscr{B}_{9.1} := \sup_{n \in \mathbb{Z}} \left(\sum_{k=n}^{\infty} \tilde{u}_k^{\frac{r_1}{s}} \tilde{v}_k \right)^{\frac{1}{r_1}} \left(\sum_{j=-\infty}^{n} \tilde{w}_j \right)^{\frac{1}{r_2}}, \mathscr{B}_{9.2} := \sup_{n \in \mathbb{Z}} \left(\sum_{k=-\infty}^{n} \tilde{v}_k \right)^{\frac{1}{r_1}} \left(\sum_{j=n}^{\infty} \tilde{u}_j^{\frac{r_2}{s}} \tilde{w}_j \right)^{\frac{1}{r_2}}.$$

For $\frac{1}{s} > \frac{1}{p_1} + \frac{1}{p_2}$,

$$\mathscr{B}_{9.1} := \left(\sum_{n=-\infty}^{\infty} \tilde{u}_n^{\frac{r_1}{s}} \tilde{v}_n \left(\sum_{k=n}^{\infty} \tilde{u}_k^{\frac{r_1}{s}} \tilde{v}_k \right)^{\frac{r_1}{p_2 - r_1}} \left(\sum_{j=-\infty}^{n} \tilde{w}_j \right)^{\frac{p_2 r_1}{r_2 (p_2 - r_1)}} \right)^{\frac{1}{r_1} - \frac{1}{p_2}},$$

$$\mathscr{B}_{9.2} := \left(\sum_{n=-\infty}^{\infty} \tilde{v}_n \left(\sum_{k=-\infty}^{n} \tilde{v}_k \right)^{\frac{r_1}{p_2 - r_1}} \left(\sum_{j=n}^{\infty} \tilde{u}_j^{\frac{r_1}{p_1}} \tilde{w}_j \right)^{\frac{p_2 r_1}{r_2 (p_2 - r_1)}} \right)^{\frac{1}{r_1} - \frac{1}{p_2}}.$$

Proof. (a) As in Theorem 8, we have

$$C \approx \sup_h \left(\sum_{j=-\infty}^{\infty} h_j^{\frac{p_1}{s}} \tilde{v}_j \right)^{-\frac{1}{p_1}} \sup_b \left(\sum_{n=-\infty}^{\infty} U_n \tilde{H}_1 h(n) \right)^{\frac{1}{s}} \|b\|_{p_2}^{-1}$$

$$\overset{(9)}{\approx} \sup_h \left(\sum_{j=-\infty}^{\infty} h_j^{\frac{p_1}{s}} \tilde{v}_j \right)^{-\frac{1}{p_1}} \left(\sum_{n=-\infty}^{\infty} u_n \tilde{H}_1 h(n) \left(\sum_{k=n}^{\infty} u_k \tilde{H}_1 h(k) \right)^{\frac{r_2}{p_2}} \mathscr{W}_n \right)^{\frac{1}{r_2}},$$

where

$$\tilde{H}_1 h(n) = \sum_{j=-\infty}^{n} \tilde{v}_j h_j, \; \mathscr{W}_n = \max_{j \leqslant n} w_j^{-\frac{r_2}{p_2}} = \sum_{j=-\infty}^{n} \tilde{w}_j.$$

Then

$$C \approx \sup_h \left(\sum_{j=-\infty}^{\infty} h_j^{\frac{p_1}{s}} \tilde{v}_j \right)^{-\frac{1}{p_1}} \left(\sum_{n=-\infty}^{\infty} \left(\sum_{k=n}^{\infty} u_k \tilde{H}_1 h(k) \right)^{\frac{r_2}{s}} \tilde{w}_n \right)^{\frac{1}{r_2}}.$$

Since, $\tilde{u}_n = \sum_{k=n}^{\infty} u_k,$ then

$$\sum_{k=n}^{\infty} u_k \tilde{H}_1 h(k) = \sum_{k=n}^{\infty} u_k \left(\sum_{j=-\infty}^{k} h_j \tilde{v}_j\right) \approx \tilde{u}_n \tilde{H}_1 h(n) + \sum_{j=n}^{\infty} h_j \tilde{w}_j \tilde{u}_j.$$

Thus, $C \approx I_1 + I_2,$ where

$$I_1^s := \sup_h \left(\sum_{n=-\infty}^{\infty} \left(\tilde{H}_1 h(n)\right)^{\frac{r_2}{s}} \tilde{u}_n^{\frac{r_2}{s}} \tilde{w}_n\right)^{\frac{s}{r_2}} \left(\sum_{j=-\infty}^{\infty} h_j^{\frac{p_1}{s}} \tilde{v}_j\right)^{-\frac{s}{p_1}},$$

$$I_2^s := \sup_h \left(\sum_{n=-\infty}^{\infty} \left(\sum_{j=n}^{\infty} h_j \tilde{w}_j \tilde{u}_j\right)^{\frac{r_2}{s}} \tilde{w}_n\right)^{\frac{s}{r_2}} \left(\sum_{j=-\infty}^{\infty} h_j^{\frac{p_1}{s}} \tilde{v}_j\right)^{-\frac{s}{p_1}}.$$

We have 2 cases

$$\text{(i)} \quad \frac{1}{s} \leqslant \frac{1}{p_1} + \frac{1}{p_2} \Leftrightarrow \frac{p_1}{s} \leqslant \frac{r_2}{s},$$
$$\text{(ii)} \quad \frac{1}{s} > \frac{1}{p_1} + \frac{1}{p_2} \Leftrightarrow \frac{p_1}{s} > \frac{r_2}{s}.$$

Since $\frac{p_1}{s} > 1,$ in (i) we apply (5) and (11), then

$$I_1 = \sup_{n \in \mathbb{Z}} \left(\sum_{k=n}^{\infty} \tilde{u}_k^{\frac{r_2}{s}} \tilde{w}_k\right)^{\frac{1}{r_2}} \left(\sum_{j=-\infty}^{n} \tilde{v}_j\right)^{\frac{1}{r_1}} = \mathscr{A}_{9.1},$$

$$I_2 = \sup_{n \in \mathbb{Z}} \left(\sum_{k=-\infty}^{n} \tilde{w}_k\right)^{\frac{1}{r_2}} \left(\sum_{j=n}^{\infty} \tilde{u}_j^{\frac{r_1}{s}} \tilde{v}_j\right)^{\frac{1}{r_1}} = \mathscr{A}_{9.2}.$$

In (ii) we apply (8) and (12), then

$$I_1 = \left(\sum_{n=-\infty}^{\infty} \tilde{u}_n^{\frac{r_2}{s}} \tilde{w}_n \left(\sum_{k=n}^{\infty} \tilde{u}_k^{\frac{r_2}{s}} \tilde{w}_k\right)^{\frac{r_2}{p_1-r_2}} \left(\sum_{j=-\infty}^{n} \tilde{v}_j\right)^{\frac{p_1 r_2}{r_1(p_1-r_2)}}\right)^{\frac{1}{r_2}-\frac{1}{p_1}} = \mathscr{A}_{9.1},$$

$$I_2 = \left(\sum_{n=-\infty}^{\infty} \tilde{w}_n \left(\sum_{k=-\infty}^{n} \tilde{w}_k\right)^{\frac{r_2}{p_1-r_2}} \left(\sum_{j=n}^{\infty} \tilde{u}_j^{\frac{r_1}{p_1}} \tilde{v}_j\right)^{\frac{p_1 r_2}{r_1(p_1-r_2)}}\right)^{\frac{1}{r_2}-\frac{1}{p_1}} = \mathscr{A}_{9.2}.$$

Similar for the case (b).

4. Auxiliary lemmas

In this section, we give some basics about q-calculus and prove some lemmas which will be used in the next section.

Let f be a function defined on $(0,b), 0<b\leqslant\infty$ and $0<q<1$. The q-differential of f is defined by

$$d_q f(x) := f(x) - f(qx)$$

and the q-derivative of f is defined by

$$D_q f(x) := \frac{d_q f(x)}{d_q x} = \frac{f(x)-f(qx)}{(1-q)x}.$$

Note that if f is differentiable, then as $q\to 1$, $D_q f(x)$ becomes the actual left derivative of f. For a positive integer n, the symbol $[n]_q$ is called the q-analogue of n which is defined as

$$[n]_q := \frac{1-q^n}{1-q}.$$

Thus it is easy to see that

$$D_q x^n = [n]_q x^{n-1}.$$

The q-analogue of the integral, usually known as the Jackson integral, of f is defined as

$$\int_0^x f(t)\, d_q t := (1-q)x \sum_{j=0}^{\infty} q^j f(q^j x) \tag{16}$$

and the improper q-integral of f is defined as

$$\int_0^{\infty} f(t)\, d_q t := (1-q) \sum_{j=-\infty}^{\infty} q^j f(q^j) \tag{17}$$

provided that the respective series on the right hand side converge. We also use

$$\int_a^b f(t)\, d_q t := \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t.$$

For any $z>0$, we have in view of (17), that

$$J_1(f,z) := \int_0^{\infty} \chi_{(0,z]}(t) f(t)\, d_q t \cong \sum_{i:q^i\leqslant z} q^i f(q^i), \tag{18}$$

$$J_2(f,z) := \int_0^{\infty} \chi_{[z,\infty)}(t) f(t)\, d_q t \cong \sum_{i:q^i\geqslant z} q^i f(q^i). \tag{19}$$

Also, for $q^n \leqslant z < q^{n-1}$, $n\in\mathbb{Z}$, we get

$$J_3(f,z) := \int_0^{\infty} \chi_{(qz,z]}(t) f(t)\, d_q t \cong q^n f(q^n)$$

and for $q^{m+1} < z \leqslant q^m$, $m \in \mathbb{Z}$,

$$J_4(f,z) := \int_0^\infty \chi_{[z,q^{-1}z)}(t) f(t)\, d_q t \cong q^m f(q^m). \tag{20}$$

It follows from (18)-(20) that for all $n \in \mathbb{Z}$

$$J_1(f,z) \cong \sum_{i \geqslant n} q^i f(q^i),\ \ q^n \leqslant z < q^{n-1}, \tag{21}$$

$$J_2(f,q^n) \cong \sum_{j \leqslant n} q^j f(q^j), \tag{22}$$

$$J_2(f,z) \cong \sum_{j \leqslant n-1} q^j f(q^j),\ \ q^n < z < q^{n-1}. \tag{23}$$

$$J_1(f,q^n) \cong \sum_{i \geqslant n} q^i f(q^i), \tag{24}$$

$$J_1(f,z) \cong \sum_{i \geqslant n+1} q^i f(q^i),\ \ q^{n+1} < z < q^n, \tag{25}$$

$$J_2(f,z) \cong \sum_{j \leqslant n} q^j f(q^j),\ \ q^{n+1} < z \leqslant q^n. \tag{26}$$

In what follows, we prove couple of lemmas which will be used in the results proved in next section. On the way and later on we use the notation $\int_0^\infty \chi_{(0,z]} f\, d_q$ instead of $\int_0^\infty \chi_{(0,z]}(t) f(t)\, d_q t$ and similar for other integrals.

LEMMA 1. *Let $f,g,h \geqslant 0$ and*

$$I(z) := \left(\int_0^\infty \chi_{(0,z]} f\, d_q\right)^\alpha \left(\int_0^\infty \chi_{(0,z]} g\, d_q\right)^\beta \left(\int_0^\infty \chi_{[z,\infty)} h\, d_q\right)^\gamma.$$

$\alpha,\beta,\gamma \in \mathbb{R}$ if $\alpha,\beta > 0$ or $\gamma > 0$, then

$$\sup_{z>0} I(z) \cong \sup_{n\in\mathbb{Z}} \left(\sum_{i=n}^\infty q^i f(q^i)\right)^\alpha \left(\sum_{j=n}^\infty q^j g(q^j)\right)^\beta \left(\sum_{l=-\infty}^n q^l h(q^l)\right)^\gamma.$$

Proof. Let $\alpha,\beta > 0$. Then

$$\begin{aligned}
\sup_{z>0} I(z) &= \sup_{n\in\mathbb{Z}} \sup_{q^{n+1}<z\leqslant q^n} J_1^\alpha(f,z) J_1^\beta(g,z) J_2^\gamma(h,z) \\
&\overset{(26)}{\cong} \sup_{n\in\mathbb{Z}} \left(\sum_{l\leqslant n} q^l h(q^l)\right)^\gamma \sup_{q^{n+1}<z\leqslant q^n} J_1^\alpha(f,z) J_1^\beta(g,z) \\
&\overset{(24),(25)}{\cong} \left(\sum_{i\geqslant n} q^i f(q^i)\right)^\alpha \left(\sum_{j\geqslant n} q^j g(q^j)\right)^\beta \left(\sum_{l\leqslant n} q^l h(q^l)\right)^\gamma.
\end{aligned}$$

If $\gamma > 0$, then

$$\begin{aligned}\sup_{z>0} I(z) &= \sup_{n\in\mathbb{Z}} \sup_{q^n\leqslant z<q^{n-1}} J_1^\alpha(f,z)J_1^\beta(g,z)J_2^\gamma(h,z)\\ &\overset{(21)}{\cong} \sup_{n\in\mathbb{Z}} \left(\sum_{i\geqslant n} q^i f(q^i)\right)^\alpha \left(\sum_{j\geqslant n} q^j g(q^j)\right)^\beta \sup_{q^n\leqslant z<q^{n-1}} J_2^\gamma(h,z)\\ &\overset{(22),(23)}{\cong} \left(\sum_{i\geqslant n} q^i f(q^i)\right)^\alpha \left(\sum_{j\geqslant n} q^j g(q^j)\right)^\beta \left(\sum_{l\leqslant n} q^l h(q^l)\right)^\gamma.\end{aligned}$$

The assertion follows in view of both cases.

LEMMA 2. *Let $f,g,h \geqslant 0$ and*

$$I(z) := \left(\int_0^\infty \chi_{(0,z]} f\, d_q\right)^\alpha \left\{\int_0^\infty \chi_{[z,\infty)}(t)g(t) \left(\int_0^\infty \chi_{[t,\infty)} g\, d_q\right)^\beta \times \left(\int_0^\infty \chi_{(0,t]} h\, d_q\right)^\gamma d_q t\right\}^\delta.$$

$\alpha,\beta,\gamma,\delta \in \mathbb{R}$ if $\alpha > 0$ or $\delta > 0$, then

$$\sup_{z>0} I(z) \cong \sup_{n\in\mathbb{Z}} \left\{\sum_{l=n}^\infty q^l f(q^l)\right\}^\alpha \left[\sum_{i=-\infty}^n q^i g(q^i) \left\{\sum_{k=-\infty}^i q^k g(q^k)\right\}^\beta \left\{\sum_{j=i}^\infty q^j h(q^j)\right\}^\gamma\right]^\delta.$$

Proof. From (18) and (19), we have

$$I(z) \cong \left(\sum_{l:q^l\leqslant z} q^l f(q^l)\right)^\alpha \left(\sum_{i:q^i\geqslant z} q^i g(q^i)\tilde{g}(q^i)\right)^\delta,$$

where

$$\tilde{g}(q^i) := \left(\int_0^\infty \chi_{[q^i,\infty)}(x)g(x)\, d_q x\right)^\beta \left(\int_0^\infty \chi_{(0,q^i]}(y)h(y)\, d_q y\right)^\gamma.$$

Since $\alpha > 0$ or $\delta > 0$, in view of Lemma 3.5 in [1], we have

$$\sup_{z>0} I(z) \cong \sup_{n\in\mathbb{Z}} \left(\sum_{l\geqslant n} q^l f(q^l)\right)^\alpha \left(\sum_{i\leqslant n} q^i g(q^i)\tilde{g}(q^i)\right)^\delta.$$

Clearly

$$\tilde{g}(q^i) \underset{(19)}{\overset{(18)}{\cong}} \left(\sum_{k\leqslant i} q^k g(q^k)\right)^\beta \left(\sum_{j\geqslant i} q^j h(q^j)\right)^\gamma.$$

Hence

$$\sup_{z>0} I(z) \cong \sup_{n\in\mathbb{Z}} \left\{\sum_{l\geqslant n} q^l f(q^l)\right\}^\alpha \left[\sum_{i\leqslant n} q^i g(q^i) \left\{\sum_{k\leqslant i} q^k g(q^k)\right\}^\beta \left\{\sum_{j\geqslant i} q^j h(q^j)\right\}^\gamma\right]^\delta.$$

5. Bilinear q-Hardy inequalities

In this section, we shall investigate q-analogue of the bilinear Hardy inequality (3). In [1] (see also [18]), the authors obtained the q-analogue of the standard Hardy inequality (1) involving the q-Hardy operator

$$H_q f(x) := \int_0^\infty \chi_{(0,x]}(t) f(t)\, d_q t, \tag{27}$$

which is defined for all $x > 0$. A natural q-analogue of the Hardy operator H seems to be

$$\int_0^x f(t)\, d_q t, \tag{28}$$

however, in view of (16), it was pointed out in [1] that (27) and (28) coincide only at the points $x = q^n, n \in \mathbb{Z}$. Therefore H_q can be considered as a true q-analogue of H.

In our case, we consider the bilinear q-Hardy operator

$$\mathscr{H}_q(f,g)(x) := H_q f(x) \cdot H_q g(x) = \left(\int_0^\infty \chi_{(0,x]} f\, d_q\right)\left(\int_0^\infty \chi_{(0,x]} g\, d_q\right)$$

and study the inequality

$$\left(\int_0^\infty \left(\mathscr{H}_q(f,g)\right)^s u\, d_q\right)^{\frac{1}{s}} \leqslant C \left(\int_0^\infty f^{p_1} v\, d_q\right)^{\frac{1}{p_1}} \left(\int_0^\infty g^{p_2} w\, d_q\right)^{\frac{1}{p_2}}, \tag{29}$$

where $0 < q < 1$ and $f, g, u, v, w \geqslant 0$ on $(0, \infty)$.

First, we prove that (29) is equivalent to discrete bilinear inequality.

THEOREM 10. *Let $0 < p_1, p_2, s < \infty$. Then (29) holds iff the inequality*

$$\left[\sum_{k=-\infty}^{\infty} \left(\sum_{i=-\infty}^{k} F_i\right)^s \left(\sum_{j=-\infty}^{k} G_j\right)^s \tilde{U}_k\right]^{\frac{1}{s}} \leqslant C \left(\sum_{k=-\infty}^{\infty} F_k^{p_1} \tilde{V}_k\right)^{\frac{1}{p_1}} \left(\sum_{k=-\infty}^{\infty} G_k^{p_2} \tilde{W}_k\right)^{\frac{1}{p_2}} \tag{30}$$

holds with $F_k := q^{-k} f(q^{-k}), G_k := q^{-k} g(q^{-k}),\ \tilde{V}_k := (1-q) q^{-(1-p_1)k} v(q^{-k})$, $\tilde{W}_k := (1-q) q^{-(1-p_2)k} w(q^{-k}),\ \tilde{U}_k := (1-q)^{2s+1} q^{-k} u(q^{-k});\ k \in \mathbb{Z}$.

Proof. The proof is similar to the proof of Theorem 3.2 in [1].

Now, we prove below theorems characterizing the inequality (29) for different combinations of the indices p_1, p_2, s.

THEOREM 11. *Let $1 < p_1, p_2 \leqslant s < \infty$. Then the inequality (29) holds iff*

$$D_1 := \sup_{z>0} \left(\int_0^\infty \chi_{[z,\infty)} u\, d_q\right)^{\frac{1}{s}} \left(\int_0^\infty \chi_{(0,z]} v^{1-p_1'} d_q\right)^{\frac{1}{p_1'}} \left(\int_0^\infty \chi_{(0,z]} w^{1-p_2'} d_q\right)^{\frac{1}{p_2'}} < \infty.$$

Proof. Using Theorem 10, the inequality (29) is equivalent with the inequality (30) which, in view of Theorem 1, holds if and only if $\tilde{\mathscr{A}_1} < \infty$, where

$$\tilde{\mathscr{A}_1} := \sup_{n\in\mathbb{Z}} \left(\sum_{i=n}^{\infty} \tilde{U}_i\right)^{\frac{1}{s}} \left(\sum_{i=-\infty}^{n} \tilde{V}_i^{1-p_1'}\right)^{\frac{1}{p_1'}} \left(\sum_{i=-\infty}^{n} \tilde{W}_i^{1-p_2'}\right)^{\frac{1}{p_2'}}.$$

Now, by using Lemma 1, we have

$$\begin{aligned}\tilde{\mathscr{A}_1} &\cong \sup_{n\in\mathbb{Z}} \left(\sum_{i=-\infty}^{n} q^i u(q^i)\right)^{\frac{1}{s}} \left[\sum_{i=n}^{\infty} \left\{q^{(1-p_1)i} v(q^i)\right\}^{1-p_1'}\right]^{\frac{1}{p_1'}} \left[\sum_{i=n}^{\infty} \left\{q^{(1-p_2)i} w(q^i)\right\}^{1-p_2'}\right]^{\frac{1}{p_2'}} \\ &= \sup_{n\in\mathbb{Z}} \left(\sum_{i=-\infty}^{n} q^i u(q^i)\right)^{\frac{1}{s}} \left(\sum_{i=n}^{\infty} q^i v^{1-p_1'}(q^i)\right)^{\frac{1}{p_1'}} \left(\sum_{i=n}^{\infty} q^i w^{1-p_2'}(q^i)\right)^{\frac{1}{p_2'}} \cong D_1\end{aligned}$$

and the result follows.

THEOREM 12. *Let $0 < p_1 \leqslant 1 < p_2 \leqslant s$. Then (29) holds iff*

$$D_{2.1} := \sup_{z>0} \left(\int_0^{\infty} \chi_{[z,\infty)} u \, d_q\right)^{\frac{1}{s}} \left(\int_0^{\infty} \chi_{(0,z]} w^{1-p_2'} d_q\right)^{\frac{1}{p_2'}} \sup_{y\leqslant z} \left(\int_0^{\infty} \chi_{[y,q^{-1}y)}(t) t^{-p_1} v(t) \, d_q t\right)^{-\frac{1}{p_1}} < \infty.$$

Proof. Using Theorem 10, the inequality (29) is equivalent to the inequality (30) which, in view of Theorem 2(a), holds if and only if $\tilde{\mathscr{A}}_{2.1} < \infty$, where

$$\tilde{\mathscr{A}}_{2.1} := \sup_{n\in\mathbb{Z}} \left(\sum_{i=n}^{\infty} \tilde{U}_i\right)^{\frac{1}{s}} \left(\sum_{i=-\infty}^{n} \tilde{W}_i^{1-p_2'}\right)^{\frac{1}{p_2'}} \sup_{k\leqslant n} \tilde{V}_k^{-\frac{1}{p_1}}.$$

Using (18), (19) and (20), we have

$$\begin{aligned}\tilde{\mathscr{A}}_{2.1} &\cong \sup_{n\in\mathbb{Z}} \left(\sum_{i=-\infty}^{n} q^i u(q^i)\right)^{\frac{1}{s}} \left(\sum_{i=n}^{\infty} q^i w^{1-p_2'}(q^i)\right)^{\frac{1}{p_2'}} \sup_{k\geqslant n} \left\{q^k \left(q^k\right)^{-p_1} v(q^k)\right\}^{-\frac{1}{p_1}} \\ &\cong \sup_{n\in\mathbb{Z}} \sup_{q^{n+1}<z\leqslant q^n} \left(\int_0^{\infty} \chi_{[z,\infty)}(t) u(t) \, d_q t\right)^{\frac{1}{s}} \left(\int_0^{\infty} \chi_{(0,z]}(t) w^{1-p_2'}(t) \, d_q t\right)^{\frac{1}{p_2'}} \\ &\quad \times \sup_{k\geqslant n} \sup_{q^{k+1}<y\leqslant q^k} \left(\int_0^{\infty} \chi_{[y,q^{-1}y)}(t) t^{-p_1} v(t) \, d_q t\right)^{-\frac{1}{p_1}} \\ &\cong \sup_{n\in\mathbb{Z}} \sup_{q^{n+1}<z\leqslant q^n} \left(\int_0^{\infty} \chi_{[z,\infty)}(t) u(t) \, d_q t\right)^{\frac{1}{s}} \left(\int_0^{\infty} \chi_{(0,z]}(t) w^{1-p_2'}(t) \, d_q t\right)^{\frac{1}{p_2'}} \\ &\quad \times \max\left\{\sup_{k\geqslant n+1} \sup_{q^{k+1}<y\leqslant q^k} \left(\int_0^{\infty} \chi_{[y,q^{-1}y)}(t) t^{-p_1} v(t) \, d_q t\right)^{-\frac{1}{p_1}}\right.\end{aligned}$$

$$\sup_{q^{n+1}<y\leqslant z}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(t)t^{-p_1}v(t)\,d_qt\right)^{-\frac{1}{p_1}}\Bigg\}$$

$$\cong \sup_{n\in\mathbb{Z}}\sup_{q^{n+1}<z\leqslant q^n}\left(\int_0^\infty \chi_{[z,\infty)}(t)u(t)\,d_qt\right)^{\frac{1}{s}}\left(\int_0^\infty \chi_{(0,z]}(t)w^{1-p_2'}(t)\,d_qt\right)^{\frac{1}{p_2'}}$$

$$\times\sup_{y\leqslant z}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(t)t^{-p_1}v(t)\,d_qt\right)^{-\frac{1}{p_1}}$$

$$\cong \sup_{z>0}\left(\int_0^\infty \chi_{[z,\infty)}(t)u(t)\,d_qt\right)^{\frac{1}{s}}\left(\int_0^\infty \chi_{(0,z]}(t)w^{1-p_2'}(t)\,d_qt\right)^{\frac{1}{p_2'}}$$

$$\times\sup_{y\leqslant z}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(t)t^{-p_1}v(t)\,d_qt\right)^{-\frac{1}{p_1}}=D_{2.1}.$$

The proofs of Theorems 13–18 are similar.

THEOREM 13. *Let $0<p_2\leqslant 1<p_1\leqslant s$. Then* (29) *holds iff and only if*

$$D_{2.2}:=\sup_{z>0}\left(\int_0^\infty \chi_{[z,\infty)}(t)u(t)\,d_qt\right)^{\frac{1}{s}}\left(\int_0^\infty \chi_{(0,z]}(t)v^{1-p_1'}(t)\,d_qt\right)^{\frac{1}{p_1'}}$$

$$\times\sup_{y\leqslant z}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(t)t^{-p_2}w(t)\,d_qt\right)^{-\frac{1}{p_2}}<\infty.$$

THEOREM 14. *Let $0<\max(p_1,p_2)\leqslant\min(1,s)<\infty$. Then* (29) *holds iff*

$$D_3:=\sup_{z>0}\left(\int_0^\infty \chi_{[z,q^{-1}z)}(t)t^{-p_1}v(t)\,d_qt\right)^{-\frac{1}{p_1}}\left(\int_0^\infty \chi_{[z,\infty)}(t)u(t)\,d_qt\right)^{\frac{1}{s}}$$

$$\times\sup_{y\leqslant z}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(t)t^{-p_2}w(t)\,d_qt\right)^{-\frac{1}{p_2}}$$

$$+\sup_{z>0}\left(\int_0^\infty \chi_{[z,q^{-1}z)}(t)t^{-p_2}w(t)\,d_qt\right)^{-\frac{1}{p_2}}\left(\int_0^\infty \chi_{[z,\infty)}(t)u(t)\,d_qt\right)^{\frac{1}{s}}$$

$$\times\sup_{y\leqslant z}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(t)t^{-p_1}v(t)\,d_qt\right)^{-\frac{1}{p_1}}<\infty.$$

Here onwards, $\dfrac{1}{r_i}=\dfrac{1}{s}-\dfrac{1}{p_i},\ i=1,2.$

THEOREM 15. *Let $1<p_1\leqslant s<p_2$. Then* (29) *holds iff*

$$D_{4.1}:=\sup_{z>0}\left(\int_0^\infty \chi_{(0,z]}(t)v^{1-p_1'}(t)\,d_qt\right)^{\frac{1}{p_1'}}\Bigg\{\int_0^\infty \chi_{[z,\infty)}(t)u(t)\times$$

$$\times\left(\int_0^\infty \chi_{[t,\infty)} u \, d_q\right)^{\frac{r_2}{p_2}} \left(\int_0^\infty \chi_{(0,t]}(y) w^{1-p_2'}(y) \, d_q y\right)^{\frac{r_2}{p_2'}} d_q t \Bigg\}^{\frac{1}{r_2}} < \infty.$$

THEOREM 16. *Let* $1 < p_2 \leqslant s < p_1$. *Then* (29) *holds iff*

$$D_{4.2} := \sup_{z>0} \left(\int_0^\infty \chi_{(0,z]}(t) w^{1-p_2'}(t) \, d_q t\right)^{\frac{1}{p_2'}} \Bigg\{\int_0^\infty \chi_{[z,\infty)}(t) u(t) \times\left(\int_0^\infty \chi_{[t,\infty)} u \, d_q\right)^{\frac{r_1}{p_1}} \left(\int_0^\infty \chi_{(0,t]} v^{1-p_1'} \, d_q\right)^{\frac{r_1}{p_1'}} d_q t \Bigg\}^{\frac{1}{r_1}} < \infty.$$

THEOREM 17. *Let* $0 < p_1 \leqslant s \leqslant 1 < p_2$. *Then* (29) *holds iff*

$$D_{5.1} := \sup_{z>0} \left(\int_0^\infty \chi_{[z,q^{-1}z)}(t) t^{-p_1} v(t) \, d_q t\right)^{-\frac{1}{p_1}} \Bigg\{\int_0^\infty \chi_{[z,\infty)}(t) u(t) \times\left(\int_0^\infty \chi_{[t,\infty)} u \, d_q\right)^{\frac{r_2}{p_2}} \left(\int_0^\infty \chi_{(0,t]} w^{1-p_2'} \, d_q\right)^{\frac{r_2}{p_2'}} d_q t \Bigg\}^{\frac{1}{r_2}} < \infty.$$

THEOREM 18. *Let* $0 < p_2 \leqslant s \leqslant 1 < p_1$. *Then* (29) *holds iff*

$$D_{5.2} := \sup_{z>0} \left(\int_0^\infty \chi_{[z,q^{-1}z)}(t) t^{-p_2} w(t) \, d_q t\right)^{-\frac{1}{p_2}} \Bigg\{\int_0^\infty \chi_{[z,\infty)}(t) u(t) \times\left(\int_0^\infty \chi_{[t,\infty)} u \, d_q\right)^{\frac{r_1}{p_1}} \left(\int_0^\infty \chi_{(0,t]} v^{1-p_1'} \, d_q\right)^{\frac{r_1}{p_1'}} d_q t \Bigg\}^{\frac{1}{r_1}} < \infty.$$

THEOREM 19. *Let* $0 < p_1 \leqslant s < p_2 \leqslant 1$. *Then* (29) *holds iff*

$$D_{6.1} := \sup_{z>0} \left(\int_0^\infty \chi_{[z,q^{-1}z)}(t) t^{-p_1} v(t) \, d_q t\right)^{-\frac{1}{p_1}} \Bigg\{\int_0^\infty \chi_{[z,\infty)}(t) u(t) \times\left(\int_0^\infty \chi_{[t,\infty)} u \, d_q\right)^{\frac{r_2}{p_2}} \sup_{y \leqslant t} \left(\int_0^\infty \chi_{[y,q^{-1}y)}(x) x^{-p_2} w(x) \, d_q x\right)^{-\frac{r_2}{p_2}} d_q t \Bigg\}^{\frac{1}{r_2}} < \infty.$$

Proof. Using Theorem 10, the inequality (29) is equivalent to the inequality (30) which, in view of Theorem 6(a), holds if and only if $\tilde{\mathscr{A}}_{6.1} < \infty$, where

$$\tilde{\mathscr{A}}_{6.1} := \sup_{n \in \mathbb{Z}} \tilde{V}_n^{-\frac{1}{p_1}} \left\{\sum_{i=n}^\infty \tilde{U}_i \left(\sum_{k=i}^\infty \tilde{U}_k\right)^{\frac{r_2}{p_2}} \max_{j \leqslant i} \tilde{W}_j^{-\frac{r_2}{p_2}}\right\}^{\frac{1}{r_2}}.$$

Using (19) and (20), we have

$$
\begin{aligned}
\tilde{\mathscr{A}}_{6.1} &\cong \sup_{n\in\mathbb{Z}} \left(q^n (q^n)^{-p_1} v(q^n)\right)^{-\frac{1}{p_1}} \\
&\quad \times \left[\sum_{i=-\infty}^{n} q^i u(q^i) \left\{\sum_{k=-\infty}^{i} q^k u(q^k)\right\}^{\frac{r_2}{p_2}} \max_{j\geqslant i} \left\{q^j (q^j)^{-p_2} w(q^j)\right\}^{-\frac{r_2}{p_2}}\right]^{\frac{1}{r_2}} \\
&= \sup_{n\in\mathbb{Z}} \left(q^n (q^n)^{-p_1} v(q^n)\right)^{-\frac{1}{p_1}} \\
&\quad \times \left[\sum_{i=-\infty}^{n} q^i u(q^i) \left\{\sum_{k:q^k\geqslant q^i} q^k u(q^k)\right\}^{\frac{r_2}{p_2}} \max_{j:q^j\leqslant q^i} \left\{q^j (q^j)^{-p_2} w(q^j)\right\}^{-\frac{r_2}{p_2}}\right]^{\frac{1}{r_2}} \\
&\cong \sup_{n\in\mathbb{Z}} \left(q^n (q^n)^{-p_1} v(q^n)\right)^{-\frac{1}{p_1}} \\
&\quad \times \left[\sum_{i=-\infty}^{n} q^i u(q^i) \left\{\int_0^\infty \chi_{[q^i,\infty)}(x) u(x)\, d_qx\right\}^{\frac{r_2}{p_2}} \times \max_{j:q^j\leqslant q^i} \left\{q^j (q^j)^{-p_2} w(q^j)\right\}^{-\frac{r_2}{p_2}}\right]^{\frac{1}{r_2}} \\
&\cong \sup_{n\in\mathbb{Z}} \sup_{q^{n+1}<z\leqslant q^n} \left(\int_0^\infty \chi_{[z,q^{-1}z)}(t) t^{-p_1} v(t)\, d_qt\right)^{-\frac{1}{p_1}} \\
&\quad \times \left[\int_0^\infty \chi_{[z,\infty)}(t) u(t) \left\{\int_0^\infty \chi_{[t,\infty)}(x) u(x)\, d_qx\right\}^{\frac{r_2}{p_2}} \max_{j:q^j\leqslant t} \left\{q^j (q^j)^{-p_2} w(q^j)\right\}^{-\frac{r_2}{p_2}} d_qt\right]^{\frac{1}{r_2}} \\
&= \sup_{z>0} \left(\int_0^\infty \chi_{[z,q^{-1}z)}(t) t^{-p_1} v(t)\, d_qt\right)^{-\frac{1}{p_1}} \\
&\quad \times \left[\int_0^\infty \chi_{[z,\infty)}(t) u(t) \left\{\int_0^\infty \chi_{[t,\infty)}(x) u(x)\, d_qx\right\}^{\frac{r_2}{p_2}} \max_{j:q^j\leqslant t} \left\{q^j (q^j)^{-p_2} w(q^j)\right\}^{-\frac{r_2}{p_2}} d_qt\right]^{\frac{1}{r_2}}.
\end{aligned}
$$

Now, let $j_0 := \sup\{j\in\mathbb{Z} : q^j \leqslant t\}$. Then

$$
\begin{aligned}
&\max_{j:q^j\leqslant t} \left\{q^j (q^j)^{-p_2} w(q^j)\right\}^{-\frac{r_2}{p_2}} \\
&= \max\left\{ \sup_{j:j\geqslant j_0} \sup_{q^{j+1}<y\leqslant q^j} \left(q^j (q^j)^{-p_2} w(q^j)\right)^{-\frac{r_2}{p_2}},\ \sup_{q^{j_0}<y\leqslant t} \left(q^j (q^j)^{-p_2} w(q^j)\right)^{-\frac{r_2}{p_2}} \right\} \\
&\cong \sup_{y\leqslant t} \left(\int_0^\infty \chi_{[y,q^{-1}y)}(x) x^{-p_2} w(x)\, d_qx\right)^{-\frac{r_2}{p_2}}.
\end{aligned}
$$

Hence

$$\tilde{\mathscr{A}}_{6.1} \cong \sup_{z>0}\left(\int_0^\infty \chi_{[z,q^{-1}z)}(t)t^{-p_1}v(t)\,d_qt\right)^{-\frac{1}{p_1}}\left[\int_0^\infty \chi_{[z,\infty)}(t)u(t)\times\right.$$
$$\left.\times\left\{\int_0^\infty \chi_{[t,\infty)}u\,d_q\right\}^{\frac{r_2}{p_2}}\sup_{y\leqslant t}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(x)x^{-p_2}w(x)\,d_qx\right)^{-\frac{r_2}{p_2}}d_qt\right]^{\frac{1}{r_2}} = D_{6.1}.$$

Analogously, we prove Theorems 20–31.

THEOREM 20. *Let* $0 < p_2 \leqslant s < p_1 \leqslant 1$. *Then* (29) *holds iff*

$$D_{6.2} := \sup_{z>0}\left(\int_0^\infty \chi_{[z,q^{-1}z)}(t)t^{-p_2}w(t)\,d_qt\right)^{-\frac{1}{p_2}}\left\{\int_0^\infty \chi_{[z,\infty)}(t)u(t)\right.$$
$$\left.\times\left(\int_0^\infty \chi_{[t,\infty)}u\,d_q\right)^{\frac{r_1}{p_1}}\sup_{y\leqslant t}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(x)x^{-p_1}v(x)\,d_qx\right)^{-\frac{r_1}{p_1}}d_qt\right\}^{\frac{1}{r_1}} < \infty.$$

THEOREM 21. *Let* $0<s<p_1<p_2<\infty,\ p_1>1,\ \frac{1}{s}\leqslant\frac{1}{p_1}+\frac{1}{p_2}$. *Then* (29) *holds iff*

$$\sup_{z>0}\left\{\int_0^\infty \chi_{[z,\infty)}(t)w^{1-p_2'}(t)\left(\int_0^\infty \chi_{(0,t]}w^{1-p_2'}\,d_q\right)^{\frac{r_2}{s'}}\left(\int_0^\infty \chi_{[t,\infty)}u\,d_q\right)^{\frac{r_2}{s}}d_qt\right\}^{\frac{1}{r_2}}$$
$$\times\left\{\int_0^\infty \chi_{(0,z]}(t)v^{1-p_1'}(t)\left(\int_0^\infty \chi_{(0,t]}v^{1-p_1'}\,d_q\right)^{\frac{r_1}{s'}}d_qt\right\}^{\frac{1}{r_1}} =: D_{7.1} < \infty$$

and

$$\sup_{z>0}\left\{\int_0^\infty \chi_{(0,z]}(t)w^{1-p_2'}(t)\left(\int_0^\infty \chi_{(0,t]}w^{1-p_2'}\,d_q\right)^{\frac{r_2}{s'}}d_qt\right\}^{\frac{1}{r_2}}$$
$$\times\left\{\int_0^\infty \chi_{[z,\infty)}(t)v^{1-p_1'}(t)\left(\int_0^\infty \chi_{[t,\infty)}u\,d_q\right)^{\frac{r_1}{s}}\left(\int_0^\infty \chi_{(0,t]}v^{1-p_1'}\,d_q\right)^{\frac{r_1}{s'}}d_qt\right\}^{\frac{1}{r_1}} =: D_{7.2} < \infty.$$

THEOREM 22. *Let* $0<s<p_1<p_2<\infty,\ p_1>1,\ \frac{1}{s}>\frac{1}{p_1}+\frac{1}{p_2}$. *Then* (29) *holds iff*

$$D_{7.3} := \left[\int_0^\infty w^{1-p_2'}(z)\left(\int_0^\infty \chi_{(0,z]}w^{1-p_2'}\,d_q\right)^{\frac{r_2}{s'}}\left(\int_0^\infty \chi_{[z,\infty)}u\,d_q\right)^{\frac{r_2}{s}}\right.$$
$$\times\left\{\int_0^\infty \chi_{[z,\infty)}(t)w^{1-p_2'}(t)\left(\int_0^\infty \chi_{(0,t]}w^{1-p_2'}\,d_q\right)^{\frac{r_2}{s'}}\right.$$

$$\times\left(\int_0^\infty \chi_{[t,\infty)} u\, d_q\right)^{\frac{r_2}{s}} d_q t\Bigg\}^{\frac{r_2}{p_1-r_2}} \Bigg\{\int_0^\infty \chi_{(0,z]}(t) v^{1-p_1'}(t)$$

$$\times\left(\int_0^\infty \chi_{(0,t]} v^{1-p_1'}\, d_q\right)^{\frac{r_1}{s'}} d_q t\Bigg\}^{\frac{p_1 r_2}{r_1(p_1-r_2)}} d_q z\Bigg]^{\frac{1}{r_2}-\frac{1}{p_1}} < \infty,$$

$$D_{7.4} := \Bigg[\int_0^\infty w^{1-p_2'}(z)\left(\int_0^\infty \chi_{(0,z]} w^{1-p_2'}\, d_q\right)^{\frac{r_2}{s'}} \Bigg\{\int_0^\infty \chi_{(0,z]}(t) w^{1-p_2'}(t)$$

$$\times\left(\int_0^\infty \chi_{(0,t]} w^{1-p_2'}\, d_q\right)^{\frac{r_2}{s'}} d_q t\Bigg\}^{\frac{r_2}{p_1-r_2}} \Bigg\{\int_0^\infty \chi_{[z,\infty)}(t) v^{1-p_1'}(t)$$

$$\times\left(\int_0^\infty \chi_{[t,\infty)} u\, d_q\right)^{\frac{r_1}{p_1}} \left(\int_0^\infty \chi_{(0,t]} v^{1-p_1'}\, d_q\right)^{\frac{r_1}{s'}} d_q t\Bigg\}^{\frac{p_1 r_2}{r_1(p_1-r_2)}} d_q z\Bigg]^{\frac{1}{r_2}-\frac{1}{p_1}} < \infty.$$

THEOREM 23. *Let* $0<s<p_2<p_1<\infty,\ p_2>1,\ \frac{1}{s}\leqslant \frac{1}{p_1}+\frac{1}{p_2}$*.Then* (29) *holds iff*

$$\sup_{z>0}\Bigg\{\int_0^\infty \chi_{[z,\infty)}(t) v^{1-p_1'}(t)\left(\int_0^\infty \chi_{(0,t]} v^{1-p_1'}\, d_q\right)^{\frac{r_1}{s'}}\left(\int_0^\infty \chi_{[t,\infty)} u\, d_q\right)^{\frac{r_1}{s}} d_q t\Bigg\}^{\frac{1}{r_1}}$$

$$\times\Bigg\{\int_0^\infty \chi_{(0,z]}(t) w^{1-p_2'}(t)\left(\int_0^\infty \chi_{(0,t]} w^{1-p_2'}\, d_q\right)^{\frac{r_2}{s'}} d_q t\Bigg\}^{\frac{1}{r_2}} =: D_{7.5} < \infty$$

and

$$\sup_{z>0}\Bigg\{\int_0^\infty \chi_{(0,z]}(t) v^{1-p_1'}(t)\left(\int_0^\infty \chi_{(0,t]} v^{1-p_1'}\, d_q\right)^{\frac{r_1}{s'}} d_q t\Bigg\}^{\frac{1}{r_1}}$$

$$\times\Bigg\{\int_0^\infty \chi_{[z,\infty)}(t) w^{1-p_2'}(t)\left(\int_0^\infty \chi_{[t,\infty)} u d_q\right)^{\frac{r_2}{s}}\left(\int_0^\infty \chi_{(0,t]} w^{1-p_2'} d_q\right)^{\frac{r_2}{s'}} d_q t\Bigg\}^{\frac{1}{r_2}} =: D_{7.6} < \infty.$$

THEOREM 24. *Let* $0<s<p_2<p_1<\infty,\ p_2>1,\ \frac{1}{s}>\frac{1}{p_1}+\frac{1}{p_2}$*.Then* (29) *holds iff*

$$D_{7.7} := \Bigg[\int_0^\infty v^{1-p_1'}(z)\left(\int_0^\infty \chi_{(0,z]} v^{1-p_1'}\, d_q\right)^{\frac{r_1}{s'}}\left(\int_0^\infty \chi_{[z,\infty)} u\, d_q\right)^{\frac{r_1}{s}}$$

$$\times\Bigg\{\int_0^\infty \chi_{[z,\infty)}(t) v^{1-p_1'}(t)\left(\int_0^\infty \chi_{(0,t]} v^{1-p_1'}\, d_q\right)^{\frac{r_1}{s'}}$$

$$\times\left(\int_0^\infty \chi_{[t,\infty)} u\, d_q\right)^{\frac{r_1}{s}} d_qt\Bigg\}^{\frac{r_1}{p_2-r_1}} \Bigg\{\int_0^\infty \chi_{(0,z]}(t) w^{1-p_2'}(t)$$

$$\times\left(\int_0^\infty \chi_{(0,t]} w^{1-p_2'} d_q\right)^{\frac{r_2}{s'}} d_qt\Bigg\}^{\frac{p_2 r_1}{r_2(p_2-r_1)}} d_qz\Bigg]^{\frac{1}{r_1}-\frac{1}{p_2}} < \infty,$$

$$D_{7.8} := \Bigg[\int_0^\infty v^{1-p_1'}(z)\left(\int_0^\infty \chi_{(0,z]} v^{1-p_1'} d_q\right)^{\frac{r_1}{s'}}$$

$$\times\Bigg\{\int_0^\infty \chi_{(0,z]}(t) v^{1-p_1'}(t)\left(\int_0^\infty \chi_{(0,t]} v^{1-p_1'} d_q\right)^{\frac{r_1}{s'}} d_qt\Bigg\}^{\frac{r_1}{p_2-r_1}}$$

$$\times\Bigg\{\int_0^\infty \chi_{[z,\infty)}(t) w^{1-p_2'}(t)\left(\int_0^\infty \chi_{[t,\infty)}(x) u(x)\, d_qx\right)^{\frac{r_2}{p_2}}$$

$$\times\left(\int_0^\infty \chi_{(0,t]} w^{1-p_2'} d_q\right)^{\frac{r_2}{s'}} d_qt\Bigg\}^{\frac{p_2 r_1}{r_2(p_2-r_1)}} d_qz\Bigg]^{\frac{1}{r_1}-\frac{1}{p_2}} < \infty.$$

THEOREM 25. *Let $0 < s < p_1 \leqslant 1 < p_2 < \infty$, $\frac{1}{s} \leqslant \frac{1}{p_1} + \frac{1}{p_2}$. Then (29) holds iff*

$$D_{8.1} := \sup_{z>0}\Bigg\{\int_0^\infty \chi_{[z,\infty)}(t) w^{1-p_2'}(t)\left(\int_0^\infty \chi_{(0,t]} w^{1-p_2'} d_q\right)^{\frac{r_2}{s'}}$$

$$\times\left(\int_0^\infty \chi_{[t,\infty)} u d_q\right)^{\frac{r_2}{s}} d_qt\Bigg\}^{\frac{1}{r_2}} \sup_{y\leqslant z}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(t) t^{-p_1} v(t)\, d_qt\right)^{-\frac{1}{p_1}} < \infty,$$

$$D_{8.2} := \sup_{z>0}\Bigg\{\int_0^\infty \chi_{(0,z]}(t) w^{1-p_2'}(t)\left(\int_0^\infty \chi_{(0,t]}(x) w^{1-p_2'}(x)\, d_qx\right)^{\frac{r_2}{s'}} d_qt\Bigg\}^{\frac{1}{r_2}}$$

$$\times\Bigg[\int_0^\infty \chi_{[z,\infty)}(t)\frac{1}{t}\left(\int_0^\infty \chi_{[t,\infty)}(x) u(x)\, d_qx\right)^{\frac{r_1}{s}} \times$$

$$\times\Bigg\{\sup_{y\leqslant t}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(x) x^{-p_1} v(x)\, d_qx\right)^{-\frac{r_1}{p_1}} -$$

$$-\sup_{y\leqslant qt}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(x) x^{-p_1} v(x)\, d_qx\right)^{-\frac{r_1}{p_1}}\Bigg\} d_qt\Bigg]^{\frac{1}{r_1}} < \infty.$$

THEOREM 26. *Let* $0 < s < p_1 \leqslant 1 < p_2 < \infty$, $\frac{1}{s} > \frac{1}{p_1} + \frac{1}{p_2}$. *Then* (29) *holds iff*

$$D_{8.3} := \Bigg[\int_0^\infty w^{1-p_2'}(z)\left(\int_0^\infty \chi_{(0,z]}(t)w^{1-p_2'}(t)\,d_qt\right)^{\frac{r_2}{s'}}\left(\int_0^\infty \chi_{[z,\infty)}(t)u(t)\,d_qt\right)^{\frac{r_2}{s}}$$
$$\times\left\{\int_0^\infty \chi_{[z,\infty)}(t)w^{1-p_2'}(t)\left(\int_0^\infty \chi_{(0,t]}w^{1-p_2'}d_q\right)^{\frac{r_2}{s'}}\left(\int_0^\infty \chi_{[t,\infty)}ud_q\right)^{\frac{r_2}{s}}d_qt\right\}^{\frac{r_2}{p_1-r_2}}$$
$$\times\sup_{y\leqslant z}\left\{\int_0^\infty \chi_{[y,q^{-1}y)}(x)x^{-p_1}v(x)\,d_qx\right\}^{-\frac{r_2}{p_1-r_2}}d_qz\Bigg]^{\frac{1}{r_2}-\frac{1}{p_1}} < \infty,$$

$$D_{8.4} := \Bigg[\int_0^\infty w^{1-p_2'}(z)\left(\int_0^\infty \chi_{(0,z]}(x)w^{1-p_2'}(x)\,d_qx\right)^{\frac{r_2}{s'}}$$
$$\times\left\{\int_0^\infty \chi_{(0,z]}(t)w^{1-p_2'}(t)\left(\int_0^\infty \chi_{(0,t]}(x)w^{1-p_2'}(x)\,d_qx\right)^{\frac{r_2}{s'}}d_qt\right\}^{\frac{r_2}{p_1-r_2}}$$
$$\times\Bigg[\int_0^\infty \chi_{[z,\infty)}(t)\frac{1}{t}\left(\int_0^\infty \chi_{[t,\infty)}(x)u(x)\,d_qx\right)^{\frac{r_1}{p_1}}\Bigg\{\sup_{y\leqslant t}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(x)\,x^{-p_1}v(x)d_qx\right)^{-\frac{r_1}{p_1}}$$
$$-\sup_{y\leqslant qt}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(x)\,x^{-p_1}v(x)\,d_qx\right)^{-\frac{r_1}{p_1}}\Bigg\}d_qt\Bigg]^{\frac{p_1r_2}{r_1(p_1-r_2)}}d_qz\Bigg]^{\frac{1}{r_2}-\frac{1}{p_1}} < \infty.$$

THEOREM 27. *Let* $0 < s < p_2 \leqslant 1 < p_1 < \infty$, $\frac{1}{s} \leqslant \frac{1}{p_1} + \frac{1}{p_2}$. *Then* (29) *holds iff*

$$D_{8.5} := \sup_{z>0}\Bigg\{\int_0^\infty \chi_{[z,\infty)}(t)v^{1-p_1'}(t)\left(\int_0^\infty \chi_{(0,t]}(x)v^{1-p_1'}(x)\,d_qx\right)^{\frac{r_1}{s'}}$$
$$\times\left(\int_0^\infty \chi_{[t,\infty)}(x)u(x)d_qx\right)^{\frac{r_1}{s}}d_qt\Bigg\}^{\frac{1}{r_1}}\sup_{y\leqslant z}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(x)x^{-p_2}w(x)d_qx\right)^{-\frac{1}{p_2}} < \infty,$$

$$D_{8.6} := \sup_{z>0}\left\{\int_0^\infty \chi_{(0,z]}(t)v^{1-p_1'}(t)\left(\int_0^\infty \chi_{(0,t]}(x)v^{1-p_1'}(x)\,d_qx\right)^{\frac{r_1}{s'}}d_qt\right\}^{\frac{1}{r_1}}$$
$$\times\left[\int_0^\infty \chi_{[z,\infty)}(t)\frac{1}{t}\left(\int_0^\infty \chi_{[t,\infty)}(x)u(x)\,d_qx\right)^{\frac{r_2}{s}}\right.$$
$$\times\left\{\sup_{y\leqslant t}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(x)\,x^{-p_2}w(x)\,d_qx\right)^{-\frac{r_2}{p_2}}\right.$$

$$- \sup_{y \leqslant qt} \left(\int_0^\infty \chi_{[y,q^{-1}y)}(x) x^{-p_2} w(x)\, d_q x \right)^{-\frac{r_2}{p_2}} \Bigg\} d_q t \Bigg]^{\frac{1}{r_2}} < \infty.$$

THEOREM 28. *Let* $0 < s < p_2 \leqslant 1 < p_1 < \infty$, $\frac{1}{s} > \frac{1}{p_1} + \frac{1}{p_2}$. *Then* (29) *holds iff*

$$D_{8.7} := \Bigg[\int_0^\infty v^{1-p_1'}(z) \left(\int_0^\infty \chi_{(0,z]}(x) v^{1-p_1'}(x)\, d_q x \right)^{\frac{r_1}{s'}} \left(\int_0^\infty \chi_{[z,\infty)}(x) u(x)\, d_q x \right)^{\frac{r_1}{s}}$$

$$\times \left\{ \int_0^\infty \chi_{[z,\infty)}(t) v^{1-p_1'}(t) \left(\int_0^\infty \chi_{(0,t]} v^{1-p_1'} d_q \right)^{\frac{r_1}{s'}} \left(\int_0^\infty \chi_{[t,\infty)} u d_q \right)^{\frac{r_1}{s}} d_q t \right\}^{\frac{r_1}{p_2 - r_1}}$$

$$\times \sup_{y \leqslant z} \left(\int_0^\infty \chi_{[y,q^{-1}y)}(x) x^{-p_2} w(x)\, d_q x \right)^{-\frac{r_1}{p_2 - r_1}} d_q z \Bigg]^{\frac{1}{r_1} - \frac{1}{p_2}} < \infty,$$

$$D_{8.8} := \Bigg[\int_0^\infty v^{1-p_1'}(z) \left(\int_0^\infty \chi_{(0,z]}(x) v^{1-p_1'}(x)\, d_q x \right)^{\frac{r_1}{s'}}$$

$$\times \left\{ \int_0^\infty \chi_{(0,z]}(t) v^{1-p_1'}(t) \left(\int_0^\infty \chi_{(0,t]}(x) v^{1-p_1'}(x)\, d_q x \right)^{\frac{r_1}{s'}} d_q t \right\}^{\frac{r_1}{p_2 - r_1}}$$

$$\times \Bigg[\int_0^\infty \chi_{[z,\infty)}(t) \frac{1}{t} \left(\int_0^\infty \chi_{[t,\infty)}(x) u(x)\, d_q x \right)^{\frac{r_2}{p_2}}$$

$$\times \Bigg\{ \sup_{y \leqslant t} \left(\int_0^\infty \chi_{[y,q^{-1}y)}(x) x^{-p_2} w(x)\, d_q x \right)^{-\frac{r_2}{p_2}}$$

$$- \sup_{y \leqslant qt} \left(\int_0^\infty \chi_{[y,q^{-1}y)}(x) x^{-p_2} w(x)\, d_q x \right)^{-\frac{r_2}{p_2}} \Bigg\} d_q t \Bigg]^{\frac{p_2 r_1}{r_2 (p_2 - r_1)}} d_q z \Bigg]^{\frac{1}{r_1} - \frac{1}{p_2}} < \infty.$$

THEOREM 29. *Let* $0 < s < p_1 \leqslant p_2 \leqslant 1$, $\frac{1}{s} \leqslant \frac{1}{p_1} + \frac{1}{p_2}$. *Then* (29) *holds iff*

$$D_{9.1} := \sup_{z > 0} \Bigg[\int_0^\infty \chi_{[z,\infty)}(t) \frac{1}{t} \left(\int_0^\infty \chi_{[t,\infty)}(x) u(x)\, d_q x \right)^{\frac{r_2}{s}}$$

$$\times \Bigg\{ \sup_{y \leqslant t} \left(\int_0^\infty \chi_{[y,q^{-1}y)}(x) x^{-p_2} w(x)\, d_q x \right)^{-\frac{r_2}{p_2}}$$

$$- \sup_{y \leqslant qt} \left(\int_0^\infty \chi_{[y,q^{-1}y)}(x) x^{-p_2} w(x)\, d_q x \right)^{-\frac{r_2}{p_2}} \Bigg\} d_q t \Bigg]^{\frac{1}{r_2}}$$

$$\times \sup_{y\leqslant z}\left\{\int_0^\infty \chi_{[y,q^{-1}y)}(x)x^{-p_1}v(x)\,d_qx\right\}^{-\frac{1}{p_1}}<\infty,$$

$$D_{9.2}:=\sup_{z>0}\left[\int_0^\infty \chi_{[z,\infty)}(t)\frac{1}{t}\left(\int_0^\infty \chi_{[t,\infty)}(x)u(x)\,d_qx\right)^{\frac{r_1}{s}}\right.$$
$$\times\left\{\sup_{y\leqslant t}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(x)x^{-p_1}v(x)\,d_qx\right)^{-\frac{r_1}{p_1}}\right.$$
$$\left.\left.-\sup_{y\leqslant qt}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(x)x^{-p_1}v(x)\,d_qx\right)^{-\frac{r_1}{p_1}}\right\}d_qt\right]^{\frac{1}{r_1}}$$
$$\times \sup_{y\leqslant z}\left\{\int_0^\infty \chi_{[y,q^{-1}y)}(x)x^{-p_2}w(x)\,d_qx\right\}^{-\frac{1}{p_2}}<\infty.$$

THEOREM 30. *Let* $0<s<p_1\leqslant p_2\leqslant 1,\ \frac{1}{s}>\frac{1}{p_1}+\frac{1}{p_2}$. *Then* (29) *holds iff*

$$D_{9.3}:=\left[\int_0^\infty \frac{1}{z}\left(\int_0^\infty \chi_{[z,\infty)}(t)u(t)\,d_qt\right)^{\frac{r_2}{s}}\left\{\sup_{y\leqslant z}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(x)x^{-p_2}w(x)\,d_qx\right)^{-\frac{r_2}{p_2}}\right.\right.$$
$$\left.-\sup_{y\leqslant qz}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(x)x^{-p_2}w(x)\,d_qx\right)^{-\frac{r_2}{p_2}}\right\}$$
$$\times\left\{\int_0^\infty \chi_{[z,\infty)}(t)\frac{1}{t}\left(\int_0^\infty \chi_{[t,\infty)}(x)u(x)\,d_qx\right)^{\frac{r_2}{s}}\right.$$
$$\times\left(\sup_{y\leqslant t}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(x)x^{-p_2}w(x)\,d_qx\right)^{-\frac{r_2}{p_2}}\right.$$
$$\left.\left.-\ \sup_{y\leqslant qt}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(x)x^{-p_2}w(x)\,d_qx\right)^{-\frac{r_2}{p_2}}\right)d_qt\right\}^{\frac{r_2}{p_1-r_2}}$$
$$\left.\times \sup_{y\leqslant z}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(x)x^{-p_1}v(x)\,d_qx\right)^{-\frac{r_2}{p_1-r_2}}d_qz\right]^{\frac{1}{r_2}-\frac{1}{p_1}}<\infty,$$

$$D_{9.4}:=\left[\int_0^\infty \frac{1}{z}\left\{\sup_{y\leqslant z}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(x)\,x^{-p_2}w(x)\,d_qx\right)^{-\frac{r_2}{p_2}}\right.\right.$$
$$\left.-\sup_{y\leqslant qz}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(x)\,x^{-p_2}w(x)\,d_qx\right)^{-\frac{r_2}{p_2}}\right\}$$
$$\times \sup_{y\leqslant z}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(x)\,x^{-p_2}w(x)\,d_qx\right)^{-\frac{r_2^2}{p_2(p_1-r_2)}}$$

$$\times \left\{ \int_0^\infty \chi_{[z,\infty)}(t) \frac{1}{t} \left(\int_0^\infty \chi_{[t,\infty)}(x) u(x)\, d_q x \right)^{\frac{r_1}{p_1}} \right.$$

$$\times \left(\sup_{y \leqslant t} \left(\int_0^\infty \chi_{[y,q^{-1}y)}(x) x^{-p_1} v(x)\, d_q x \right)^{-\frac{r_1}{p_1}} \right.$$

$$\left. \left. - \sup_{y \leqslant qt} \left(\int_0^\infty \chi_{[y,q^{-1}y)}(x) x^{-p_1} v(x)\, d_q x \right)^{-\frac{r_1}{p_1}} \right) d_q t \right\}^{\frac{p_1 r_2}{r_1(p_1-r_2)}} d_q z \Bigg]^{\frac{1}{r_2}-\frac{1}{p_1}} < \infty.$$

THEOREM 31. *Let* $0 < s < p_2 < p_1 \leqslant 1$, $\frac{1}{s} \leqslant \frac{1}{p_1} + \frac{1}{p_2}$. *Then* (29) *holds iff*

$$D_{9.5} := \sup_{z>0} \left[\int_0^\infty \chi_{[z,\infty)}(t) \frac{1}{t} \left(\int_0^\infty \chi_{[t,\infty)}(x) u(x)\, d_q x \right)^{\frac{r_1}{s}} \right.$$

$$\times \left\{ \sup_{y \leqslant t} \left(\int_0^\infty \chi_{[y,q^{-1}y)}(x) x^{-p_1} v(x)\, d_q x \right)^{-\frac{r_1}{p_1}} \right.$$

$$\left. \left. - \sup_{y \leqslant qt} \left(\int_0^\infty \chi_{[y,q^{-1}y)}(x) x^{-p_1} v(x)\, d_q x \right)^{-\frac{r_1}{p_1}} \right\} d_q t \right]^{\frac{1}{r_1}}$$

$$\times \sup_{y \leqslant z} \left\{ \int_0^\infty \chi_{[y,q^{-1}y)}(x) x^{-p_2} w(x)\, d_q x \right\}^{-\frac{1}{p_2}} < \infty,$$

$$D_{9.6} := \sup_{z>0} \left[\int_0^\infty \chi_{[z,\infty)}(t) \frac{1}{t} \left(\int_0^\infty \chi_{[t,\infty)}(x) u(x)\, d_q x \right)^{\frac{r_2}{s}} \times \right.$$

$$\times \left\{ \sup_{y \leqslant t} \left(\int_0^\infty \chi_{[y,q^{-1}y)}(x) x^{-p_2} w(x)\, d_q x \right)^{-\frac{r_2}{p_2}} \right.$$

$$\left. \left. - \sup_{y \leqslant qt} \left(\int_0^\infty \chi_{[y,q^{-1}y)}(x) x^{-p_2} w(x)\, d_q x \right)^{-\frac{r_2}{p_2}} \right\} d_q t \right]^{\frac{1}{r_2}}$$

$$\times \sup_{y \leqslant z} \left\{ \int_0^\infty \chi_{[y,q^{-1}y)}(x) x^{-p_1} v(x)\, d_q x \right\}^{-\frac{1}{p_1}} < \infty.$$

THEOREM 32. *Let* $0 < s < p_2 < p_1 \leqslant 1$, $\frac{1}{s} > \frac{1}{p_1} + \frac{1}{p_2}$. *Then* (29) *holds iff*

$$D_{9.7} := \left[\int_0^\infty \frac{1}{z} \left(\int_0^\infty \chi_{[z,\infty)}(t) u(t)\, d_q t \right)^{\frac{r_1}{s}} \left\{ \sup_{y \leqslant z} \left(\int_0^\infty \chi_{[y,q^{-1}y)}(x) x^{-p_1} v(x)\, d_q x \right)^{-\frac{r_1}{p_1}} \right. \right.$$

$$\left. - \sup_{y \leqslant qz} \left(\int_0^\infty \chi_{[y,q^{-1}y)}(x) x^{-p_1} v(x)\, d_q x \right)^{-\frac{r_1}{p_1}} \right\}$$

$$\times\left\{\int_0^\infty \chi_{[z,\infty)}(t)\frac{1}{t}\left(\int_0^\infty \chi_{[t,\infty)}(x)u(x)\,d_qx\right)^{\frac{r_1}{s}}\left(\sup_{y\leqslant t}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(x)x^{-p_1}v(x)d_qx\right)^{-\frac{r_1}{p_1}}\right.\right.$$
$$\left.\left.-\sup_{y\leqslant qt}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(x)x^{-p_1}v(x)\,d_qx\right)^{-\frac{r_1}{p_1}}\right)d_qt\right\}^{\frac{r_1}{p_2-r_1}}$$
$$\left.\times\sup_{y\leqslant z}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(x)x^{-p_2}w(x)\,d_qx\right)^{-\frac{r_1}{p_2-r_1}}d_qz\right]^{\frac{1}{r_1}-\frac{1}{p_2}}<\infty,$$

$$D_{9.8}:=\left[\int_0^\infty \frac{1}{z}\left\{\sup_{y\leqslant z}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(x)\,x^{-p_1}v(x)\,d_qx\right)^{-\frac{r_1}{p_1}}\right.\right.$$
$$\left.-\sup_{y\leqslant qz}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(x)\,x^{-p_1}v(x)\,d_qx\right)^{-\frac{r_1}{p_1}}\right\}$$
$$\times\sup_{y\leqslant z}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(x)\,x^{-p_1}v(x)\,d_qx\right)^{-\frac{r_1^2}{p_1(p_2-r_1)}}$$
$$\times\left\{\int_0^\infty \chi_{[z,\infty)}(t)\frac{1}{t}\left(\int_0^\infty \chi_{[t,\infty)}(x)u(x)d_qx\right)^{\frac{r_2}{p_2}}\left(\sup_{y\leqslant t}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(x)\,x^{-p_2}w(x)\,d_qx\right)^{-\frac{r_2}{p_2}}\right.\right.$$
$$\left.\left.\left.-\sup_{y\leqslant qt}\left(\int_0^\infty \chi_{[y,q^{-1}y)}(x)\,x^{-p_2}w(x)\,d_qx\right)^{-\frac{r_2}{p_2}}\right)d_qt\right\}^{\frac{p_2r_1}{r_2(p_2-r_1)}}d_qz\right]^{\frac{1}{r_1}-\frac{1}{p_2}}<\infty.$$

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