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BOHR PHENOMENON ON THE UNIT BALL OF A COMPLEX BANACH SPACE

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Abstract. Let $\mathbb{B}_X$ be the unit ball of a complex Banach space X. In this paper, we will generalize several results related to the Bohr radius for analytic functions or harmonic functions on the unit disc $\mathbb{U}$ in $\mathbb{C}$ to holomorphic mappings or pluriharmonic mappings on $\mathbb{B}_X$. We will establish Bohr's inequality for the class of holomorphic mappings which are subordinate to convex mappings on $\mathbb{B}_X$. Next, we will establish Bohr's inequality for pluriharmonic mappings on $\mathbb{B}_X$. We will also obtain the p-Bohr radius for bounded pluriharmonic functions on $\mathbb{B}_X$. Finally, we will determine the Bohr radius for a class of holomorphic functions on $\mathbb{B}_X$ which contains odd holomorphic functions on $\mathbb{B}_X$.

1. Introduction

Bohr's inequality says that if

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

is analytic in the unit disc $\mathbb{U}$ in $\mathbb{C}$ and $|f(z)| < 1$ holds for all $z \in \mathbb{U}$, then the inequality

$$\sum_{k=0}^{\infty} |a_k z^k| \leqslant 1 \quad \text{for } |z| \leqslant \frac{1}{3}$$

holds. Bohr [5] originally obtained the above inequality for $|z| \leqslant 1/6$. In fact, the inequality is actually true for $|z| \leqslant 1/3$. The constant $1/3$ is best possible and it is called the Bohr radius (e.g. [13], [14]).

A class of analytic or harmonic functions f in the unit disc $\mathbb{U}$ is said to have Bohr's phenomenon if an inequality of this type holds in the disc $\{z : |z| < \rho_0\}$ for some $\rho_0 \in (0,1]$ and all such functions with $\|f\| \leqslant 1$. Since not every class of functions has Bohr's phenomenon [4], it is of interest to know when a class does have it, and it is also natural to consider an extension of Bohr's inequality to more general domains or higher dimensional spaces.

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Using homogeneous polynomial expansions of holomorphic functions, Aizenberg [3, Theorem 8] obtained a generalization of Bohr's inequality to holomorphic functions on bounded balanced domains in $\mathbb{C}^n$. Liu and Wang [12] gave a generalization of Bohr's inequality to holomorphic mappings of B into itself, where B is one of the four classical domains in $\mathbb{C}^n$. Hamada, Honda and Kohr [7] generalized the above results to holomorphic mappings from a bounded balanced domain in a complex Banach space to a homogeneous unit ball of a complex Banach space.

Recently, Abu Muhanna [1] established Bohr's inequality for the class of analytic functions which are subordinate to univalent functions on the unit disc $\mathbb{U}$ in $\mathbb{C}$. He [1] also established two types of Bohr's inequality for harmonic functions from $\mathbb{U}$ into $\mathbb{U}$. Abu Muhanna, Ali, Ng and Hansi [2] generalized the above results for harmonic functions to harmonic functions from $\mathbb{U}$ to a general bounded domain in $\mathbb{C}$. Kayumov and Ponnusamy [11] determined the Bohr radius for a class of analytic functions in the unit disc $\mathbb{U}$ which contains odd analytic functions on $\mathbb{U}$. They also obtained the p-Bohr radius for bounded harmonic functions on $\mathbb{U}$. As a corollary, they improve one of the results on harmonic functions obtained in [1].

In this paper, we will generalize several results related to the Bohr radius for analytic functions or harmonic functions on $\mathbb{U}$ in [1], [2] and [11] to holomorphic mappings or pluriharmonic mappings on the unit ball $\mathbb{B}_X$ of a complex Banach space X. In section 2, we will establish Bohr's inequality for the class of holomorphic mappings which are subordinate to convex mappings on $\mathbb{B}_X$. In section 3, we will establish Bohr's inequalities for pluriharmonic mappings on $\mathbb{B}_X$. We also obtain the p-Bohr radius for bounded pluriharmonic mappings from $\mathbb{B}_X$ to the Euclidean unit ball of $\mathbb{C}^n$. As a corollary, we obtain that the holomorphic part and the anti-holomorphic part of bounded pluriharmonic mappings on $\mathbb{B}_X$ with values in $\mathbb{C}^n$ have the homogeneous polynomial expansions which converge uniformly on each ball $r\mathbb{B}_X$ with $r \in (0,1)$. Further, we show that a generalization of [2, Theorem 4.4] can be obtained as a corollary of a generalization of [1, Theorem 2]. In section 4, we will determine the Bohr radius for a class of holomorphic functions on $\mathbb{B}_X$ which contains odd holomorphic functions on $\mathbb{B}_X$. To prove the main result in this section, we first prove a lemma which was used in [11] without proof.

2. Subordination classes

Let $\mathbb{B}_X$ be the unit ball of a complex Banach space X. For a holomorphic mapping $f : \mathbb{B}_X \to X$, let $D^k f(z)$ denote the k-th Fréchet derivative of f at $z \in \mathbb{B}_X$. A holomorphic mapping $f : \mathbb{B}_X \to X$ is said to be normalized if $f(0) = 0$ and $Df(0) = I$, where I is the identity operator on X. A holomorphic mapping $f : \mathbb{B}_X \to X$ is said to be convex if f maps $\mathbb{B}_X$ onto a convex domain in X biholomorphically.

Let $f : \mathbb{B}_X \to X$ and $g : \mathbb{B}_X \to X$ be two holomorphic mappings. We say that g is subordinate to f if there exists a Schwarz mapping v on $\mathbb{B}_X$ (i.e. v is a holomorphic mapping from $\mathbb{B}_X$ to $\mathbb{B}_X$ and $\|v(z)\| \leqslant \|z\|$, $z \in \mathbb{B}_X$) such that $g = f \circ v$. Consequently, when g is subordinate to f, we have $\|Dg(0)\| \leqslant \|Df(0)\|$. Let $S(f)$ denote the class of all mappings $g : \mathbb{B}_X \to X$ which are subordinate to f.

Let X^* be the dual space of X. For each $a \in X \setminus \{0\}$, we define

$$T(a) = \{l_a \in X^* : \|l_a\| = 1, l_a(a) = \|a\|\}.$$

By the Hahn-Banach theorem, $T(a)$ is nonempty.

DEFINITION 2.1. Let X and Y be complex Banach spaces. Let k be a positive integer. A mapping $P : X \to Y$ is called a homogeneous polynomial of degree k if there exists a k-linear mapping u from X^k into Y such that

$$P(x) = u(x, \dots, x)$$

for every $x \in X$.

Throughout of this paper, the degree of a homogeneous polynomial is denoted by a subscript. Namely, if P_m is a homogeneous polynomial, then the degree of P_m is m. We note that if P_m is an m-homogeneous polynomial from X into Y, there uniquely exists a symmetric m-linear mapping u with $P_m(x) = u(x, \dots, x)$.

The following theorem is a generalization of [1, Lemma 3 and Theorem 1] to convex mappings f on $\mathbb{B}_X$ (see also [1, Remark 1]).

THEOREM 2.2. *Let $f : \mathbb{B}_X \to X$ be a convex mapping on $\mathbb{B}_X$ and $g : \mathbb{B}_X \to X$ be a holomorphic mapping with*

$$g(z) = \sum_{k=0}^{\infty} Q_k(z), \quad \textit{near the origin},$$

where Q_k is a homogeneous polynomial mapping of degree k. If $g \in S(f)$, then we have

(i) $\|Q_k(w)\| \leqslant \|Df(0)\|$ *for* $k \geqslant 1$, $\|w\| = 1$,

(ii)

$$\sum_{k=1}^{\infty} \|Q_k(z)\| \leqslant \frac{1}{2}\|Df(0)\| \tag{2.1}$$

for $\|z\| \leqslant 1/3$. When $\mathbb{B}_X$ is the Hilbert ball, $1/3$ is sharp for the convex mapping $f(z) = z/(1 - l_a(z))$, where $l_a \in T(a)$, $a \neq 0$.

Proof. (i) For a fixed positive integer k, let

$$g_k(z) = \sum_{j=1}^{k} \frac{g(e^{i2\pi j/k}z)}{k}, \quad z \in \mathbb{B}_X.$$

From the homogeneous expansion of g, we have

$$g_k(z) = g(0) + \frac{1}{k}\left(\sum_{j=1}^{k}\left(\sum_{l=1}^{\infty} e^{i2\pi jl/k} Q_l(z)\right)\right)$$

for z sufficiently close to the origin. Since

$$\frac{1}{k}\sum_{j=1}^{k} e^{i2\pi jl/k} = \begin{cases} 1 \text{ if } l \equiv 0 \pmod{k}, \\ 0 \text{ otherwise,} \end{cases}$$

we have

$$g_k(z) = g(0) + \sum_{l=1}^{\infty} Q_{lk}(z)$$

for z sufficiently close to the origin. Since f is convex, $g_k \in S(f)$. Let $h(z) = f^{-1}(g_k(z))$ for $z \in \mathbb{B}_X$. Then $h : \mathbb{B}_X \to X$ is holomorphic, $h(0) = 0$ and $h(\mathbb{B}_X) \subset \mathbb{B}_X$. Since

$$f^{-1}(z) = [Df(0)]^{-1}(z - f(0)) + O(\|z - f(0)\|^2)$$

in a neighbourhood of $f(0)$, we have

$$h(z) = f^{-1}(g_k(z)) = [Df(0)]^{-1}Q_k(z) + O(\|z\|^{k+1}) \tag{2.2}$$

in a neighbourhood of 0. By the well-known Cauchy estimates for Schwarz mapping, we have

$$\left\| \frac{1}{m!} D^m h(0)(w^m) \right\| \leqslant 1, \quad \|w\| = 1, m \geqslant 1. \tag{2.3}$$

By (2.2) and (2.3), we have

$$\|[Df(0)]^{-1}Q_k(w)\| \leqslant 1 \tag{2.4}$$

for $\|w\| = 1$. Therefore, we have $\|Q_k(w)\| \leqslant \|Df(0)\|$ for $\|w\| = 1$.

(ii) For fixed $z \in \mathbb{B}_X \setminus \{0\}$ with $\|z\| \leqslant 1/3$, let $w = z/\|z\|$. Then, by (i), we have

$$\begin{aligned} \sum_{k=1}^{\infty} \|Q_k(z)\| &= \sum_{k=1}^{\infty} \|Q_k(\|z\| w)\| \\ &\leqslant \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k \|Q_k(w)\| \\ &\leqslant \|Df(0)\| \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k \\ &= \frac{1}{2}\|Df(0)\|. \end{aligned}$$

This implies (2.1) as desired.

Finally, we prove the sharpness of the constant $1/3$ in the case $\mathbb{B}_X$ is the Hilbert ball. Indeed, for any fixed $a \in X \setminus \{0\}$, let

$$f(z) = \frac{z}{1 - l_a(z)} = \frac{z}{1 - \langle z, u \rangle}, \quad z \in \mathbb{B}_X,$$

where $\langle \cdot, \cdot \rangle$ is the inner product on X and $u = a/\|a\|$. Then f is a normalized convex mapping on the Hilbert ball $\mathbb{B}_X$ by [8, Remark 2.2]. Let $g(z) = f(z)$. Since $\|Q_k(ru)\| =$

r^k for $k \geqslant 1$ and $r \in (0,1)$ and $\|Df(0)\| = 1$, (2.1) holds if and only if $r \leqslant 1/3$. This completes the proof. $\square$

As a corollary of the above theorem, we obtain that every holomorphic mapping on $\mathbb{B}_X$ which is subordinate to a convex mapping on $\mathbb{B}_X$ has the homogeneous polynomial expansion which converges uniformly on each ball $r\mathbb{B}_X$ with $r \in (0,1)$.

COROLLARY 2.3. *Let $f : \mathbb{B}_X \to X$ be a convex mapping on $\mathbb{B}_X$ and $g : \mathbb{B}_X \to X$ be a holomorphic mapping such that $g \in S(f)$. Then g has the homogeneous polynomial expansion*

$$g(z) = \sum_{k=0}^{\infty} Q_k(z), \quad z \in \mathbb{B}_X,$$

which converges uniformly on each ball $r\mathbb{B}_X$ with $r \in (0,1)$.

For a point $z \in X$ and a subset E in X, let $d(z,E)$ denote the distance between z and E. The following theorem is another version.

THEOREM 2.4. *Let $f : \mathbb{B}_X \to X$ be a convex mapping on $\mathbb{B}_X$ and $g : \mathbb{B}_X \to X$ be a holomorphic mapping with*

$$g(z) = \sum_{k=0}^{\infty} Q_k(z), \quad \textit{near the origin},$$

where Q_k is a homogeneous polynomial mapping of degree k. If $g \in S(f)$, then we have

(i) $\|[Df(0)]^{-1}Q_k(w)\| \leqslant 1$ *for* $k \geqslant 1$, $\|w\| = 1$,

(ii)

$$\sum_{k=1}^{\infty} \|[Df(0)]^{-1}Q_k(z)\| \leqslant \frac{1}{2} \leqslant d([Df(0)]^{-1}f(0), \partial\Omega^*) \tag{2.5}$$

for $\|z\| \leqslant 1/3$, where $\Omega^* = [Df(0)]^{-1}\Omega$ and $\Omega = f(\mathbb{B}_X)$. When $\mathbb{B}_X$ is the Hilbert ball, $1/3$ is sharp for the convex mapping $f(z) = z/(1 - l_a(z))$, where $l_a \in T(a)$, $a \neq 0$.

Proof. (i) We have already obtained in (2.4).

(ii) For fixed $z \in \mathbb{B}_X \setminus \{0\}$ with $\|z\| \leqslant 1/3$, let $w = z/\|z\|$. Using (i), we have

$$\begin{aligned}\sum_{k=1}^{\infty} \|[Df(0)]^{-1}Q_k(z)\| &= \sum_{k=1}^{\infty} \left\|[Df(0)]^{-1}Q_k(\|z\|w)\right\| \\ &\leqslant \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k \|[Df(0)]^{-1}Q_k(w)\| \\ &\leqslant \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k\end{aligned}$$

$$= \frac{1}{2}.$$

This implies the first inequality in (2.5) as desired.

About the second inequality in (2.5), we set

$$F(z) = [Df(0)]^{-1}(f(z) - f(0)).$$

Then, F is a (normalized) convex mapping from $\mathbb{B}_X$ to X.

By [8, Theorem 2.1] (cf.[6]), $F(\mathbb{B}_X)$ contains the ball with center 0 and radius $\frac{1}{2}$. That is,

$$\frac{1}{2} \leqslant d([Df(0)]^{-1}f(0), \partial[Df(0)]^{-1}(f(\mathbb{B}_X))).$$

The proof of the sharpness of the constant $1/3$ is similar to those in the proof of Theorem 2.2. This completes the proof. $\square$

REMARK 2.5. When $\dim X = 1$, then $\mathbb{B}_X = \mathbb{U}$ and $d(f(0), \partial\Omega) \geqslant \frac{1}{2}|f'(0)|$ by [1, Lemma 2]. Therefore, Theorem 2.2 reduces [1, Theorem 1] in the case f is a convex function on $\mathbb{U}$.

3. Bounded pluriharmonic mappings

Let $\mathbb{B}_X$ be the unit ball of a complex Banach space X. A continuous mapping $f : \mathbb{B}_X \to \mathbb{C}^n$ is said to be pluriharmonic if there exist holomorphic mappings h, g from $\mathbb{B}_X$ to $\mathbb{C}^n$ such that $f = h + \overline{g}$. We may assume that $g(0) = 0$. Let B^n be the unit ball of $\mathbb{C}^n$ with respect to an arbitrary norm on $\mathbb{C}^n$.

The following lemma is a generalization of [1, Lemma 4] (see also [9, Theorem 4.2] in the case $k = 1$).

LEMMA 3.1. *Let $f = h + \overline{g} : \mathbb{B}_X \to B^n$ be a pluriharmonic mapping and let*

$$h(z) = \sum_{k=0}^{\infty} P_k(z),$$

and

$$g(z) = \sum_{k=1}^{\infty} Q_k(z),$$

be the homogeneous polynomial expansions near $0 \in \mathbb{B}_X$. Then, we have

$$\|P_k(w) + \overline{Q_k(w)}\| \leqslant \frac{4}{\pi}, \quad k \geqslant 1, \|w\|_X = 1. \tag{3.1}$$

Proof. For a fixed positive integer k and a fixed $w \in \partial\mathbb{B}_X$, let $a = P_k(w) + \overline{Q_k(w)}$. If $a = 0$, then (3.1) holds. So, we may assume that $a \neq 0$. In this case, let

$$f_k(z) = \sum_{j=1}^{k} \frac{f(e^{i2\pi j/k}z)}{k}, \quad z \in \mathbb{B}_X.$$

Then, we have $f_k(\mathbb{B}_X) \subset B^n$ and

$$f_k(\zeta w) = f(0) + \sum_{l=1}^{\infty} (P_{kl}(\zeta w) + \overline{Q_{kl}(\zeta w)}), \quad \zeta \in \mathbb{U}. \tag{3.2}$$

Let

$$\phi_w(\zeta) = l_a\left(f(0) + \sum_{l=1}^{\infty} (P_{kl}(w)\zeta^l + \overline{Q_{kl}(w)\zeta^l})\right), \quad \zeta \in \mathbb{U},$$

where $l_a \in T(a)$. Using (3.2), it follows that ϕ_w is a harmonic mapping from $\mathbb{U}$ into $\mathbb{U}$. By applying the harmonic Schwarz-Pick lemma to ϕ_w, we have

$$\|a\| = l_a(a) = \left|\frac{\partial \phi_w}{\partial \zeta}(0) + \frac{\partial \phi_w}{\partial \overline{\zeta}}(0)\right| \leqslant \frac{4}{\pi}.$$

This completes the proof. □

Using the above lemma, we obtain the following theorem. The following theorem is a generalization of [1, Theorem 2].

THEOREM 3.2. *Let $f = h + \overline{g} : \mathbb{B}_X \to B^n$ be a pluriharmonic mapping and let*

$$h(z) = \sum_{k=0}^{\infty} P_k(z),$$

and

$$g(z) = \sum_{k=1}^{\infty} Q_k(z),$$

be the homogeneous polynomial expansions near $0 \in \mathbb{B}_X$. Then, for $\|z\| \leqslant 1/3$, we have

$$\sum_{k=1}^{\infty} \|P_k(z) + \overline{Q_k(z)}\| \leqslant \frac{2}{\pi}. \tag{3.3}$$

Proof. For fixed $z \in \mathbb{B}_X \setminus \{0\}$ with $\|z\| \leqslant 1/3$, let $w = z/\|z\|$. Then, by Lemma 3.1, we have

$$\begin{aligned}
\sum_{k=1}^{\infty} \|P_k(z) + \overline{Q_k(z)}\| &= \sum_{k=1}^{\infty} \left\|P_k(\|z\|w) + \overline{Q_k(\|z\|w)}\right\| \\
&\leqslant \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k \|P_k(w) + \overline{Q_k(w)}\| \\
&\leqslant \frac{4}{\pi} \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k \\
&= \frac{2}{\pi}.
\end{aligned}$$

This implies (3.3) as desired. This completes the proof. □

Next, we consider the p-Bohr radius for bounded pluriharmonic mappings from $\mathbb{B}_X$ to $\mathbb{B}^n$, where $\mathbb{B}^n$ is the Euclidean unit ball of $\mathbb{C}^n$. First, we obtain the following generalization of [11, Theorem 3].

THEOREM 3.3. *Let* $f = h + \overline{g} : \mathbb{B}_X \to \mathbb{B}^n$ *be a pluriharmonic mapping and let*

$$h(z) = \sum_{k=0}^{\infty} P_k(z),$$

and

$$g(z) = \sum_{k=1}^{\infty} Q_k(z),$$

be the homogeneous polynomial expansions near $0 \in \mathbb{B}_X$. *Then, for any* $p \geqslant 1$ *and* $\|z\| = r \in (0,1)$, *we have*

$$\sum_{k=1}^{\infty} (\|P_k(z)\|^p + \|Q_k(z)\|^p)^{1/p} \leqslant \max\{2^{(1/p)-1/2}, 1\}\sqrt{1-\|P_0\|^2}\frac{r}{\sqrt{1-r^2}}. \tag{3.4}$$

Proof. For fixed $w \in \partial\mathbb{B}_X$, we set $z = re^{i\theta}w$. Then, for any $r \in (0,1)$, we have

$$h(re^{i\theta}w) = \sum_{k=0}^{\infty} P_k(re^{i\theta}w), \quad 0 \leqslant \theta \leqslant 2\pi$$

$$g(re^{i\theta}w) = \sum_{k=1}^{\infty} Q_k(re^{i\theta}w), \quad 0 \leqslant \theta \leqslant 2\pi$$

and

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \|f(re^{i\theta}w)\|^2 d\theta &= \frac{1}{2\pi}\int_0^{2\pi} \|h(re^{i\theta}w) + \overline{g(re^{i\theta}w)}\|^2 d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left(\|h(re^{i\theta}w)\|^2 + \|\overline{g(re^{i\theta}w)}\|^2\right) d\theta \\
&= \|P_0\|^2 + \sum_{k=1}^{\infty} \left(\|P_k(w)\|^2 + \|Q_k(w)\|^2\right) r^{2k}.
\end{aligned}$$

Since $f(\mathbb{B}_X) \subset \mathbb{B}^n$, we have

$$\|P_0\|^2 + \sum_{k=1}^{\infty} \left(\|P_k(w)\|^2 + \|Q_k(w)\|^2\right) r^{2k} \leqslant 1.$$

Letting $r \to 1$, we obtain

$$\|P_0\|^2 + \sum_{k=1}^{\infty} \left(\|P_k(w)\|^2 + \|Q_k(w)\|^2\right) \leqslant 1.$$

It follows from this, the Cauchy-Hölder-Schwarz inequality and the inequality $a^{p/2}+b^{p/2}\leqslant(a+b)^{p/2}$ for $a,b\geqslant 0$ and $p>2$ that

$$\begin{aligned}
&\sum_{k=1}^{\infty}(\|P_k(z)\|^p+\|Q_k(z)\|^p)^{1/p}\\
&\leqslant\sqrt{\sum_{k=1}^{\infty}(\|P_k(w)\|^p+\|Q_k(w)\|^p)^{2/p}}\sqrt{\sum_{k=1}^{\infty}r^{2k}}\\
&\leqslant\sqrt{\max(2^{\frac{2}{p}-1},1)\sum_{k=1}^{\infty}(\|P_k(w)\|^2+\|Q_k(w)\|^2)}\frac{r}{\sqrt{1-r^2}}\\
&\leqslant\max(2^{(1/p)-1/2},1)\sqrt{1-\|P_0\|^2}\frac{r}{\sqrt{1-r^2}}
\end{aligned}$$

This completes the proof. □

It is well known that every bounded holomorphic mapping on $\mathbb{B}_X$ has the homogeneous polynomial expansion which converges uniformly on each ball $r\mathbb{B}_X$ with $r\in(0,1)$. As a corollary of the above theorem, it can be extended to bounded pluriharmonic mappings with values in $\mathbb{C}^n$.

COROLLARY 3.4. *Let $f=h+\overline{g}:\mathbb{B}_X\to\mathbb{C}^n$ be a bounded pluriharmonic mapping. Then f and g have the homogeneous polynomial expansions*

$$h(z)=\sum_{k=0}^{\infty}P_k(z),\quad z\in\mathbb{B}_X$$

and

$$g(z)=\sum_{k=1}^{\infty}Q_k(z),\quad z\in\mathbb{B}_X,$$

which converge uniformly on each ball $r\mathbb{B}_X$ with $r\in(0,1)$.

Putting $p=1$ and $r\leqslant 1/3$ in Theorem 3.3, we obtain the following result (cf. [11, p.867]).

COROLLARY 3.5. *Let $f=h+\overline{g}:\mathbb{B}_X\to\mathbb{B}^n$ be a pluriharmonic mapping and let*

$$h(z)=\sum_{k=0}^{\infty}P_k(z),\quad z\in\mathbb{B}_X$$

and

$$g(z)=\sum_{k=1}^{\infty}Q_k(z),\quad z\in\mathbb{B}_X$$

be the homogeneous polynomial expansions on $\mathbb{B}_X$. Then, for $\|z\|\leqslant 1/3$, we have

$$\sum_{k=1}^{\infty}(\|P_k(z)\|+\|Q_k(z)\|)\leqslant\frac{\sqrt{1-\|P_0\|^2}}{2}.\tag{3.5}$$

We also have the following generalizations of [11, Corollaries 2, 3 and 4].

COROLLARY 3.6. *Let $f = h + \overline{g} : \mathbb{B}_X \to \mathbb{B}^n$ be a pluriharmonic mapping with $f(0) = 0$ and let*

$$h(z) = \sum_{k=1}^{\infty} P_k(z), \quad z \in \mathbb{B}_X$$

and

$$g(z) = \sum_{k=1}^{\infty} Q_k(z), \quad z \in \mathbb{B}_X$$

be the homogeneous polynomial expansions on $\mathbb{B}_X$. If $p \geqslant 2$, then we have

$$\sum_{k=1}^{\infty} (\|P_k(z)\|^p + \|Q_k(z)\|^p)^{1/p} \leqslant 1, \quad \|z\| \leqslant \frac{1}{\sqrt{2}}. \tag{3.6}$$

The number $1/\sqrt{2}$ is sharp.

Proof. Considering the case $P_0 = 0$, $r \leqslant 1/\sqrt{2}$ and $p \geqslant 2$ in Theorem 3.3, we obtain (3.6). Sharpness is given by the holomorphic mapping

$$f(z) = \left(\frac{l_u(z)\left(\frac{1}{\sqrt{2}} - l_u(z)\right)}{1 - \frac{1}{\sqrt{2}} l_u(z)}, 0, \ldots, 0 \right),$$

where $l_u \in T(u)$ and $u \in X \setminus \{0\}$ are arbitrary. $\square$

COROLLARY 3.7. *Let $f = h + \overline{g} : \mathbb{B}_X \to \mathbb{B}^n$ be a pluriharmonic mapping and let*

$$h(z) = \sum_{k=0}^{\infty} P_k(z), \quad z \in \mathbb{B}_X$$

and

$$g(z) = \sum_{k=1}^{\infty} Q_k(z), \quad z \in \mathbb{B}_X$$

be the homogeneous polynomial expansions on $\mathbb{B}_X$.

(i) If $p \in [1, 2]$, then we have

$$\|P_0\| + \sum_{k=1}^{\infty} (\|P_k(z)\|^p + \|Q_k(z)\|^p)^{1/p} \leqslant 1, \quad \|z\| \leqslant r_p(\|P_0\|), \tag{3.7}$$

where

$$r_p(\|P_0\|) = \sqrt{\frac{1 - \|P_0\|}{2^{(2/p)-1} + 1 + (2^{(2/p)-1} - 1)\|P_0\|}}.$$

(ii) If $p \geqslant 2$, then

$$\|P_0\| + \sum_{k=1}^{\infty} (\|P_k(z)\|^p + \|Q_k(z)\|^p)^{1/p} \leqslant 1, \quad \|z\| \leqslant \sqrt{\frac{1 - \|P_0\|}{2}}.$$

COROLLARY 3.8. *Let* $f = h + \overline{g} : \mathbb{B}_X \to \mathbb{B}^n$ *be a pluriharmonic mapping and let*

$$h(z) = \sum_{k=0}^{\infty} P_k(z), \quad z \in \mathbb{B}_X$$

and

$$g(z) = \sum_{k=1}^{\infty} Q_k(z), \quad z \in \mathbb{B}_X$$

be the homogeneous polynomial expansions on $\mathbb{B}_X$. *If* $p \in [1,2]$ *and*

$$\|P_0\| \leqslant A(p) = \frac{8 - 2^{(2/p)-1}}{8 + 2^{(2/p)-1}},$$

then we have

$$\|P_0\| + \sum_{k=1}^{\infty} (\|P_k(z)\|^p + \|Q_k(z)\|^p)^{1/p} \leqslant 1, \quad \|z\| \leqslant \frac{1}{3}. \tag{3.8}$$

Let $\mathbb{U}$ be the unit disc in $\mathbb{C}$. For pluriharmonic functions from $\mathbb{B}_X$ to $\mathbb{U}$, we have the following results.

The following lemma is a generalization of [1, Lemma 4].

LEMMA 3.9. *Let* $f = h + \overline{g} : \mathbb{B}_X \to \mathbb{U}$ *be a pluriharmonic function and let*

$$h(z) = \sum_{k=0}^{\infty} P_k(z), \quad z \in \mathbb{B}_X$$

and

$$g(z) = \sum_{k=1}^{\infty} Q_k(z), \quad z \in \mathbb{B}_X$$

be the homogeneous polynomial expansions on $\mathbb{B}_X$. *Then, we have*

$$|e^{i\mu} P_k(w) + e^{-i\mu} Q_k(w)| \leqslant 2(1 - |\Re(e^{i\mu} P_0)|),$$

for any $\mu \in \mathbb{R}$, $k \geqslant 1$, $\|w\|_X = 1$.

Proof. Let

$$\phi_\mu(z) = e^{i\mu} h(z) + e^{-i\mu} g(z), \quad \mu \in \mathbb{R}.$$

Then, $\phi_\mu(0) = e^{i\mu} P_0$ and $|\Re(\phi_\mu(z))| = |\Re(e^{i\mu} f(z))| < 1$ for $z \in \mathbb{B}_X$.

Let $\Gamma = \{\zeta \in \mathbb{C}; |\Re(\zeta)| < 1\}$, and let

$$\psi(\zeta) = \frac{2i}{\pi} \log \frac{1+\zeta}{1-\zeta}.$$

Since $\phi_\mu(0) \in \Gamma$ and ψ conformally maps $\mathbb{U}$ onto Γ, there exists $\eta_0 \in \mathbb{U}$ such that $\psi(\eta_0) = \phi_\mu(0)$. We set the function

$$\varphi(\zeta) = \frac{\zeta + \eta_0}{1 + \overline{\eta_0}\zeta} : \mathbb{U} \to \mathbb{U}.$$

Then $\psi \circ \varphi(0) = \phi_\mu(0)$ and for each fixed $w \in X$ with $\|w\| = 1$, the mapping : $\zeta \mapsto \phi_\mu(\zeta w)$ is subordinate to $\psi \circ \varphi$. Since $\psi \circ \varphi$ is convex, by [1, Lemma 3], we have

$$|e^{i\mu} P_k(w) + e^{-i\mu} Q_k(w)| \leqslant 2d(\psi \circ \varphi(0), \partial\Gamma) = 2(1 - |\Re(e^{i\mu} P_0)|).$$

This completes the proof. □

Using the above lemma, we obtain the following theorem, which is a generalization of [1, Theorem 2].

THEOREM 3.10. *Let $f = h + \overline{g} : \mathbb{B}_X \to \mathbb{U}$ be a pluriharmonic function and let*

$$h(z) = \sum_{k=0}^{\infty} P_k(z), \quad z \in \mathbb{B}_X$$

and

$$g(z) = \sum_{k=1}^{\infty} Q_k(z), \quad z \in \mathbb{B}_X$$

be the homogeneous polynomial expansions on $\mathbb{B}_X$. Then, for $\|z\| \leqslant 1/3$, we have

$$\sum_{k=1}^{\infty} |e^{i\mu} P_k(z) + e^{-i\mu} Q_k(z)| + |\Re(e^{i\mu} P_0)| \leqslant 1 \tag{3.9}$$

for any $\mu \in \mathbb{R}$. The bound $1/3$ is sharp. The sharpness is shown by the functions φ_w, $w \in \partial\mathbb{B}_X$, where

$$\varphi_w(z) = \frac{l_w(z) + a}{1 + a l_w(z)}$$

for some $a \in (0,1)$.

Proof. For fixed $z \in \mathbb{B}_X \setminus \{0\}$ with $\|z\| \leqslant 1/3$, let $w = z/\|z\|$. Then, by Lemma 3.9, we have

$$\begin{aligned}
\sum_{k=1}^{\infty} |e^{i\mu} P_k(z) + e^{-i\mu} Q_k(z)| &= \sum_{k=1}^{\infty} \left| e^{i\mu} P_k(\|z\| w) + e^{-i\mu} Q_k(\|z\| w) \right| \\
&\leqslant \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k |e^{i\mu} P_k(w) + e^{-i\mu} Q_k(w)| \\
&\leqslant 2(1 - |\Re(e^{i\mu} P_0)|) \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k \\
&= 1 - |\Re(e^{i\mu} P_0)|.
\end{aligned}$$

Thus, we obtain (3.9) as desired.

Finally, we prove the sharpness of the bound $1/3$. For fixed $w \in \partial\mathbb{B}_X$, let

$$f(z) = \varphi_w(z) = \frac{l_w(z) + a}{1 + a l_w(z)},$$

where $a \in (0,1)$. Then for $r \in (0,1)$, we have $P_k(rw) = (1-a^2)(-a)^{k-1}r^k$, $Q_k(rw) = 0$ for $k \geqslant 1$ and $P_0 = a$. Therefore

$$\sum_{k=1}^{\infty} |P_k(rw)| + |P_0(rw)| > 1$$

if and only if $a + (1-2a^2)r > 1 - ar$. This is equivalent to $a > (1/2)(1/r-1)$. Therefore, for any r with $r > 1/3$, there exists a such that $1 > a > (1/2)(1/r-1)$. Thus, the bound $1/3$ is sharp. This completes the proof. □

Let D be a bounded set in $\mathbb{C}$ and denote by $\overline{D}$ the closure of D. Let $\overline{D}_{min}$ be the smallest closed disk containing $\overline{D}$. As a corollary of Theorem 3.10, we obtain the following generalization of [2, Theorem 4.4] by using a simple proof.

THEOREM 3.11. *Let $f = h + \overline{g} : \mathbb{B}_X \to \mathbb{C}$ be a pluriharmonic function and let*

$$h(z) = \sum_{k=0}^{\infty} P_k(z), \quad z \in \mathbb{B}_X$$

and

$$g(z) = \sum_{k=1}^{\infty} Q_k(z), \quad z \in \mathbb{B}_X$$

be the homogeneous polynomial expansions on $\mathbb{B}_X$. If $f(\mathbb{B}_X) \subset D$ for some bounded domain D in $\mathbb{C}$, then, for $\|z\| \leqslant 1/3$, we have

$$\sum_{k=1}^{\infty} |e^{i\mu} P_k(z) + e^{-i\mu} Q_k(z)| + |\Re e^{i\mu}(P_0 - w_0)| \leqslant \rho \tag{3.10}$$

for any $\mu \in \mathbb{R}$, where ρ and w_0 are respectively the radius and center of $\overline{D}_{min}$. If D is a disc in $\mathbb{C}$, then the bound $1/3$ is sharp.

Proof. Let $F = \rho^{-1}(f - w_0)$. Then $F = H + \overline{G} : \mathbb{B}_X \to \mathbb{C}$ satisfies the assumptions of Theorem 3.10, where

$$H(z) = \rho^{-1}(P_0(z) - w_0) + \sum_{k=1}^{\infty} \rho^{-1} P_k(z), \quad z \in \mathbb{B}_X$$

and

$$G(z) = \sum_{k=1}^{\infty} \rho^{-1} Q_k(z), \quad z \in \mathbb{B}_X.$$

By applying Theorem 3.10, we obtain (3.10) as desired. Sharpness also follows from Theorem 3.10. This completes the proof. □

4. Special family of holomorphic mappings

First, we give a lemma which will be used in the proof of Theorem 4.2. The following result was used in [11, p.861] without proof and (ii) is noted in [11, p.862]. However, for the proof of it, they use an increasing property of the Bohr radius before proving that $r_{p,m}$ is the (sharp) Bohr radius. So, we give a direct and elementary proof here.

LEMMA 4.1. *Let $p \in \mathbb{N}$, $m \in \mathbb{Z}$ with $0 \leqslant m \leqslant p$, and let $r_{p,m}$ be the maximal positive root of the equation*

$$-6r^{p-m} + r^{2(p-m)} + 8r^{2p} + 1 = 0$$

in $(0,1)$.

(i) If $m = 0$, then $r_{p,0} = 1/\sqrt[p]{3}$;

(ii) if $m \geqslant 1$, then $1/3 < r_{p,m}^p$ holds.

Proof. (i) By direct computation, we obtain the unique solution $r_{p,0} = 1/\sqrt[p]{3}$.
(ii) Let

$$\varphi(r) = -6r^{p-m} + r^{2(p-m)} + 8r^{2p} + 1.$$

Since $\varphi(1) = 4 > 0$, it suffices to show that $\varphi(1/\sqrt[p]{3}) < 0$. We have

$$\varphi(1/\sqrt[p]{3}) = -2 \cdot 3^{m/p} + \frac{1}{9}(3^{m/p})^2 + \frac{17}{9}.$$

Since the function

$$\psi(x) = -2x + \frac{1}{9}x^2 + \frac{17}{9}$$

is decreasing on the interval $[0,9]$ and $\psi(1) = 0$, we obtain $\varphi(1/\sqrt[p]{3}) = \psi(3^{m/p}) < 0$. This completes the proof. $\square$

The following theorem is a generalization of [11, Theorem 1].

THEOREM 4.2. *Let $p \in \mathbb{N}$, $m \in \mathbb{Z}$ with $0 \leqslant m \leqslant p$, and $f : \mathbb{B}_X \to \mathbb{U}$ be a holomorphic function with the homogeneous polynomial expansion*

$$f(z) = \sum_{k=0}^{\infty} P_{pk+m}(z), \quad z \in \mathbb{B}_X.$$

Then, for $\|z\| \leqslant r_{p,m}$, we have

$$\sum_{k=0}^{\infty} |P_{pk+m}(z)| \leqslant 1, \tag{4.1}$$

where $r_{p,m}$ is the maximal positive root of the equation

$$-6r^{p-m} + r^{2(p-m)} + 8r^{2p} + 1 = 0 \tag{4.2}$$

in $(0,1)$. The number $r_{p,m}$ is sharp.

Proof. Let $z \in \mathbb{B}_X \setminus \{0\}$ with $\|z\| \leqslant r_{p,m}$ be fixed and let $w = z/\|z\|$. Let

$$F(\zeta) = f(\zeta w), \quad \zeta \in \mathbb{U}.$$

Then $F : \mathbb{U} \to \mathbb{U}$ is holomorphic and

$$F(\zeta) = \zeta^m \sum_{k=0}^{\infty} P_{pk+m}(w)\zeta^{pk}, \quad \zeta \in \mathbb{U}.$$

By [11, Theorem 1], we have

$$r^m \sum_{k=0}^{\infty} |P_{pk+m}(w)| r^{pk} \leqslant 1, \quad \text{for } r \leqslant r_{p,m}.$$

Taking $r = \|z\|$, we obtain (4.1) as desired.

Next, we prove the sharpness. First, we consider the case $m \geqslant 1$. In this case, let $w \in \partial\mathbb{B}_X$ be arbitrarily fixed, $r = r_{p,m}$ and let

$$f(z) = l(z)^m \left(\frac{l(z)^p - a}{1 - al(z)^p} \right) \quad \text{with } a = r^{-p} \left(1 - \frac{\sqrt{1 - r^{2p}}}{\sqrt{2}} \right),$$

where $l \in T(w)$. Note that $0 < a < 1$, since $m \geqslant 1$ implies that $1/3 < r_{p,m}^p < 1$ by Lemma 4.1. We have $\sum_{k=0}^{\infty} |P_{pk+m}(rw)| = 1$ as in the proof of [11, Theorem 1]. This implies that $r_{p,m}$ is sharp in the case $m \geqslant 1$.

Finally, we consider the case $m = 0$. In this case $r_{p,0} = 1/\sqrt[p]{3}$. Let $z_0 \in \mathbb{B}_X$ with $r = \|z_0\| > 1/\sqrt[p]{3}$ be fixed. Then there exists $\lambda \in (0,1)$ such that $r^p > \frac{1}{1+2\lambda}$. Let

$$f(z) = \frac{l(z)^p - \lambda}{1 - \lambda l(z)^p},$$

where $l \in T(w)$ and $w = z_0/\|z_0\|$. Then $f : \mathbb{B}_X \to \mathbb{U}$ is holomorphic and

$$\begin{aligned}
\sum_{k=0}^{\infty} |P_{pk}(z_0)| &= \sum_{k=0}^{\infty} |P_{pk}(rw)| \\
&= \lambda + (1 - \lambda^2) \frac{r^p}{1 - \lambda r^p} \\
&> \lambda + (1 - \lambda^2) \frac{\frac{1}{1+2\lambda}}{1 - \lambda \frac{1}{1+2\lambda}} \\
&= 1.
\end{aligned}$$

This implies that the constant $1/\sqrt[p]{3}$ is best possible. This completes the proof. □

As a corollary of the above theorem, we also have the following generalizations of [10, Corollary 1] and [11, Corollary 1]. We obtain the Bohr radius for odd holomorphic functions on $\mathbb{B}_X$ in Corollary 4.3.

COROLLARY 4.3. *Let $f : \mathbb{B}_X \to \mathbb{U}$ be a holomorphic function with the homogeneous polynomial expansion*

$$f(z) = \sum_{k=0}^{\infty} P_{2k+1}(z), \quad z \in \mathbb{B}_X.$$

Then, for $\|z\| \leqslant r_2 = r_{2,1}$, we have

$$\sum_{k=0}^{\infty} |P_{2k+1}(z)| \leqslant 1, \tag{4.3}$$

where $r_2 = 0.789991\cdots$ is the maximal positive root of the equation

$$-6r^1 + r^2 + 8r^4 + 1 = 0 \tag{4.4}$$

in $(0,1)$. The number r_2 is sharp.

COROLLARY 4.4. *Let $p \geqslant 1$ and let $f : \mathbb{B}_X \to \mathbb{U}$ be a holomorphic function with the homogeneous polynomial expansion*

$$f(z) = \sum_{k=0}^{\infty} P_{pk}(z), \quad z \in \mathbb{B}_X.$$

Then, for $\|z\| \leqslant 1/\sqrt[p]{3}$, we have

$$\sum_{k=0}^{\infty} |P_{pk}(z)| \leqslant 1. \tag{4.5}$$

The radius $1/\sqrt[p]{3}$ is best possible.

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