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WEIGHTED COMPOSITION OPERATORS FROM DIRICHLET–TYPE SPACES INTO STEVIĆ–TYPE SPACES

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Abstract. The boundedness and compactness of weighted composition operators from Dirichlet-type spaces into Stević-type spaces are investigated in this paper. Some estimates for the essential norm of weighted composition operators are also given.

1. Introduction

Let $\mathbb{N}$ be the set of all positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $H(\mathbb{D})$ the space of all analytic functions on the open unit disk $\mathbb{D}$. Let φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. The weighted composition operator uC_φ, which is induced by φ and u, is defined on $H(\mathbb{D})$ by

$$(uC_\varphi f)(z) = u(z)f(\varphi(z)), \qquad f \in H(\mathbb{D}).$$

When $u \equiv 1$, we get the composition operator C_φ. We refer the readers to [7, 32] for the theory of composition operators and weighted composition operators.

For $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space A^p_α is the set of all $f \in H(\mathbb{D})$ such that (see, e.g., [32])

$$\|f\|^p_{A^p_\alpha} = (\alpha+1)\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dA(z) < \infty,$$

where $dA(z)$ is the normalized Lebesgue area measure. We say that an $f \in H(\mathbb{D})$ belongs to the Dirichlet-type space, denoted by $\mathscr{D}^p_\alpha$, if

$$\|f\|^p_{\mathscr{D}^p_\alpha} = |f(0)|^p + (\alpha+1)\int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^\alpha dA(z) < \infty.$$

When $p > 1$ and $\alpha = p-2$, the Dirichlet-type space $\mathscr{D}^p_{p-2}$ is just the Besov space, which is denoted by B_p. Denote by $H^\infty = H^\infty(\mathbb{D})$ the space of bounded analytic functions on $\mathbb{D}$.

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A positive and continuous function μ is called *weight*. It is radial if $\mu(z) = \mu(|z|)$, for every $z \in \mathbb{D}$. Let μ be a radial weight. The Bloch-type space $\mathscr{B}_\mu$ consists of all $f \in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}} \mu(z)|f'(z)| < \infty.$$

When $\mu(z) = (1-|z|^2)^\alpha$, the space $\mathscr{B}_\mu$ becomes the Bloch-type space $\mathscr{B}^\alpha$ ([31]), which for $\alpha = 1$ reduces to the Bloch space $\mathscr{B}$. It is a simple consequence of the Schwartz-Pick lemma that any analytic self-mapping φ of $\mathbb{D}$ induces a bounded composition operator C_φ on the Bloch space ([14]).

Let $n \in \mathbb{N}_0$. In [17] Stević introduced the space consisting of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathscr{W}_\mu^n} = \sum_{k=0}^{n-1} |f^{(k)}(0)| + \sup_{z\in\mathbb{D}} \mu(z)|f^{(n)}(z)| < \infty,$$

the, so called, Stević-type space, which he called the n-th weighted-type space and denoted by $\mathscr{W}_\mu^n$. It is easy to check that $\mathscr{W}_\mu^n$ is a Banach space with the above norm. Recently, there has been a great interest in studying weighted composition and other concrete product-type operators (see, for example, [1]-[5], [7]-[34] and the related references therein), many of which study mappings from or into Bloch-type spaces or Stević-type spaces such as [1, 2, 3, 4, 5, 8, 10, 11, 12, 13, 14, 17, 18, 19, 20, 21, 23, 24, 25, 26, 30, 33, 34].

In 2009, Stević firstly studied composition operators from A_α^p to $\mathscr{W}_\mu^n$ in [17]. Composition followed by differentiation from H^∞ and the Bloch space to $\mathscr{W}_\mu^n$ was studied by Stević in [20]. The corresponding space on the upper half-plane was introduced in [21], where the composition operators from the Hardy space to the Stević-type spaces on the unit disk and the half-plane were studied. The weighted differentiation composition operators from H^∞ and the Bloch space $\mathscr{B}$ to $\mathscr{W}_\mu^n$ was studied in [23] by Stević. The weighted differentiation composition operators from the mixed-norm space to the Stević-type spaces on the unit disk was studied in [24]. The corresponding space on the unit ball was introduced by Stević in [25], where he studied the weighted radial operator from the mixed-norm space to the Stević-type space. In this way the series of papers [17, 20, 21, 23, 24, 25] introduced the basic notions and established the ground for further investigations of concrete operators from or to Stević-type spaces on various domains of the complex plane or the vector space $\mathbb{C}^n$. Motivated by [17, 20, 21, 23, 24, 25] some other authors continue the line of investigations. For example, in [1], Abbasi et al gave some new characterizations for the boundedness, compactness and essential norm of the operators $uC_\varphi : H^\infty \to \mathscr{W}_\mu^n$. In [34], the author of this paper and Du studied weighted composition operators from weighted Bergman spaces A_ω^p with doubling weight to $\mathscr{W}_\mu^n$. Abbasi and the author of this paper in [2] studied the boundedness and compactness of weighted composition operators from Besov spaces $B_p = \mathscr{D}_{p-2}^p$ (when $\alpha = p-2$) to $\mathscr{W}_\mu^n$.

Another topic of recent interest is studying essential norms of operators. Some classical results can be found in [7], while some recent ones can be found, for example, in [1, 5, 8, 9, 10, 12, 13, 18, 19, 22, 26, 27, 29, 33]. Related to the spaces studing in the present paper, we would like to mention that essential norm of some extensions of the generalized composition operators between Stević-type spaces was studied in [26], whereas essential norm of an integral-type operator from the Dirichlet space to the Bloch-type space on the unit ball was studied in [22].

Recall that the essential norm of a bounded linear operator $uC_\varphi : \mathscr{D}^p_\alpha \to \mathscr{W}^n_\mu$ is its distance to the set of compact operators K mapping $\mathscr{D}^p_\alpha$ into $\mathscr{W}^n_\mu$, that is,

$$\|uC_\varphi\|_{e,\mathscr{D}^p_\alpha \to \mathscr{W}^n_\mu} = \inf\left\{ \|uC_\varphi - K\|_{\mathscr{D}^p_\alpha \to \mathscr{W}^n_\mu} : K \text{ is a compact operator } \right\}.$$

Here $\|\cdot\|_{\mathscr{D}^p_\alpha \to \mathscr{W}^n_\mu}$ denotes the operator norm.

Motivated by [2, 17, 20, 21, 23, 24, 34], here we study the boundedness and compactness of weighted composition operators uC_φ from $\mathscr{D}^p_\alpha(-1 < \alpha < p-2)$ to $\mathscr{W}^n_\mu$. This was done by employing Stević's idea of using Bell polynomials ([17, 20, 21, 23, 24, 25]). We also give some estimates for $\|uC_\varphi\|_{e,\mathscr{D}^p_\alpha \to \mathscr{W}^n_\mu}$, the essential norm of $uC_\varphi : \mathscr{D}^p_\alpha \to \mathscr{W}^n_\mu$.

Throughout the paper, we denote by C a positive constant which may differ from one occurrence to the next. In addition, we say that $A \lesssim B$ if there exists a constant C such that $A \leqslant CB$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

2. Main results and proofs

In this section we formulate and prove our main results in this paper. For this purpose, we state some lemmas which will be used in this paper. The first one is folklore, hence we omit its proof.

LEMMA 1. *Let $\alpha > -1$, $\alpha + 2 < p < \infty$ and k be a positive integer. Then there exists a positive constant C such that*

$$|f(z)| \leqslant C\|f\|_{\mathscr{D}^p_\alpha},$$

and

$$\left|f^{(k)}(z)\right| \leqslant \frac{C\|f\|_{\mathscr{D}^p_\alpha}}{(1-|z|^2)^{k-1+(\alpha+2)/p}} \tag{1}$$

for every $f \in \mathscr{D}^p_\alpha$.

LEMMA 2. *Let $\alpha > -1$, $\alpha + 2 < p < \infty$ and $0 \neq a \in \mathbb{D}$. For any $j \in \{1, 2, \ldots, n+1\}$, set*

$$f_{j,a}(z) = \frac{(1-|a|^2)^j}{(1-\overline{a}z)^{j+\frac{\alpha+2}{p}-1}}, \quad z \in \mathbb{D}. \tag{2}$$

Then, $f_{j,a}$ converge to 0 uniformly in $\overline{\mathbb{D}}$ as $|a| \to 1$,

$$f_{j,a} \in \mathscr{D}^p_\alpha \quad \text{and} \quad \sup_{a \in \mathbb{D}} \|f_{j,a}\|_{\mathscr{D}^p_\alpha} < \infty.$$

Proof. Using Lemma 3.10 in [32], after some simple calculations, we get that $f_{j,a} \in \mathscr{D}^p_\alpha$ and $\sup_{a\in\mathbb{D}} \|f_{j,a}\|_{\mathscr{D}^p_\alpha} < \infty$. In addition, since $\frac{\alpha+2}{p} - 1 < 0$, it is easy to see that $f_{j,a}$ converge to 0 uniformly in $\overline{\mathbb{D}}$ as $|a| \to 1$.

The following lemma is proved similar to the corresponding ones in Stević's papers [17, 20, 21, 23, 24]. Hence we omit the details of the proof.

LEMMA 3. *Let $\alpha > -1$, $\alpha+2 < p < \infty$ and $0 \neq a \in \mathbb{D}$. For any $i \in \{1,...,n\}$, there exist constants $c^i_1,...,c^i_{n+1}$, which are independent of the choice of a, such that*

$$v_{i,a} = \sum_{j=1}^{n+1} c^i_j f_{j,a} \in \mathscr{D}^p_\alpha, \qquad v_{i,a}(a) = 0,$$

and for $k \in \{1,...,n\}$,

$$v^{(k)}_{i,a}(a) = \begin{cases} \dfrac{\bar{a}^i}{(1-|a|^2)^{i+\frac{\alpha+2}{p}-1}}, & k = i, \\ 0, & k \neq i. \end{cases}$$

Moreover, $v_{i,a}$ converge to 0 uniformly in $\overline{\mathbb{D}}$ as $|a| \to 1$.

LEMMA 4. *Let $\alpha > -1$, $\alpha+2 < p < \infty$. If $f \in \mathscr{D}^p_\alpha$, then for all $t \in (0,1)$ and $z \in \mathbb{D}\setminus\{0\}$, there exists a positive constant C such that*

$$\left| f(z) - f\left(\frac{t}{|z|}z\right)\right| \leqslant C\|f\|_{\mathscr{D}^p_\alpha}(1-|z|)^{1-\frac{\alpha+2}{p}}.$$

Proof. Fix $f \in \mathscr{D}^p_\alpha$. Let $t \in (0,1)$ and $z \in \mathbb{D}\setminus\{0\}$. Then by Lemma 1, we get

$$\begin{aligned} \left| f(z) - f\left(\frac{t}{|z|}z\right)\right| &\leqslant \left| \int_1^{t/|z|} zf'(sz)ds\right| \leqslant \int_1^{1/|z|} |z||f'(sz)|ds \\ &\leqslant C\|f\|_{\mathscr{D}^p_\alpha} \int_1^{1/|z|} \frac{|z|}{(1-s^2|z|^2)^{\frac{\alpha+2}{p}}} ds \leqslant C\|f\|_{\mathscr{D}^p_\alpha}(1-|z|)^{1-\frac{\alpha+2}{p}}, \end{aligned}$$

as desired.

By using Lemma 1 and Lemma 4, similarly, for example, to the proofs of Lemma 4 and Lemma 6 in [28], we get the following lemma.

LEMMA 5. *Let $\alpha > -1$, $\alpha+2 < p < \infty$. Then, every norm bounded sequence in $\mathscr{D}^p_\alpha$ has a subsequence which converges uniformly in $\overline{\mathbb{D}}$ to a function in $\mathscr{D}^p_\alpha$.*

LEMMA 6. [4] *Let X be a Banach space that is continuously contained in the disk algebra, and let Y be any Banach space of analytic functions on $\mathbb{D}$. Suppose that*

(1) The point evaluation functionals on Y are continuous.

(2) For every sequence $\{f_n\}$ in the unit ball of X there exists $f \in X$ and a subsequence $\{f_{n_j}\}$ such that $f_{n_j} \to f$ uniformly on $\overline{\mathbb{D}}$.

(3) The operator $T : X \to Y$ is continuous if X has the supremum norm and Y is given the topology of uniform convergence on compact sets.

Then, T is a compact operator if and only if, given a bounded sequence $\{f_n\}$ in X such that $f_n \to 0$ uniformly on $\overline{\mathbb{D}}$, then the sequence $\|Tf_n\|_Y \to 0$ as $n \to \infty$.

The following Schwartz-type result ([15]) is a direct consequence of Lemmas 5 and 6.

LEMMA 7. *Let $\alpha > -1$, $\alpha + 2 < p < \infty$, $n \in \mathbb{N}$ and μ be a weight. If $T : \mathscr{D}^p_\alpha \to \mathscr{W}^n_\mu$ is bounded, then T is compact if and only if $\|Tf_k\|_{\mathscr{W}^n_\mu} \to 0$ as $k \to \infty$ for any sequence $\{f_k\}$ in $\mathscr{D}^p_\alpha$ bounded in norm which converge to 0 uniformly in $\overline{\mathbb{D}}$.*

Let $n, k \in \mathbb{N}_0$ with $k \leqslant n$. The partial Bell polynomials are defined as follows

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum \frac{n!}{j_1! j_2! \dots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}},$$

where the sum is taken over all sequences $j_1, j_2, \dots, j_{n-k+1}$ of nonnegative integers such that the following two conditions hold

$$j_1 + j_2 + \cdots + j_{n-k+1} = k \quad \text{and} \quad j_1 + 2j_2 + \cdots + (n-k+1)j_{n-k+1} = n.$$

See [6] for more information about Bell polynomials.

Now we are in a position to state and prove the main results in this paper.

THEOREM 1. *Let $\alpha > -1$, $\alpha + 2 < p < \infty$, $n \in \mathbb{N}$ and μ be a weight. Let φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. Then the following statements are equivalent.*

(i) *The operator $uC_\varphi : \mathscr{D}^p_\alpha \to \mathscr{W}^n_\mu$ is bounded.*

(ii) $u \in \mathscr{W}^n_\mu$,

$$\sum_{j=1}^{n+1} \sup_{a \in \mathbb{D}} \|uC_\varphi f_{j,a}\|_{\mathscr{W}^n_\mu} < \infty \qquad \textit{and} \qquad \sum_{i=1}^{n} \sup_{z \in \mathbb{D}} \mu(z) |I^n_i(z)| < \infty.$$

(iii) *$u \in \mathscr{W}_\mu^n$ and*

$$\sum_{i=1}^{n} \sup_{z\in\mathbb{D}} \frac{\mu(z)|I_i^n(z)|}{(1-|\varphi(z)|^2)^{i+\frac{\alpha+2}{p}-1}} < \infty.$$

Here $f_{j,a}$ are defined in (2) and

$$I_i^n(z) = \sum_{l=i}^{n} \binom{n}{l} u^{(n-l)}(z) B_{l,i}(\varphi'(z), \varphi''(z), \ldots, \varphi^{(l-i+1)}(z)). \tag{3}$$

Proof. (iii)$\Rightarrow$(i). Let $f \in \mathscr{D}_\alpha^p$. Since $B_{0,0}(\varphi'(z)) = 1$ and

$$B_{l,0}(\varphi'(z), \ldots, \varphi^{(l+1)}(z)) = 0 (l \in \mathbb{N}),$$

we get that $I_0^n(z) = u^{(n)}(z)$. By a known formula and Lemma 1, we obtain

$$\begin{aligned}
&\mu(z)|(uC_\varphi f)^{(n)}(z)| \\
=&\mu(z)\left|\sum_{i=0}^{n} f^{(i)}(\varphi(z)) \sum_{l=i}^{n} \binom{n}{l} u^{(n-l)}(z) B_{l,i}(\varphi'(z), \ldots, \varphi^{(l-i+1)}(z))\right| && (4)\\
\leqslant&\mu(z)|f(\varphi(z))I_0^n(z)| + \mu(z)\sum_{i=1}^{n} \left|f^{(i)}(\varphi(z))\right|\left|I_i^n(z)\right| && (5)\\
\lesssim&\|f\|_{\mathscr{D}_\alpha^p}\|u\|_{\mathscr{W}_\mu^n} + \|f\|_{\mathscr{D}_\alpha^p} \sum_{i=1}^{n} \sup_{z\in\mathbb{D}} \frac{\mu(z)|I_i^n(z)|}{(1-|\varphi(z)|^2)^{i+\frac{\alpha+2}{p}-1}} \\
\lesssim&\|f\|_{\mathscr{D}_\alpha^p}. && (6)
\end{aligned}$$

From (5) for each $j \in \{0, 1, \ldots, n-1\}$,

$$\begin{aligned}
|(uC_\varphi f)^{(j)}(0)| &\leqslant |f(\varphi(0))||u^{(j)}(0)| + \sum_{i=1}^{j} |f^{(i)}(\varphi(0))||I_i^j(0)| \\
&\lesssim \|f\|_{\mathscr{D}_\alpha^p}|u^{(j)}(0)| + \|f\|_{\mathscr{D}_\alpha^p} \sum_{i=1}^{j} \frac{|I_i^j(0)|}{(1-|\varphi(0)|^2)^{i+\frac{\alpha+2}{p}-1}} \lesssim \|f\|_{\mathscr{D}_\alpha^p}. \qquad (7)
\end{aligned}$$

So, by (6) and (7) we see that $uC_\varphi : \mathscr{D}_\alpha^p \to \mathscr{W}_\mu^n$ is bounded.

(i)$\Rightarrow$(ii). Assume that $uC_\varphi : \mathscr{D}_\alpha^p \to \mathscr{W}_\mu^n$ is bounded. By Lemma 2, it is clear that

$$\sum_{j=1}^{n+1} \sup_{a\in\mathbb{D}} \|uC_\varphi f_{j,a}\|_{\mathscr{W}_\mu^n} < \infty. \tag{8}$$

Applying the operator uC_φ to $h_0(z) = 1$, we obtain $u \in \mathscr{W}_\mu^n$. Applying the operator uC_φ to $h_1(z) = z$, by (3) and (4) we obtain

$$\sup_{z\in\mathbb{D}} \mu(z)|I_0^n(z)\varphi(z) + I_1^n(z)| = \sup_{z\in\mathbb{D}} \mu(z)|(uC_\varphi h_1)^{(n)}(z)| \leqslant \|uC_\varphi h_1\|_{\mathscr{W}_\mu^n} < \infty.$$

By the boundedness of φ, $u \in \mathcal{W}_\mu^n$ and the triangle inequality, we have

$$\sup_{z\in\mathbb{D}} \mu(z)|I_1^n(z)| < \infty.$$

Now assume that for $1 \leqslant i \leqslant j-1 (j \leqslant n)$, $\sup_{z\in\mathbb{D}} \mu(z)|I_i^n(z)| < \infty$. To get the desired result, we only need to show that

$$\sup_{z\in\mathbb{D}} \mu(z)|I_j^n(z)| < \infty.$$

Applying the operator uC_φ to $h_j(z) = z^j$, we obtain

$$\sup_{z\in\mathbb{D}} \mu(z)\left|\varphi^j(z)I_0^n(z) + \sum_{k=1}^{j} j(j-1)\cdots(j-k+1)(\varphi(z))^{j-k}\, I_k^n(z)\right| \leqslant \|uC_\varphi h_j\|_{\mathcal{W}_\mu^n} < \infty.$$

Hence, from the boundedness of φ and by using the triangle inequality again, we get the desired result.

(ii)$\Rightarrow$(iii). Assume that (ii) holds. For any $i \in \{1,...,n\}$ and $\varphi(a) \neq 0$, by Lemma 3, we obtain

$$\begin{aligned}\frac{\mu(a)|\varphi(a)|^i|I_i^n(a)|}{(1-|\varphi(a)|^2)^{i+\frac{\alpha+2}{p}-1}} &\leqslant \sup_{a\in\mathbb{D}} \|uC_\varphi v_{i,\varphi(a)}\|_{\mathcal{W}_\mu^n} \\ &\leqslant \sum_{j=1}^{n+1} |c_j^i| \sup_{a\in\mathbb{D}} \|uC_\varphi f_{j,a}\|_{\mathcal{W}_\mu^n} < \infty,\end{aligned}$$

where c_j^i are independent of the choice of a. From the last inequality, we get

$$\sum_{i=1}^{n} \sup_{|\varphi(a)|>\frac{1}{2}} \frac{\mu(a)|I_i^n(a)|}{(1-|\varphi(a)|^2)^{i+\frac{\alpha+2}{p}-1}} \lesssim \sum_{i=1}^{n}\sum_{j=1}^{n+1} |c_j^i| \sup_{a\in\mathbb{D}} \|uC_\varphi f_{j,a}\|_{\mathcal{W}_\mu^n} < \infty.$$

From the assumption that $\sum_{i=1}^n \sup_{z\in\mathbb{D}} \mu(z)|I_i^n(z)| < \infty$, we obtain

$$\sum_{i=1}^{n} \sup_{|\varphi(a)|\leqslant\frac{1}{2}} \frac{\mu(a)|I_i^n(a)|}{(1-|\varphi(a)|^2)^{i+\frac{\alpha+2}{p}-1}} \lesssim \sum_{i=1}^{n} \sup_{|\varphi(a)|\leqslant\frac{1}{2}} \mu(a)|I_i^n(a)| < \infty.$$

From the last two estimates the implication follows. The proof is complete.

Let $n=1$. We get the following corollary.

COROLLARY 1. *Let $\alpha > -1$, $\alpha+2 < p < \infty$ and μ be a weight. Let φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. Then the operator $uC_\varphi : \mathcal{D}_\alpha^p \to \mathcal{B}_\mu$ is bounded if and only if $u \in \mathcal{B}_\mu$ and*

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} < \infty.$$

Next, we give some estimates for the essential norm of $uC_\varphi : \mathcal{D}_\alpha^p \to \mathcal{W}_\mu^n$.

THEOREM 2. *Let $\alpha > -1$, $\alpha + 2 < p < \infty$, $n \in \mathbb{N}$ and μ be a weight. Let φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$ such that $uC_\varphi : \mathscr{D}^p_\alpha \to \mathscr{W}^n_\mu$ is bounded. Then*

$$\|uC_\varphi\|_{e,\mathscr{D}^p_\alpha \to \mathscr{W}^n_\mu} \approx \sum_{j=1}^{n+1} \limsup_{|a|\to 1} \|uC_\varphi f_{j,a}\|_{\mathscr{W}^n_\mu} \approx \sum_{i=1}^{n} B_i,$$

where

$$B_i = \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|I^n_i(z)|}{(1-|\varphi(z)|^2)^{i+\frac{\alpha+2}{p}-1}}.$$

Proof. First we prove that

$$\|uC_\varphi\|_{e,\mathscr{D}^p_\alpha \to \mathscr{W}^n_\mu} \lesssim \sum_{j=1}^{n+1} \limsup_{|a|\to 1} \|uC_\varphi f_{j,a}\|_{\mathscr{W}^n_\mu} \quad \text{and} \quad \|uC_\varphi\|_{e,\mathscr{D}^p_\alpha \to \mathscr{W}^n_\mu} \lesssim \sum_{i=1}^{n} B_i.$$

Let $r \in [0,1)$ and define $K_r f(z) = f_r(z) = f(rz)$. Then $K_r : \mathscr{D}^p_\alpha \to \mathscr{D}^p_\alpha$ is compact and $\|K_r\|_{\mathscr{D}^p_\alpha \to \mathscr{D}^p_\alpha} \leqslant 1$. It is clear that $f_r \to f$ uniformly on compact subsets of $\mathbb{D}$ as $r \to 1$. Let $\{r_j\} \subset (0,1)$ be a sequence such that $r_j \to 1$ as $j \to \infty$. Then for any $j \in \mathbb{N}$, $uC_\varphi K_{r_j} : \mathscr{D}^p_\alpha \to \mathscr{W}^n_\mu$ is compact. So,

$$\|uC_\varphi\|_{e,\mathscr{D}^p_\alpha \to \mathscr{W}^n_\mu} \leqslant \limsup_{j\to\infty} \|uC_\varphi - uC_\varphi K_{r_j}\|_{\mathscr{D}^p_\alpha \to \mathscr{W}^n_\mu}.$$

Hence, it is sufficient to show that

$$\limsup_{j\to\infty} \|uC_\varphi - uC_\varphi K_{r_j}\|_{\mathscr{D}^p_\alpha \to \mathscr{W}^n_\mu} \lesssim \min\left\{ \sum_{j=1}^{n+1} \limsup_{|a|\to 1} \|uC_\varphi f_{j,a}\|_{\mathscr{W}^n_\mu}, \sum_{i=1}^{n} B_i \right\}. \tag{9}$$

For any $f \in \mathscr{D}^p_\alpha$ such that $\|f\|_{\mathscr{D}^p_\alpha} \leqslant 1$,

$$\begin{aligned} &\|(uC_\varphi - uC_\varphi K_{r_j})f\|_{\mathscr{W}^n_\mu} \\ \leqslant &\sum_{t=0}^{n-1}\sum_{i=0}^{t} \left|(f-f_{r_j})^{(i)}(\varphi(0))\right| |I^t_i(0)| + \sup_{z\in\mathbb{D}} \mu(z) \sum_{i=0}^{n} \left|(f-f_{r_j})^{(i)}(\varphi(z))\right| |I^n_i(z)| \\ \leqslant &\Omega_0 + \Omega_1 + \Omega_2 + \Omega_3. \end{aligned} \tag{10}$$

Here

$$\Omega_0 = \sum_{t=0}^{n-1}\sum_{i=0}^{t} \left|(f-f_{r_j})^{(i)}(\varphi(0)) I^t_i(0)\right|, \qquad \Omega_1 = \sup_{z\in\mathbb{D}} \mu(z)|(f-f_{r_j})(z)| \left|u^{(n)}(z)\right|,$$

$$\Omega_2 = \sup_{|\varphi(z)|\leqslant r_N} \mu(z) \sum_{i=1}^{n} \left|(f-f_{r_j})^{(i)}(\varphi(z))\right| |I^n_i(z)|$$

and

$$\Omega_3 = \sup_{|\varphi(z)|>r_N} \mu(z) \sum_{i=1}^{n} \left|(f - f_{r_j})^{(i)}(\varphi(z))\right| |I_i^n(z)|,$$

where $N \in \mathbb{N}$ is such that $r_j \geqslant \frac{2}{3}$ for all $j \geqslant N$.

Since for any nonnegative integer s, $(f - f_{r_j})^{(s)} \to 0$, uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$. It is clear that

$$\limsup_{j\to\infty} \Omega_0 = 0 \qquad \text{and} \qquad \limsup_{j\to\infty} \Omega_2 = 0. \tag{11}$$

Assume that $\{g_j\}$ is a bounded sequence in $\mathscr{D}_\alpha^p$ satisfying $g_j \to 0$ uniformly on any compact subset of $\mathbb{D}$. For any $\varepsilon > 0$, there exists $0 < \eta < 1$ such that $(1-\eta)^{1-\frac{\alpha+2}{p}} < \varepsilon$. By Lemma 4, there exists a $C > 0$ such that

$$\left| g_j(z) - g_j\left(\frac{\eta}{|z|} z\right)\right| \leqslant C\|g_j\|_{\mathscr{D}_\alpha^p} (1-\eta)^{1-\frac{\alpha+2}{p}} \leqslant C\|g_j\|_{\mathscr{D}_\alpha^p} \varepsilon,$$

when $\eta < |z| < 1$.

Hence

$$\sup_{\eta<|z|<1} |g_j(z)| \leqslant C\|g_j\|_{\mathscr{D}_\alpha^p} \varepsilon + \sup_{|w|=\eta} |g_j(w)|.$$

From this and by the assumption that $g_j \to 0$ uniformly on any compact subset of $\mathbb{D}$, we easily get that $\lim_{j\to\infty} \sup_{z\in\mathbb{D}} \left|g_j(z)\right| = 0$.

Hence

$$\lim_{j\to\infty} \Omega_1 \leqslant \|u\|_{\mathscr{W}_\mu^n} \lim_{j\to\infty} \sup_{z\in\mathbb{D}} \left|(f - f_{r_j})(z)\right| = 0. \tag{12}$$

While

$$\begin{aligned} \Omega_3 &\leqslant \sum_{i=1}^{n} \sup_{|\varphi(z)|>r_N} \mu(z) \left|f^{(i)}(\varphi(z))\right| |I_i^n(z)| + \sum_{i=1}^{n} \sup_{|\varphi(z)|>r_N} \mu(z) \left|r_j^i f^{(i)}(r_j\varphi(z))\right| |I_i^n(z)| \\ &:= \sum_{i=1}^{n} P_i(N) + \sum_{i=1}^{n} Q_i(N). \end{aligned} \tag{13}$$

Here

$$P_i(N) = \sup_{|\varphi(z)|>r_N} \mu(z) \left|f^{(i)}(\varphi(z))\right| |I_i^n(z)|, \quad Q_i(N) = \sup_{|\varphi(z)|>r_N} \mu(z) \left|r_j^i f^{(i)}(r_j\varphi(z))\right| |I_i^n(z)|.$$

For any $i \in \{1, ..., n\}$, by Lemma 1,

$$\begin{aligned} P_i(N) &= \sup_{|\varphi(z)|>r_N} \mu(z) \frac{(1-|\varphi(z)|^2)^{i+\frac{\alpha+2}{p}-1} |f^{(i)}(\varphi(z))|}{|\varphi(z)|^i} \frac{|\varphi(z)|^i |I_i^n(z)|}{(1-|\varphi(z)|^2)^{i+\frac{\alpha+2}{p}-1}} \\ &\lesssim \|f\|_{\mathscr{D}_\alpha^p} \sup_{|\varphi(z)|>r_N} \|uC_\varphi v_{i,\varphi(z)}\|_{\mathscr{W}_\mu^n} \\ &\lesssim \sum_{k=1}^{n+1} \sup_{|a|>r_N} \|uC_\varphi f_{k,a}\|_{\mathscr{W}_\mu^n}. \end{aligned} \tag{14}$$

Taking the limit as $N \to \infty$, we get

$$\limsup_{j\to\infty} P_i(N) \lesssim \sum_{k=1}^{n+1} \limsup_{|a|\to 1} \|uC_\varphi f_{k,a}\|_{\mathscr{W}_\mu^n}. \tag{15}$$

By Lemma 1 and (14), we also get

$$\limsup_{j\to\infty} P_i(N) \lesssim B_i. \tag{16}$$

Similarly, since the function $k(t) = t^i/(1-t^2)^{i+\frac{\alpha+2}{p}-1}$ is increasing, we have

$$\begin{aligned}
Q_i(N) &= \sup_{|\varphi(z)|>r_N} \mu(z) \frac{(1-|r_j\varphi(z)|^2)^{i+\frac{\alpha+2}{p}-1}\left|f^{(i)}(r_j\varphi(z))\right|}{|\varphi(z)|^i} \frac{|r_j\varphi(z)|^i|I_i^n(z)|}{(1-|r_j\varphi(z)|^2)^{i+\frac{\alpha+2}{p}-1}} \\
&\lesssim \sup_{|\varphi(z)|>r_N} \mu(z) \frac{(1-|r_j\varphi(z)|^2)^{i+\frac{\alpha+2}{p}-1}\left|f^{(i)}(r_j\varphi(z))\right|}{|\varphi(z)|^i} \frac{|\varphi(z)|^i|I_i^n(z)|}{(1-|\varphi(z)|^2)^{i+\frac{\alpha+2}{p}-1}} \\
&\lesssim \|f\|_{\mathscr{D}_\alpha^p} \sup_{|\varphi(z)|>r_N} \|uC_\varphi v_{i,\varphi(z)}\|_{\mathscr{W}_\mu^n} \\
&\lesssim \sum_{k=1}^{n+1} \sup_{|a|>r_N} \|uC_\varphi f_{k,a}\|_{\mathscr{W}_\mu^n}.
\end{aligned}$$

Taking the limit as $N \to \infty$, we also get

$$\limsup_{j\to\infty} Q_i(N) \lesssim \sum_{k=1}^{n+1} \limsup_{|a|\to 1} \|uC_\varphi f_{k,a}\|_{\mathscr{W}_\mu^n} \quad \text{and} \quad \limsup_{j\to\infty} Q_i(N) \lesssim B_i. \tag{17}$$

Hence, by (10), (11), (12), (13), (15), (16) and (17), we obtain

$$\begin{aligned}
&\limsup_{j\to\infty} \|uC_\varphi - uC_\varphi K_{r_j}\|_{\mathscr{D}_\alpha^p \to \mathscr{W}_\mu^n} \\
&= \limsup_{j\to\infty} \sup_{\|f\|_{\mathscr{D}_\alpha^p}\leqslant 1} \|(uC_\varphi - uC_\varphi K_{r_j})f\|_{\mathscr{W}_\mu^n} \lesssim \sum_{k=1}^{n+1} \limsup_{|a|\to 1} \|uC_\varphi f_{k,a}\|_{\mathscr{W}_\mu^n}
\end{aligned}$$

and

$$\limsup_{j\to\infty} \|uC_\varphi - uC_\varphi K_{r_j}\|_{\mathscr{D}_\alpha^p \to \mathscr{W}_\mu^n} \lesssim \sum_{i=1}^{n} B_i.$$

So we obtain (9).

Next we prove that

$$\sum_{j=1}^{n+1} \limsup_{|a|\to 1} \|uC_\varphi f_{j,a}\|_{\mathscr{W}_\mu^n} \lesssim \|uC_\varphi\|_{e,\mathscr{D}_\alpha^p \to \mathscr{W}_\mu^n}.$$

It is clear that for all $a\in\mathbb{D}$ and $j\in\{1,...,n+1\}$, $\|f_{j,a}\|_{\mathscr{D}^p_\alpha}\lesssim 1$. Moreover, $f_{j,a}$ converge to 0 uniformly on $\overline{\mathbb{D}}$. Therefore, for any compact operator $K:\mathscr{D}^p_\alpha\to\mathscr{W}^n_\mu$, by Lemmas 2 and 7 we have $\lim_{|a|\to 1}\|Kf_{j,a}\|_{\mathscr{W}^n_\mu}=0$. Thus,

$$\begin{aligned}\|uC_\varphi-K\|_{\mathscr{D}^p_\alpha\to\mathscr{W}^n_\mu}&\gtrsim\limsup_{|a|\to 1}\|(uC_\varphi-K)f_{j,a}\|_{\mathscr{W}^n_\mu}\\&\geqslant\limsup_{|a|\to 1}\|uC_\varphi f_{j,a}\|_{\mathscr{W}^n_\mu}-\limsup_{|a|\to 1}\|Kf_{j,a}\|_{\mathscr{W}^n_\mu}.\end{aligned}$$

Hence,

$$\|uC_\varphi\|_{e,\mathscr{D}^p_\alpha\to\mathscr{W}^n_\mu}=\inf_K\|uC_\varphi-K\|_{\mathscr{D}^p_\alpha\to\mathscr{W}^n_\mu}\gtrsim\sum_{j=1}^{n+1}\limsup_{|a|\to 1}\|uC_\varphi f_{j,a}\|_{\mathscr{W}^n_\mu}.$$

Finally, we prove that

$$\sum_{i=1}^{n}B_i\lesssim\|uC_\varphi\|_{e,\mathscr{D}^p_\alpha\to\mathscr{W}^n_\mu}.$$

Without loss of generality, we assume that $\sup_{z\in\mathbb{D}}|\varphi(z)|=1$. Let $\{z_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)|\to 1$ as $j\to\infty$. Since $uC_\varphi:\mathscr{D}^p_\alpha\to\mathscr{W}^n_\mu$ is bounded, for any compact operator $K:\mathscr{D}^p_\alpha\to\mathscr{W}^n_\mu$ and $i\in\{1,...,n\}$, by using Lemmas 3 and 7 we obtain

$$\begin{aligned}\|uC_\varphi-K\|_{\mathscr{D}^p_\alpha\to\mathscr{W}^n_\mu}&\gtrsim\limsup_{j\to\infty}\|uC_\varphi v_{i,\varphi(z_j)}\|_{\mathscr{W}^n_\mu}-\limsup_{j\to\infty}\|Kv_{i,\varphi(z_j)}\|_{\mathscr{W}^n_\mu}\\&\gtrsim\limsup_{j\to\infty}\frac{\mu(z_j)|\varphi(z_j)|^i|I^n_i(z_j)|}{(1-|\varphi(z_j)|^2)^{i+\frac{\alpha+2}{p}-1}}.\end{aligned}$$

Hence,

$$\|uC_\varphi\|_{e,\mathscr{D}^p_\alpha\to\mathscr{W}^n_\mu}=\inf_K\|uC_\varphi-K\|_{\mathscr{D}^p_\alpha\to\mathscr{W}^n_\mu}\gtrsim\sum_{i=1}^{n}\limsup_{j\to\infty}\frac{\mu(z_j)|\varphi(z_j)|^i|I^n_i(z_j)|}{(1-|\varphi(z_j)|^2)^{i+\frac{\alpha+2}{p}-1}}=\sum_{i=1}^{n}B_i,$$

which imply the desired result. The proof is complete.

From Theorem 2 and the well-known result that $\|T\|_{e,X\to Y}=0$ if and only if $T:X\to Y$ is compact, we obtain the following corollary.

COROLLARY 2. *Let $\alpha>-1$, $\alpha+2<p<\infty$, $n\in\mathbb{N}$ and μ be a weight. Let φ be an analytic self-map of $\mathbb{D}$ and $u\in H(\mathbb{D})$ such that $uC_\varphi:\mathscr{D}^p_\alpha\to\mathscr{W}^n_\mu$ is bounded. Then the following statements are equivalent.*

(i) *$uC_\varphi:\mathscr{D}^p_\alpha\to\mathscr{W}^n_\mu$ is compact.*

(ii) $\sum_{j=1}^{n+1}\limsup_{|a|\to 1}\|uC_\varphi f_{j,a}\|_{\mathscr{W}^n_\mu}=0.$

(iii)

$$\sum_{i=1}^{n} \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)\left|\sum_{l=i}^{n}\binom{n}{l}u^{(n-l)}(z)B_{l,i}(\varphi'(z),\varphi''(z),\ldots,\varphi^{(l-i+1)}(z))\right|}{(1-|\varphi(z)|^2)^{i+\frac{\alpha+2}{p}-1}} = 0.$$

In particular, when $n=1$, we get the following result.

COROLLARY 3. *Let $\alpha > -1$, $\alpha+2 < p < \infty$ and μ be a weight. Let φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$ such that $uC_\varphi : \mathscr{D}_\alpha^p \to \mathscr{B}_\mu$ is bounded. Then $uC_\varphi : \mathscr{D}_\alpha^p \to \mathscr{B}_\mu$ is compact if and only if*

$$\limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} = 0.$$

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