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n–DERIVATIONS AND FUNCTIONAL INEQUALITIES WITH APPLICATIONS

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Abstract. We prove that every bounded n-derivation of a commutative factorizable Banach algebra maps into its radical. Also, the nilpotency of eigenvectors of any bounded n-derivation corresponding to its eigenvalues is derived. We introduce the notion of approximate n-derivations on a Banach algebra $\mathscr{A}$ and show that the separating space of an approximate n-derivation ($n > 2$) is not necessarily an ideal, unless the Banach algebra $\mathscr{A}$ is factorizable. From this and some results on bounded n-derivations, we prove that every approximate n-derivation of a semisimple factorizable Banach algebra is automatically continuous and every approximate n-derivation of a commutative semisimple factorizable Banach algebra is identically zero. Some applications of our results are also provided.

1. Introduction

Let $\mathscr{A}$ be an algebra and $n \geqslant 2$ be a fixed integer. A linear mapping $f: \mathscr{A} \longrightarrow \mathscr{A}$ is called an n-derivation provided

$$f\left(\prod_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} \prod_{\imath=1}^{i-1} x_\imath f(x_i) \prod_{\imath=i+1}^{n} x_\imath \tag{1}$$

for all $x_1, x_2, \ldots, x_n \in \mathscr{A}$, where $\prod_{\imath=l+1}^{l} x_\imath = 1 \in \mathbb{C}$ with $l \in \{0, n\}$.

Letting $x_i = x \ (i = 1, \ldots, n)$ in (1), we see that f satisfies the nth power property (see [5, 18]); that is,

$$f(x^n) = \sum_{i=1}^{n} x^{i-1} f(x) x^{n-i}$$

for all x in $\mathscr{A}$. For more details of the nth power property and other applications, see, e.g., [4, 9, 13, 19, 32, 33].

Note that a 2-derivation is a derivation, in the usual sense, on an algebra. It is easy to show that if f is a derivation, then it has the nth power property and is an n-derivation for all $n > 2$, but the converse is not true, in general. For instance, let

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us consider the algebra of 3×3 matrices $\mathscr{A} = \left\{ \begin{bmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{C} \right\}$. Then the mapping $f \colon \mathscr{A} \longrightarrow \mathscr{A}$, defined by

$$f\left(\begin{bmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & \alpha & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(or the identity map on $\mathscr{A}$) is a 3-derivation, while is not a derivation. Also, in unital algebras, it is easy to see that each n-derivation is a derivation. Some properties of the n-derivations were investigated in [12, 14, 27].

Singer and Wermer [30] obtained a fundamental result which initiated investigation into the ranges of derivations on Banach algebras. The result, which is called the Singer-Wermer theorem, states that every continuous linear derivation of a commutative Banach algebra maps into its radical. In particular, there is no nonzero continuous derivation on a commutative semisimple Banach algebra. Thomas in [31] has shown that the assumption of continuity is unnecessary in the Singer-Wermer theorem.

Chang et al. [15, 16, 26] and Park and Rassias [21]–[24] examined the functional inequalities related to derivations and multipliers and their stability. The topic of approximate homomorphisms and approximate derivations in the field of functional equations and inequalities was taken up by several mathematicians, see [1]–[3], [6]–[8], [20, 25] and references therein.

The remainder of this article is organized as follows. In Section 2, using the Kaplansky's trick, we extend Singer and Wermer's result [30] to n-derivations. Also, the nilpotency of eigenvectors of any bounded n-derivation corresponding to its eigenvalues is derived.

The separating space of every approximate derivation on a Banach algebra is a separating ideal; see [15]. In Section 3, we introduce the notion of approximate n-derivations and show that this result is not necessarily true for approximate n-derivations $(n > 2)$, unless the Banach algebra is factorizable. From this and some results on bounded n-derivations, we prove that every approximate n-derivation of a semisimple factorizable Banach algebra is automatically continuous, and also prove that, if the Banach algebra is commutative, then the approximate n-derivation is identically zero. Our results in this section generalize the main results of Kim, Chang and Roh [15, 26].

Finally, in Section 4, from the above results and a result of Brzdęk and Fošner [7], we present some applications of approximate n-derivations related to the stability theory and functional inequalities.

From now on, $\mathscr{A}$ stands for a complex Banach algebra with radical $rad(\mathscr{A})$ and n is a fixed integer greater than 2, unless explicitly stated otherwise. We write $Q(\mathscr{A})$ for the set of all quasinilpotent elements in $\mathscr{A}$, that is, $Q(\mathscr{A}) = \{a \in \mathscr{A} : \sigma(a) = 0\}$, where $\sigma(a)$ is the spectrum of $a \in \mathscr{A}$. It is known that $rad(\mathscr{A}) = \{a \in \mathscr{A} : a\mathscr{A} \subseteq Q(\mathscr{A})\}$ and in the commutative case $rad(\mathscr{A}) = Q(\mathscr{A})$.

2. Bounded n-derivations

In this section, we discuss and extend some important results of bounded derivations to bounded n-derivations.

LEMMA 1. *Let $\mathscr{A}$ be an algebra, $D\colon \mathscr{A} \longrightarrow \mathscr{A}$ be an n-derivation and $D^2(a) = 0$. Then $D^k(a^n) = k!(D(a))^k$ holds for each $k \geqslant n$.*

Proof. Since $D^2(a) = 0$, $D^3(a) = D(D^2(a)) = 0$ and by the same way $D^4(a) = D^5(a) = \cdots = 0$. By Leibnitz rule for n-derivations,

$$D^k(a^n) = \sum_{k_1+k_2+\cdots+k_n=k} \frac{k!}{k_1!k_2!\ldots!k_n!} D^{k_1}(a)D^{k_2}(a)\cdots D^{k_n}(a).$$

In each summand, $D^{k_i}(a) = 0$, except when $k_1 = k_2 = \cdots = k_n = 1$. So $D^k(a^n) = k!(D(a))^k$ for each $k \geqslant n$.

THEOREM 1. *Let $D\colon \mathscr{A} \longrightarrow \mathscr{A}$ be a bounded n-derivation and $D^2(a) = 0$. Then $D(a)$ is quasinilpotent.*

Proof. By Lemma 1, for each $k \geqslant n$ we have $D^k(a^n) = k!(D(a))^k$. So

$$\left\|k!(D(a))^k\right\| = \left\|D^k(a^n)\right\| \leqslant \|D\|^k \|a\|^n. \tag{2}$$

Thus

$$\left\|(D(a))^k\right\| \leqslant \frac{1}{k!}\|D\|^k\|a\|^n \qquad (k \geqslant n). \tag{3}$$

So

$$\lim_{k\to\infty} \left\|(D(a))^k\right\|^{\frac{1}{k}} \leqslant \lim_{k\to\infty} \frac{1}{\sqrt[k]{k!}} \|D\| \, \|a\|^{\frac{n}{k}} = 0.$$

Hence $r(D(a)) = 0$. Therefore $D(a)$ is quasinilpotent.

In the sequel, the bracket $[a,b]$ stands for the commutator $ab - ba$ for all $a, b \in \mathscr{A}$.

In [30], Singer and Wermer proved that for a commutative Banach algebra $\mathscr{A}$, the range of a bounded derivation $D\colon \mathscr{A} \longrightarrow \mathscr{A}$ is contained in its radical. Using the Kaplansky's trick, we extend this result to n-derivations.

THEOREM 2. *Let $\mathscr{A}$ be a commutative Banach algebra and $D\colon \mathscr{A} \longrightarrow \mathscr{A}$ be a bounded n-derivation. Then $D(\mathscr{A}^n) \subseteq rad(\mathscr{A})$, where $\mathscr{A}^n = lin \underbrace{\mathscr{A}\cdot\mathscr{A}\cdots\mathscr{A}}_{n\text{-times}}$.*

Proof. Since $\mathscr{A}$ is commutative, so for all $a \in \mathscr{A}$ we have $\left[a^{n-1}, (n-1)a^{n-2}D(a)\right] = 0$ and then

$$\left[L_{a^{n-1}}, L_{(n-1)a^{n-2}D(a)}\right] = 0.$$

So for each $x \in \mathscr{A}$, we obtain

$$\begin{aligned}[L_{a^{n-1}}, -D](x) &= [D, L_{a^{n-1}}](x) \\ &= D \circ L_{a^{n-1}}(x) - L_{a^{n-1}} \circ D(x) \\ &= D(a^{n-1}x) - a^{n-1}D(x) \\ &= (n-1)a^{n-2}D(a)x = L_{(n-1)a^{n-2}D(a)}(x).\end{aligned}$$

Hence we get

$$[L_{a^{n-1}}, -D] = L_{(n-1)a^{n-2}D(a)}. \tag{4}$$

Thus $[L_{a^{n-1}}, [L_{a^{n-1}}, -D]] = 0$ and by the Kleinecke-Shirikov Theorem [17, 28], the commutator $[L_{a^{n-1}}, -D]$ is quasinilpotent and so $L_{(n-1)a^{n-2}D(a)}$ is quasinilpotent (i.e., $r(L_{(n-1)a^{n-2}D(a)}) = 0$ in $B(\mathscr{A})$, where $B(\mathscr{A})$ is the space of all bounded linear maps on $\mathscr{A}$). That is, $r((n-1)a^{n-2}D(a)) = 0$. By [11, Proposition 1.5.32], we get

$$a^{n-1}D(a) = a(a^{n-2})D(a) \in Q(\mathscr{A}) = rad(\mathscr{A}). \tag{5}$$

Also by [11, Corollary 2.6.32] we obtain that $na^{n-1}D(a) \in rad(\mathscr{A})$. On the other hand, by the commutativity of $\mathscr{A}$, we have $D(a^n) = na^{n-1}D(a)$ and so $D(a^n) \in rad(\mathscr{A})$. According to this fact that every element of $\mathscr{A}^n$ could be written as a linear combination of the nth power of elements of $\mathscr{A}$; that is,

$$\prod_{i=1}^{n} a_i = \frac{1}{n!2^{n-1}} \sum_{i_1=1}^{n-1} \cdots \sum_{i_{n-1}=1}^{n-1} (-1)^{\sum_{k=1}^{n-1} i_k} \left(a_1 + \sum_{k=1}^{n-1} (-1)^{i_k} a_{k+1} \right)^n,$$

hence $D(\mathscr{A}^n) \subseteq rad(\mathscr{A})$.

We remark that a Banach algebra $\mathscr{A}$ is called factorizable if for each a in $\mathscr{A}$ there are b and c in $\mathscr{A}$ such that $a = bc$. For a factorizable Banach algebra $\mathscr{A}$, we have $\mathscr{A} = \mathscr{A}^n$. As a prompt result, we obtain the following.

COROLLARY 1. *Let $\mathscr{A}$ be a commutative factorizable Banach algebra and $D\colon \mathscr{A} \longrightarrow \mathscr{A}$ be a bounded n-derivation. Then $D(\mathscr{A}) \subseteq rad(\mathscr{A})$. Moreover, if $\mathscr{A}$ is semisimple, then D is identically zero.*

By a classical theorem due to Cohen [10], every Banach algebra with a bounded approximate identity is factorizable. Every C^*-algebra and the group algebra $L^1(G)$, for a locally compact group G with a unique left Haar measure, are relevant examples of this algebras. In the following, we characterize bounded n-derivations on $L^1(G)$ and also on C^*-algebras.

COROLLARY 2. *Suppose that (i) G is an abelian locally compact group and $D\colon L^1(G) \longrightarrow L^1(G)$ is a bounded n-derivation or (ii) $\mathscr{A}$ is a commutative C^*-algebra and $D\colon \mathscr{A} \longrightarrow \mathscr{A}$ is a bounded n-derivation. Then D is identically zero.*

THEOREM 3. *Let D be a bounded n-derivation on a Banach algebra $\mathscr{A}$ and let $a \in \mathscr{A}$. If a commutes with $D(a)$, then $D(a^n)$ is quasinilpotent.*

Proof. Since $aD(a) = D(a)a$, we can conclude that

$$[L_{a^{n-1}}, [L_{a^{n-1}}, -D]] = \left[L_{a^{n-1}}, L_{(n-1)a^{n-2}D(a)}\right] = L_{\left[a^{n-1},(n-1)a^{n-2}D(a)\right]} = 0. \tag{6}$$

So by a similar argument as in the proof of Theorem 2 we get

$$r((n-1)a^{n-2}D(a)) = r(L_{(n-1)a^{n-2}D(a)}) = 0.$$

Moreover, for each $k \in \mathbb{N}$ we have

$$\left\|\left(a^{n-1}D(a)\right)^k\right\| = \left\|a^k\left(a^{n-2}D(a)\right)^k\right\| \leqslant \left\|a^k\right\|\left\|\left(a^{n-2}D(a)\right)^k\right\|.$$

Hence

$$\left\|\left(a^{n-1}D(a)\right)^k\right\|^{\frac{1}{k}} \leqslant \|a\|\left\|\left(a^{n-2}D(a)\right)^k\right\|^{\frac{1}{k}}$$

holds for all positive integers k. Thus, $\left\|\left(a^{n-1}D(a)\right)^k\right\|^{\frac{1}{k}} \to 0$, as $k \to \infty$. Hence, we have $D(a^n)$ is quasinilpotent.

Finally, in this section, we give a result concerning n-derivations where nilpotency is implied.

THEOREM 4. *Let $D\colon \mathscr{A} \longrightarrow \mathscr{A}$ be a bounded n-derivation on a Banach algebra $\mathscr{A}$. The eigenvectors of D corresponding to nonzero eigenvalues are nilpotent.*

Proof. If $\lambda \in \mathbb{C}\setminus\{0\}$ and $D(a) = \lambda a$, then

$$D(a^{kn-(k-1)}) = (kn-(k-1))\lambda a^{kn-(k-1)}, \quad \text{for all } k \in \mathbb{N}.$$

To prove this, we proceed by induction as follows:

For $k = 1$, $D(a^n) = n\lambda a^n$. So we prove the condition for $k = 2$.

$$\begin{aligned} D(a^{2n-1}) &= D(a^n a^{n-1}) = D(a^n)a^{n-1} + a^n D(a)a^{n-2} + \ldots + a^n a^{n-2}D(a) \\ &= n\lambda a^{2n-1} + \underbrace{\lambda a^{2n-1} + \ldots + \lambda a^{2n-1}}_{(n-1)\text{-times}} \qquad (\text{by } k=1) \\ &= \lambda(2n-1)a^{2n-1}. \end{aligned}$$

Therefore the assertion is true for $k = 2$. Suppose the assertion is true for $k-1$. Then

$$\begin{aligned} D(a^{kn-(k-1)}) &= D(a^{(k-1)(n-1)+1}a^{n-1}) \\ &= D(a^{(k-1)(n-1)+1})a^{n-1} + a^{(k-1)(n-1)+1}D(a)a^{n-2} \\ &\quad + \ldots + a^{(k-1)(n-1)+1}a^{n-2}D(a) \\ &= ((k-1)(n-1)+1)\lambda a^{kn-(k-1)} + \underbrace{\lambda a^{kn-(k-1)} + \ldots + \lambda a^{kn-(k-1)}}_{(n-1)\text{-times}} \\ &\quad (\text{by hypothesis}) \\ &= \lambda(kn-(k-1))\lambda a^{kn-(k-1)}. \end{aligned}$$

Thus we conclude that $D(a^{kn-(k-1)}) = (kn-(k-1))\lambda a^{kn-(k-1)}$, for all $k \in \mathbb{N}$. Since the spectrum is a compact set, we can choose $k \in \mathbb{N}$ such that $k = k'n-(k'-1)$ for some $k' \in \mathbb{N}$, and also $k > \frac{r(D)}{|\lambda|}$ in which case $|\lambda| k > r(D)$. Therefore $\lambda k \notin \sigma(D)$, so that $D-(\lambda k)I$ is an invertible linear operator. Thus $D(a^k) = (\lambda k)a^k = \lambda k I(a^k)$, I as identity operator. We can deduce

$$D(a^k) - \lambda k I(a^k) = (D-\lambda k I)(a^k) = 0.$$

Hence we have $(D-\lambda kI)^{-1}(D-\lambda kI)(a^k) = 0$, which yields $a^k = 0$. Therefore, a is nilpotent.

3. Automatic continuity of approximate n-derivations

In this section, we introduce the notion of approximate n-derivations that extends the notion of approximate derivations, which was investigated in [15]. Now we deal with automatic continuity of approximate n-derivations.

DEFINITION 1. We say that a mapping $f\colon \mathscr{A} \longrightarrow \mathscr{A}$ is an approximate n-derivation if f is linear and satisfying

$$\left\| f\left(\prod_{i=1}^{n} x_i\right) - \sum_{i=1}^{n}\prod_{\iota=1}^{i-1} x_\iota f(x_i) \prod_{\iota=i+1}^{n} x_\iota \right\| \leqslant \varepsilon \tag{7}$$

for all $x_1, x_2, \ldots, x_n \in \mathscr{A}$ and for some $\varepsilon > 0$, where $\prod_{\iota=l+1}^{l} x_\iota = 1 \in \mathbb{C}$ with $l \in \{0, n\}$.

Let $\mathscr{A}$, $\mathscr{B}$ be normed linear spaces and $f\colon \mathscr{A} \longrightarrow \mathscr{B}$ be a linear mapping. The separating space of f is defined by

$$\mathfrak{S}(f) = \{b \in \mathscr{B} : \exists \text{sequence } (a_n) \text{ in } \mathscr{A} \text{ such that } a_n \to 0 \text{ and } f(a_n) \to b\}.$$

The separating space $\mathfrak{S}(f)$ is a closed linear subspace of $\mathscr{B}$; moreover, if $\mathscr{A}$ and $\mathscr{B}$ are F-spaces, then, by the closed graph theorem, f is continuous if and only if $\mathfrak{S}(f) = \{0\}$ [11, Proposition 5.1.2].

DEFINITION 2. Let $\mathscr{A}$ be a Banach algebra and I is a closed ideal in $\mathscr{A}$. We say I is a separating ideal if for every sequence $\{a_n\}$ in $\mathscr{A}$, there exists $N \in \mathbb{N}$ such that $\overline{(Ia_n \cdots a_1)} = \overline{(Ia_N \cdots a_1)}$ for all $n \geqslant N$.

It is known that the separating space of every derivation on a Banach algebra is a separating ideal; see [29]. This result has been extended in [15] for approximate derivations. In the sequel, we show that the above result is not necessarily true for approximate n-derivations $(n > 2)$ unless the Banach algebra is factorizable.

THEOREM 5. *Let $\mathscr{A}$ be a factorizable Banach algebra and $f\colon \mathscr{A} \longrightarrow \mathscr{A}$ be an approximate n-derivation. Then $\mathfrak{S}(f)$ is a separating ideal.*

Proof. Let $a \in \mathscr{A}$ and $y \in \mathfrak{S}(f)$. Then there exists a sequence $\{x_n\} \in \mathscr{A}$ with $x_n \to 0$ and $f(x_n) \to y$. Since $\mathscr{A}$ is factorizable, there exist $a_1, a_2, \ldots, a_{n-1} \in \mathscr{A}$ such that $a = \prod_{i=1}^{n-1} a_i$. As $ax_n \to 0$ we have

$$\begin{aligned}
\|f(nax_n) - nay\| &= \left\| f\left(\prod_{i=1}^{n-1} na_i x_n\right) - n\prod_{i=1}^{n-1} a_i y\right\| \\
&\leqslant \left\| f\left(\prod_{i=1}^{n-1} na_i x_n\right) - n\sum_{i=1}^{n-1}\prod_{\imath=1}^{i-1} a_\imath f(a_i) \prod_{\imath=i+1}^{n-1} a_\imath x_n - n\prod_{i=1}^{n-1} a_i f(x_n)\right\| \\
&\quad + n\left\|\sum_{i=1}^{n-1}\prod_{\imath=1}^{i-1} a_\imath f(a_i) \prod_{\imath=i+1}^{n-1} a_\imath\right\| \|x_n\| + n\left\|\prod_{i=1}^{n-1} a_i\right\| \|f(x_n) - y\|.
\end{aligned}$$

for all natural numbers n. Hence

$$\begin{aligned}
\|f(ax_n) - ay\| \leqslant \frac{\varepsilon}{n} &+ \left(\sum_{i=1}^{n-1}\prod_{\imath=1}^{i-1} \|a_\imath\| \, \|f(a_i)\| \prod_{\imath=i+1}^{n-1} \|a_\imath\|\right) \|x_n\| \\
&+ \left(\prod_{i=1}^{n-1} \|a_i\|\right) \|f(x_n) - y\| \longrightarrow 0
\end{aligned}$$

as $n \to \infty$. Therefore, $f(ax_n) \to ay$ and we get $ay \in \mathfrak{S}(f)$. Similarly $ya \in \mathfrak{S}(f)$. Next for an arbitrary sequence $\{b_n\} \in \mathscr{A}$, we define $\mathscr{R}_n = \mathscr{T}_n$ by $\mathscr{R}_n(y) = \mathscr{T}_n(y) = yb_n$. By the factorizability of $\mathscr{A}$ we obtain that $b_n = \prod_{i=1}^{m-1} c_i^n$ for some $c_1^n, c_2^n, \ldots, c_{m-1}^n \in \mathscr{A}$. So, for each natural number k,

$$\begin{aligned}
\|(f \circ \mathscr{T}_n - \mathscr{R}_n \circ f)(ky)\| &= \|f(kyb_n) - f(ky)b_n\| \\
&= \left\| f\left(ky\prod_{i=1}^{m-1} c_i^n\right) - f(ky)\prod_{i=1}^{m-1} c_i^n\right\| \\
&\leqslant \left\| f\left(ky\prod_{i=1}^{m-1} c_i^n\right) - f(ky)\prod_{i=1}^{m-1} c_i^n - ky\sum_{i=1}^{n-1}\prod_{\imath=1}^{i-1} c_\imath^n f(c_i^n) \prod_{\imath=i+1}^{n-1} c_\imath^n\right\| \\
&\quad + \|ky\| \left\|\sum_{i=1}^{n-1}\prod_{\imath=1}^{i-1} c_\imath^n f(c_i^n) \prod_{\imath=i+1}^{n-1} c_\imath^n\right\| \\
&\leqslant \varepsilon + \|ky\| \left(\sum_{i=1}^{n-1}\prod_{\imath=1}^{i-1} \|c_\imath^n\| \, \|f(c_i^n)\| \prod_{\imath=i+1}^{n-1} \|c_\imath^n\|\right),
\end{aligned}$$

which implies that

$$\|(f \circ \mathscr{T}_n - \mathscr{R}_n \circ f)(y)\| \leqslant \frac{\varepsilon}{k} + \|y\| \left(\sum_{i=1}^{n-1}\prod_{\imath=1}^{i-1} \|c_\imath^n\| \, \|f(c_i^n)\| \prod_{\imath=i+1}^{n-1} \|c_\imath^n\|\right).$$

Then we have $f \circ \mathscr{T}_n - \mathscr{R}_n \circ f$ is continuous for each $n \in \mathbb{N}$. Consequently $\mathfrak{S}(f)$ is a separating ideal by [29, Lemma 1.6].

The next example indicates that Theorem 5 is not true without the assumption that $\mathscr{A}$ ia a factorizable Banach algebra.

EXAMPLE 1. Let $\mathscr{A} = \left\{ \begin{bmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{C} \right\}$ be a subalgebra of $\mathbb{M}_3(\mathbb{C})$ with the operator norm. Then $\mathscr{A}$ is not a factorizable Banach algebra. Define the mapping $f\colon \mathscr{A} \longrightarrow \mathscr{A}$ by

$$f\left(\begin{bmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & \alpha & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

From the fact that, for all $X \in \mathscr{A}$, we have $X^3 = 0$, it is easy to show that f is an approximate 3-derivation. We want to show that $\mathfrak{S}(f)$ is not an ideal in $\mathscr{A}$. To this end, suppose that $A \in \mathscr{A}$ and $Y \in \mathfrak{S}(f)$. Then

$$A = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad Y = \begin{bmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{bmatrix},$$

for some $a, b, c, \alpha, \beta, \gamma \in \mathbb{C}$, and

$$AY = \begin{bmatrix} 0 & 0 & a\gamma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Given a sequence $\{X_n\} \in \mathscr{A}$ such that $X_n \to 0$ and $f(X_n) \to AY$. The following equalities

$$\begin{aligned} \|f(X_n) - AY\| &= \left\| \begin{bmatrix} 0 & x_n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & a\gamma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0 & x_n & -a\gamma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\| \\ &= \sup \left\{ \left\| \begin{bmatrix} 0 & x_n & -a\gamma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \right\| : \left\| \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \right\| \leqslant 1 \right\} \\ &= \sup \left\{ \left\| \begin{bmatrix} x_n t_2 - a\gamma t_3 \\ 0 \\ 0 \end{bmatrix} \right\| : |t_1|^2 + |t_2|^2 + |t_3|^2 \leqslant 1 \right\} \\ &= \sup \left\{ |x_n t_2 - a\gamma t_3| : |t_1|^2 + |t_2|^2 + |t_3|^2 \leqslant 1 \right\}, \end{aligned}$$

hold. Letting $t_1 = t_2 = 0$ and $t_3 = 1$ in the above equality, we have

$$\|f(X_n) - AY\| \geqslant |a\gamma| > 0. \tag{8}$$

We obtain by (8) that $AY \notin \mathfrak{S}(f)$. Therefore $\mathfrak{S}(f)$ is not a separating ideal.

THEOREM 6. *Let $\mathscr{A}$ be a semisimple factorizable Banach algebra. Then every approximate n-derivation $f\colon \mathscr{A} \longrightarrow \mathscr{A}$ is automatically continuous.*

Proof. By virtue of Theorem 5, $\mathfrak{S}(f)$ is a separating ideal. Since $\mathscr{A}$ is semisimple, we have $\mathfrak{S}(f)$ is finite-dimensional [11, Corollary 5.2.28]. Also, by [11, Theorem 1.5.4], we deduce that

$$\operatorname{rad}(\mathfrak{S}(f)) = \mathfrak{S}(f) \cap \operatorname{rad}(\mathscr{A}) = \{0\}.$$

Then $\mathfrak{S}(f)$ is a semisimple finite-dimensional algebra. From the Wedderburn Structure Theorem [11, Theorem 1.5.9], $\mathfrak{S}(f)$ has an identity element e. Then there exists a sequence $\{x_m\}$ in $\mathscr{A}$ such that $x_m \to 0$ and $f(x_m) \to e$. Hence we get

$$\lim_{m\to\infty} f(x_m)\, e = e. \tag{9}$$

Moreover, since $\mathfrak{S}(f)$ is finite-dimensional and $x_m e \to 0$ in $\mathfrak{S}(f)$, we have $f(x_m e) \to 0$. On the other hand,

$$\begin{aligned}
\|f(x_m)\cdot me\| &= \left\|f(x_m)\cdot me^{n-1}\right\| \\
&\leqslant \left\| f\left(x_m \cdot me^{n-1}\right) - f(x_m)\cdot me^{n-1} - x_m \cdot m \sum_{i=1}^{n-1} e^{i-1} f(e)\, e^{n-i-1} \right\| \\
&\quad + \left\| f\left(x_m\cdot me^{n-1}\right)\right\| + m\,\|x_m\| \sum_{i=1}^{n-1} \|f(e)\, e\| \\
&\leqslant \varepsilon + \|f(x_m\cdot me)\| + m(n-1)\,\|x_m\|\,\|f(e)\|\,\|e\|,
\end{aligned}$$

which implies that

$$\|f(x_m)\, e\| \leqslant \frac{\varepsilon}{m} + \|f(x_m e)\| + (n-1)\,\|x_m\|\,(\|f(e)\, e\|).$$

By letting $m\to\infty$, it follows that $f(x_m)\, e \to 0$. Thus, by equation (9), we obtain $e = 0$; that is, $x = xe = 0$ for all $x \in \mathfrak{S}(f)$. Therefore, $\mathfrak{S}(f) = \{0\}$ and so f is continuous.

COROLLARY 3. *Suppose that (i) $\mathscr{A}$ is a semisimple Banach algebra with a bounded left (right) approximate identity and $f\colon \mathscr{A} \longrightarrow \mathscr{A}$ is an approximate n-derivation, (ii) $\mathscr{A}$ is a C^*-algebra and $f\colon \mathscr{A} \longrightarrow \mathscr{A}$ is an approximate n-derivation or (iii) G is a locally compact group and $f\colon L^1(G) \longrightarrow L^1(G)$ is an approximate n-derivation. Then f is automatically continuous.*

From Corollary 1 and Theorem 6, we prove the following theorem.

THEOREM 7. *Let $\mathscr{A}$ be a commutative semisimple factorizable Banach algebra. Then every approximate n-derivation on $\mathscr{A}$ is an n-derivation and so vanishes.*

Proof. Suppose that $f\colon \mathscr{A} \to \mathscr{A}$ is an approximate n-derivation satisfying inequality (7). Let $c, x_1, \ldots, x_n$ be arbitrary elements of $\mathscr{A}$. Since $\mathscr{A}$ is factorizable thus we can write $c=\prod_{i=1}^{n-1} c_i$ for some $c_1,c_2,\ldots,c_{n-1}\in\mathscr{A}$. For every character $\varphi\in\Delta(\mathscr{A})$ ($\Delta(\mathscr{A})$ denotes the set of all nonzero multiplicative linear functionals on $\mathscr{A}$), we have

$$\begin{aligned}
&\Big|\varphi(c)\ \varphi\Big(f(\prod_{i=1}^{n}x_i)-\sum_{i=1}^{n}\prod_{\iota=1}^{i-1}x_\iota f(x_i)\prod_{\iota=i+1}^{n}x_\iota\Big)\Big|\\
&\quad=\Big|\varphi\Big(\prod_{i=1}^{n-1}c_i\Big[f(\prod_{i=1}^{n}x_i)-\sum_{i=1}^{n}\prod_{\iota=1}^{i-1}x_\iota f(x_i)\prod_{\iota=i+1}^{n}x_\iota\Big]\Big)\Big|\\
&\quad\leqslant\Big|\varphi\Big(\prod_{i=1}^{n}c_i f(\prod_{i=1}^{n}x_i)+\sum_{i=1}^{n-1}\prod_{\iota=1}^{i-1}c_\iota f(c_i)\prod_{\iota=i+1}^{n-1}c_\iota\prod_{j=1}^{n}x_j-f\Big(\prod_{i=1}^{n-1}c_i\prod_{i=1}^{n}x_i\Big)\Big)\Big|\\
&\qquad+\Big|\varphi\Big(\sum_{i=1}^{n-1}\prod_{\iota=1}^{i-1}x_\iota f(x_i)\prod_{\iota=i+1}^{n}x_\iota\prod_{j=1}^{n-1}c_j-f\Big(x_n\prod_{i=1}^{n-1}c_i\Big)\prod_{i=1}^{n-1}x_i+f\Big(\prod_{i=1}^{n-1}c_i\prod_{i=1}^{n}x_i\Big)\Big)\Big|\\
&\qquad+\Big|\varphi\Big(f(x_n)\prod_{i=1}^{n-1}c_i\prod_{i=1}^{n-1}x_i-\sum_{i=1}^{n-1}\prod_{\iota=1}^{i-1}c_\iota f(c_i)\prod_{\iota=i+1}^{n}c_\iota\prod_{j=1}^{n}x_j+f\Big(x_n\prod_{i=1}^{n-1}c_i\Big)\prod_{i=1}^{n-1}x_i\Big)\Big|\\
&\quad\leqslant 2\varepsilon+\varepsilon\Big(\prod_{i=1}^{n-1}\|x_i\|\Big).
\end{aligned}$$

Since φ is a character so we can choose a sequence $c_n\in\mathscr{A}$ such that $\lim_{n\mapsto\infty}\varphi(c_n)=\infty$. Dividing the above inequality by $|\varphi(c_n)|$, we conclude that

$$f(\prod_{i=1}^{n}x_i)-\sum_{i=1}^{n}\prod_{\iota=1}^{i-1}x_\iota f(x_i)\prod_{\iota=i+1}^{n}x_\iota\in\bigcap_{\varphi\in\Delta(\mathscr{A})}\ker\varphi=rad(\mathscr{A})=\{0\},$$

and so f is an n-derivation. On the other hand, from Theorem 6, f is continuous. Therefore, by Corollary 1, the proof of this theorem is complete.

Recall that the above theorem applied to commutative C^*-algebras and the group algebras associated with abelian locally compact groups.

4. Applications in stability theory and functional inequalities

In this section, we present some applications of the results, presented in the previous sections, to the stability theory and functional inequalities.

We introduce a useful result that can be easily derived from Brzdęk and Fošner [7, Lemma 1].

LEMMA 2. *Let $\mathscr{A}$ be a Banach algebra and $S\subset\mathbb{U}:=\{z\in\mathbb{C}:|z|=1\}$ be a connected set containing at least two points. Let $f\colon\mathscr{A}\to\mathscr{A}$ be an additive mapping such that $f(\mu x)=\mu f(x)$ for all $x\in\mathscr{A}$ and $\mu\in S$. Then f is $\mathbb{C}$-linear.*

In the rest of this section, S stands for a connected subset of $\mathbb{U}$ such that $1\in S$ and $S\setminus\{1\}\neq\emptyset$.

LEMMA 3. *Suppose $\mathscr{A}$ is a vector space and $\mathscr{B}$ is a Banach space. Let $f\colon \mathscr{A} \to \mathscr{B}$ and $\mathscr{T}\colon \mathscr{A}^n \to [0,\infty)$ be mappings such that*

$$\left\| \sum_{i=1}^{n} f\left(x_i + \frac{\sum_{j=1, j\neq i}^{n} x_j}{n-1} \right) \right\| \leqslant \left\| 2f\left(\sum_{i=1}^{n} x_i \right) \right\| + \mathscr{T}(x_1, x_2, \ldots, x_n), \tag{10}$$

$$\lim_{m\to\infty} 2^{-\kappa m} \mathscr{T}(2^{\kappa m}x_1, 2^{\kappa m}x_2, \ldots, 2^{\kappa m}x_n) = 0, \tag{11}$$

$$\begin{aligned} \exists \kappa \in \{-1,1\}, \quad \sum_{l=\frac{1-\kappa}{2}}^{\infty} 2^{-l\kappa} \Bigg[2\mathscr{T}\left(\frac{2^{\kappa l}(n-1)}{n-2}x, \frac{2^{\kappa l}(1-n)}{n-2}x, 0, \ldots, 0 \right) \\ + \mathscr{T}\left(\frac{2^{l\kappa+1}(n-1)}{n-2}x, \frac{2^{\kappa l}(1-n)}{n-2}x, \frac{2^{\kappa l}(1-n)}{n-2}x, 0, \ldots, 0 \right) \Bigg] < \infty \end{aligned} \tag{12}$$

for all $x, x_1, x_2, \ldots, x_n \in \mathscr{A}$. Then there exists a unique additive mapping $D\colon \mathscr{A} \to \mathscr{B}$ with

$$\begin{aligned} \|f(x) - D(x)\| \leqslant \sum_{l=\frac{1-\kappa}{2}}^{\infty} 2^{-\kappa l - 1} \Bigg[& 2\mathscr{T}\left(\frac{2^{\kappa l}(n-1)}{n-2}x, \frac{2^{\kappa l}(1-n)}{n-2}x, 0, \ldots, 0 \right) \\ & + \mathscr{T}\left(\frac{2^{\kappa l+1}(n-1)}{n-2}x, \frac{2^{\kappa l}(1-n)}{n-2}x, \frac{2^{\kappa l}(1-n)}{n-2}x, 0, \ldots, 0 \right) \\ & + \frac{(3n-1)(1+\kappa)}{2n-4} \mathscr{T}(0, 0, \ldots, 0) \Bigg], \qquad x \in \mathscr{A}. \end{aligned} \tag{13}$$

Proof. Letting $x_1 = x_2 = \cdots = x_n = 0$ in (10), we obtain

$$\|f(0)\| \leqslant \frac{1}{n-2} \mathscr{T}(0, 0, \ldots, 0). \tag{14}$$

If $\kappa = -1$, since $\mathscr{T}(0,0,\ldots,0) = 0$, $f(0) = 0$. Setting $x_1 = \frac{n-1}{n-2}x$, $x_2 = \frac{1-n}{n-2}x$ and $x_3 = \cdots = x_n = 0$ in (10), we see that

$$\|f(x) + f(-x) + (n-2)f(0)\| \leqslant \mathscr{T}\left(\frac{n-1}{n-2}x, \frac{1-n}{n-2}x, 0, \ldots, 0 \right) + \|2f(0)\| \tag{15}$$

for all $x \in \mathscr{A}$. It follows from (14) and (15) that

$$\|f(x) + f(-x)\| \leqslant \mathscr{T}\left(\frac{n-1}{n-2}x, \frac{1-n}{n-2}x, 0, \ldots, 0 \right) + \frac{n(1+\kappa)}{2n-4} \mathscr{T}(0, 0, \ldots, 0) \tag{16}$$

for all $x \in \mathscr{A}$. Putting $x_1 = \frac{2n-2}{n-2}x$, $x_2 = x_3 = \frac{1-n}{n-2}x$ and $x_4 = \cdots = x_n = 0$ in (10), we get

$$\begin{aligned} \|f(2x) + 2f(-x) + (n-3)f(0)\| \leqslant & \mathscr{T}\left(\frac{2n-2}{n-2}x, \frac{1-n}{n-2}x, \frac{1-n}{n-2}x, 0, \ldots, 0 \right) \\ & + \|2f(0)\| \end{aligned} \tag{17}$$

for all $x \in \mathscr{A}$. It follows from (16) and (17) that

$$\|f(2x)+2f(-x)\| \leqslant \mathscr{T}\left(\frac{2n-2}{n-2}x, \frac{1-n}{n-2}x, \frac{1-n}{n-2}x, 0, \ldots, 0\right) + \frac{(n-1)(1+\kappa)}{2n-4}\mathscr{T}(0,0,\ldots,0) \tag{18}$$

for all $x \in \mathscr{A}$. We deduce from (16) and (18) that

$$\begin{aligned}\|f(2x)-2f(x)\| \leqslant & 2\mathscr{T}\left(\frac{n-1}{n-2}x, \frac{1-n}{n-2}x, 0, \ldots, 0\right) \\ & + \mathscr{T}\left(\frac{2n-2}{n-2}x, \frac{1-n}{n-2}x, \frac{1-n}{n-2}x, 0, \ldots, 0\right) \\ & + \frac{(3n-1)(1+\kappa)}{2n-4}\mathscr{T}(0,0,\ldots,0)\end{aligned} \tag{19}$$

for all $x \in \mathscr{A}$. Hence

$$\begin{aligned}&\left\|2^{-\kappa(m+1)}f\left(2^{\kappa(m+1)}x\right) - 2^{-\kappa k}f\left(2^{\kappa k}x\right)\right\| \\ &\leqslant \sum_{l=k}^{m}\left\|2^{-\kappa(l+1)}f\left(2^{\kappa(l+1)}x\right) - 2^{-\kappa l}f\left(2^{\kappa l}x\right)\right\| \\ &\leqslant \sum_{l=k+\frac{1-\kappa}{2}}^{m} 2^{-\kappa l-1}\left[2\mathscr{T}\left(\frac{2^{\kappa l}(n-1)}{n-2}x, \frac{2^{\kappa l}(1-n)}{n-2}x, 0, \ldots, 0\right)\right. \\ &\quad + \mathscr{T}\left(\frac{2^{l\kappa+1}(n-1)}{n-2}x, \frac{2^{\kappa l}(1-n)}{n-2}x, \frac{2^{\kappa l}(1-n)}{n-2}x, 0, \ldots, 0\right) \\ &\quad \left. + \frac{(3n-1)(1+\kappa)}{2n-4}\mathscr{T}(0,0,\ldots,0)\right]\end{aligned} \tag{20}$$

for all $k,m \in \mathbb{N}$ with $m > k$ and all $x \in \mathscr{A}$.Thus the sequence $\{2^{-\kappa m}f(2^{\kappa m}x)\}$ is Cauchy and, since $\mathscr{B}$ is a Banach space, there exists a limit mapping $D : \mathscr{A} \to \mathscr{B}$ by $D(x) := \lim_{m\to\infty} 2^{-\kappa m}f(2^{\kappa m}x)$ for all $x \in \mathscr{A}$ such that (13) holds.

The next step is to show that D is additive. Letting $(2^{\kappa m}x_1, 2^{\kappa m}x_2, \ldots, 2^{\kappa m}x_n)$ for $(x_1, x_2, \ldots, x_n)$ in (10), we obtain

$$\left\|\sum_{i=1}^{n} 2^{-\kappa m}f\left(2^{\kappa m}x_i + \frac{\sum_{j=1, j\neq i}^{n} 2^{\kappa m}x_j}{n-1}\right)\right\| \leqslant \left\|2^{-\kappa m+1}f\left(\sum_{i=1}^{n} 2^{\kappa m}x_i\right)\right\| + 2^{-\kappa m}\mathscr{T}(2^{\kappa m}x_1, 2^{\kappa m}x_2, \ldots, 2^{\kappa m}x_n) \tag{21}$$

for all $m \in \mathbb{N}$ and all $x_1, x_2, \ldots, x_n \in \mathscr{A}$. Letting $m \to \infty$ and using (11), we observe that

$$\left\|\sum_{i=1}^{n} D\left(x_i + \frac{\sum_{j=1, j\neq i}^{n} x_j}{n-1}\right)\right\| \leqslant \left\|2D\left(\sum_{i=1}^{n} x_i\right)\right\| \tag{22}$$

for all $x_1, x_2, \ldots, x_n \in \mathscr{A}$. Setting $x_1 = x_2 = \cdots = x_n = 0$ in (22), we have

$$\|nD(0)\| \leqslant \|2D(0)\|.$$

Since $n \geqslant 3$, $D(0) = 0$. Putting $x_1 = \frac{n-1}{n-2}x$, $x_2 = \frac{1-n}{n-2}x$ and $x_3 = \cdots = x_n = 0$ in (22), we obtain

$$\begin{aligned}\|D(x) + D(-x) + (n-2)D(0)\| &= \|D(x) + D(-x)\| \\ &\leqslant \|2D(0)\| = 0\end{aligned}$$

for all $x \in \mathscr{A}$. Hence $D(-x) = -D(x)$ for all $x \in \mathscr{A}$. Letting $x_1 = \frac{n-1}{n-2}(x+y)$, $x_2 = \frac{1-n}{n-2}x$, $x_3 = \frac{1-n}{n-2}y$ and $x_4 = \cdots = x_n = 0$ in (22), we get

$$\begin{aligned}\|D(x+y) + D(-x) + D(-y) + (n-3)D(0)\| &= \|D(x+y) - D(x) - D(y)\| \\ &\leqslant \|2D(0)\| = 0\end{aligned}$$

for all $x, y \in \mathscr{A}$. Thus we have $D(x+y) = D(x) + D(y)$ for all $x, y \in \mathscr{A}$; that is, D is additive.

Our next goal is to show that D is unique. Suppose $D' : \mathscr{A} \to \mathscr{B}$ is another additive mapping satisfying the inequality (13). Then, for every $m \in \mathbb{N}$ and all $x \in \mathscr{A}$, we get

$$\begin{aligned}\|D(x) - D'(x)\| &= 2^{-\kappa m}\left\|D(2^{\kappa m}x) - f(2^{\kappa m}x) + f(2^{\kappa m}x) - D'(2^{\kappa m}x)\right\| \\ &\leqslant 2^{-\kappa m}\left\{\|f(2^{\kappa m}x) - D(2^{\kappa m}x)\| + \left\|f(2^{\kappa m}x) - D'(2^{\kappa m}x)\right\|\right\} \\ &\leqslant \sum_{l=m+\frac{1-\kappa}{2}}^{\infty} 2^{-\kappa l}\left[2\mathscr{T}\left(\frac{2^{\kappa l}(n-1)}{n-2}x, \frac{2^{\kappa l}(1-n)}{n-2}x, 0, \ldots, 0\right)\right. \\ &\quad + \mathscr{T}\left(\frac{2^{\kappa l+1}(n-1)}{n-2}x, \frac{2^{\kappa l}(1-n)}{n-2}x, \frac{2^{\kappa l}(1-n)}{n-2}x, 0, \ldots, 0\right) \\ &\quad \left. + \frac{(3n-1)(1+\kappa)}{2n-4}\mathscr{T}(0, 0, \ldots, 0)\right],\end{aligned}$$

whence, letting $m \to \infty$ and using (12), we have $D(x) - D'(x) = 0$. Since this is true for all $x \in \mathscr{A}$, we obtain $D = D'$, as desired.

THEOREM 8. *Let $\mathscr{A}$ be a Banach algebra. Suppose a mapping $\mathscr{T} : \mathscr{A}^n \to [0, \infty)$ satisfies* (12) *and*

$$\lim_{m\to\infty} 2^{\frac{-m}{2}(1-n+\kappa+\kappa n)}\mathscr{T}(2^{\kappa m}x_1, 2^{\kappa m}x_2, \ldots, 2^{\kappa m}x_n) = 0 \tag{23}$$

for all $x_1, x_2, \ldots, x_n \in \mathscr{A}$. If a mapping $f : \mathscr{A} \to \mathscr{A}$ satisfies the inequalities

$$\begin{aligned}&\left\|\mu f\left(x_1 + \frac{\sum_{j=2}^{n} x_j}{n-1}\right) + \sum_{i=2}^{n} f\left(\mu x_i + \frac{\sum_{j=1, j\neq i}^{n} \mu x_j}{n-1}\right)\right\| \\ &\qquad \leqslant \left\|2f\left(\sum_{i=1}^{n} \mu x_i\right)\right\| + \mathscr{T}(x_1, x_2, \ldots, x_n),\end{aligned} \tag{24}$$

$$\left\| f\left(\prod_{i=1}^{n} x_i\right) - \sum_{i=1}^{n}\prod_{\imath=1}^{i-1} x_\imath f(x_i) \prod_{\imath=i+1}^{n} x_\imath \right\| \leqslant \mathscr{T}(x_1, x_2, \ldots, x_n) \tag{25}$$

for all $x_1, x_2, \ldots, x_n \in \mathscr{A}$ *and all* $\mu \in S$*, then there exists a unique* n*-derivation* $D\colon \mathscr{A} \to \mathscr{A}$ *satisfying* (13)*. Moreover, if* $\lim_{m\to\infty} 2^{-\kappa m}\mathscr{T}(2^{\kappa m}x_1, x_2, \ldots, x_n) = 0$*, then*

$$\sum_{i=2}^{n}\prod_{\imath=1}^{i-1} x_\imath (D(x_i) - f(x_i)) \prod_{\imath=i+1}^{n} x_\imath = 0 \tag{26}$$

for all $x_1, x_2, \ldots, x_n \in \mathscr{A}$*. Moreover, if* $\mathscr{A}$ *is semisimple factorizable, then* D *is continuous.*

Proof. Letting $\mu = 1$ in (8), we observe that f satisfies (10) with

$$\|f(0)\| \leqslant \frac{1+\kappa}{2n-4}\mathscr{T}(0,0,\ldots,0). \tag{27}$$

From Lemma 3, it follows that there exists a unique additive mapping $D\colon \mathscr{A} \to \mathscr{A}$ satisfying (13), where $D(x) := \lim_{m\to\infty} 2^{-\kappa m} f(2^{\kappa m}x)$ for all $x \in \mathscr{A}$.

Setting $x_1 = \frac{n-1}{n-2}x$, $x_2 = \frac{1-n}{n-2}x$ and $x_3 = \cdots = x_n = 0$ in (8), we obtain

$$\|\mu f(x) + f(-\mu x) + (n-2)f(0)\| \leqslant \|2f(0)\| + \mathscr{T}\left(\frac{n-1}{n-2}x, \frac{1-n}{n-2}x, 0, \ldots, 0\right),$$

which by setting $x = 2^{\kappa m}x$ and using (27) yields

$$\begin{aligned} &\left\|\mu 2^{-\kappa m} f(2^{\kappa m}x) + 2^{-\kappa m} f(-2^{\kappa m}\mu x)\right\| \\ &\quad\leqslant 2^{-\kappa m}\left[\mathscr{T}\left(\frac{2^{\kappa m}(n-1)}{n-2}x, \frac{2^{\kappa m}(1-n)}{n-2}x, 0, \ldots, 0\right) + \frac{n(1+\kappa)}{2n-4}\mathscr{T}(0,0,\ldots,0)\right] \end{aligned}$$

for all $m \in \mathbb{N}$, all $x \in \mathscr{A}$ and all $\mu \in S$. Allowing m tending to infinity and using the fact that D is additive, it is easy to see that

$$\mu D(x) - D(\mu x) = \mu D(x) + D(-\mu x) = 0$$

for all $x \in \mathscr{A}$ and all $\mu \in S$. Hence, $\mu D(x) = D(\mu x)$ for all $x \in \mathscr{A}$ and all $\mu \in S$. So by Lemma 2, the mapping $D\colon \mathscr{A} \to \mathscr{A}$ is $\mathbb{C}$-linear.

Also, we see from the inequality (25) that

$$\begin{aligned} 2^{-\kappa nm}&\left\| f\left(\prod_{i=1}^{n} 2^{\kappa nm}x_i\right) - \sum_{i=1}^{n}\prod_{\imath=1}^{i-1} 2^{\kappa m}x_\imath f(2^{\kappa m}x_i) \prod_{\imath=i+1}^{n} 2^{\kappa m}x_\imath \right\| \\ &\leqslant 2^{-\kappa nm}\mathscr{T}(2^{\kappa m}x_1, 2^{\kappa m}x_2, \ldots, 2^{\kappa m}x_n) \end{aligned}$$

whence, letting $m \to \infty$ and using (23), we observe that D satisfies (1). Therefore, $D\colon \mathscr{A} \to \mathscr{A}$ is a unique n-derivation satisfying (13).

Now, replacing $x_1 = 2^{\kappa m}x_1$ in (25), one finds

$$\left\| 2^{-\kappa m} f\left(2^{\kappa m}\prod_{i=1}^{n} x_i\right) - 2^{-\kappa m} f\left(2^{\kappa m}x_1\right)\prod_{i=2}^{n} x_i - \sum_{i=2}^{n}\prod_{\imath=1}^{i-1} x_\imath f(x_i)\prod_{\imath=i+1}^{n} x_\imath \right\| \leqslant 2^{-\kappa m}\mathscr{T}\left(2^{\kappa m}x_1, x_2, \ldots, x_n\right),$$

and since $2^{-\kappa m}\mathscr{T}\left(2^{\kappa m}x_1, x_2, \ldots, x_n\right) \to 0$ as $m \to \infty$, we have

$$D\left(\prod_{i=1}^{n} x_i\right) - D(x_1)\prod_{i=2}^{n} x_i - \sum_{i=2}^{n}\prod_{\imath=1}^{i-1} x_\imath f(x_i)\prod_{\imath=i+1}^{n} x_\imath = 0 \tag{28}$$

for all $x_1, x_2, \ldots, x_n \in \mathscr{A}$. Since D is a n-derivation, we get (26) from (28).

Furthermore, if $\mathscr{A}$ is semisimple factorizable, then Theorem 6 guarantees that D is continuous.

COROLLARY 4. *Let $\mathscr{A}$ be a Banach algebra. If a mapping $f\colon \mathscr{A} \to \mathscr{A}$ satisfies the inequality* (7) *and*

$$\left\| \mu f\left(x_1 + \frac{\sum_{j=2}^{n} x_j}{n-1}\right) + \sum_{i=2}^{n} f\left(\mu x_i + \frac{\sum_{j=1, j\neq i}^{n} \mu x_j}{n-1}\right) \right\| \leqslant \left\| 2f\left(\sum_{i=1}^{n} \mu x_i\right) \right\| + \varepsilon$$

for all $x_1, x_2, \ldots, x_n \in \mathscr{A}$ and all $\mu \in S$, then there exists a unique n-derivation $D\colon \mathscr{A} \to \mathscr{A}$ satisfying (26) *and*

$$\|f(x) - D(x)\| \leqslant \frac{6n-7}{n-2}\varepsilon \tag{29}$$

for all $x \in \mathscr{A}$. Moreover, if $\mathscr{A}$ is unital semisimple, then f is a continuous derivation.

Proof. It follows from Theorem 8 that there exists a unique n-derivation $D\colon \mathscr{A} \to \mathscr{A}$ satisfying (26) and (29), where $D(x) := \lim_{m\to\infty} 2^{-m} f(2^m x)$ for all $x \in \mathscr{A}$.

By letting $x_1 = x_2 = \cdots = x_n = e$ in (26), we get $(n-1)(D(e) - f(e)) = 0$, and so $f(e) = D(e)$. Next, by letting $x_1 = x_2 = \cdots = x_{n-1} = e$ and $x_n = x$ in (26), we obtain $f(x) = D(x)$ for all $x \in \mathscr{A}$. Thus f is an n-derivation. Since $\mathscr{A}$ is unital, we can conclude that f is an derivation, and since $\mathscr{A}$ is semisimple, f is a continuous.

THEOREM 9. *Let $\mathscr{A}$ be a semisimple factorizable Banach algebra. Suppose a mapping $\mathscr{T}\colon \mathscr{A}^n \to [0,\infty)$ satisfies the relations*

$$\mathscr{T}\left(\frac{n-1}{n-2}x, \frac{1-n}{n-2}x, 0, \ldots, 0\right) = \mathscr{T}\left(\frac{2n-2}{n-2}x, \frac{1-n}{n-2}x, \frac{1-n}{n-2}x, 0, \ldots, 0\right) = 0$$

and

$$\exists \kappa \in \{-1,1\}, \quad \lim_{m\to\infty} 2^{-\kappa m}\mathscr{T}\left(\frac{n-1}{n-2}2^{\kappa m}(x+y), \frac{1-n}{n-2}2^{\kappa m}x, \frac{1-n}{n-2}2^{\kappa m}y, 0, \ldots, 0\right) = 0 \tag{30}$$

for all $x, y \in \mathscr{A}$. If a mapping $f\colon \mathscr{A} \to \mathscr{A}$ satisfies (7) *and* (8)*, then f is continuous.*

Proof. Setting $x_1 = x_2 = \cdots = x_n = 0$ in (8) gives $f(0) = 0$. Putting $\mu = 1$ in (8) yields that f satisfies (10). Letting $x_1 = \frac{n-1}{n-2}x$, $x_2 = \frac{1-n}{n-2}x$ and $x_3 = \cdots = x_n = 0$ in (10), we see that $f(x) = -f(x)$ for all $x \in \mathscr{A}$. Putting $x_1 = \frac{2n-2}{n-2}x$, $x_2 = x_3 = \frac{1-n}{n-2}x$ and $x_4 = \cdots = x_n = 0$ in (10), we have $f(2x) = 2f(x)$ for all $x \in \mathscr{A}$. Thus $f(x) = 2^{-\kappa} f(2^{\kappa}x) = \cdots = 2^{-\kappa m} f(2^{\kappa m}x)$ for all $m \in \mathbb{N}$ and all $x \in \mathscr{A}$. Therefore, we can define $f(x) := \lim_{m\to\infty} 2^{-\kappa m} f(2^{\kappa m}x)$ for all $x \in \mathscr{A}$. It follows from (10) and (30) that

$$\begin{aligned}
&\|f(x+y) - f(x) - f(y)\| \\
&\quad = \lim_{m\to\infty} 2^{-\kappa m} \|f(2^{\kappa m}(x+y)) + f(-2^{\kappa m}x)) + f(-2^{\kappa m}y)\| \\
&\quad \leqslant \lim_{m\to\infty} 2^{-\kappa m} \mathscr{T}\left(\frac{n-1}{n-2}2^{\kappa m}(x+y), \frac{1-n}{n-2}2^{\kappa m}x, \frac{1-n}{n-2}2^{\kappa m}y, 0, \ldots, 0\right) \\
&\quad = 0,
\end{aligned}$$

and so $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathscr{A}$.

Setting $x_1 = \frac{n-1}{n-2}x$, $x_2 = \frac{1-n}{n-2}x$ and $x_3 = \cdots = x_n = 0$ in (8), we obtain $\mu f(x) + f(-\mu x) = \mu f(x) - f(\mu x) = 0$, and so $\mu f(x) = f(\mu x)$ for all $x \in \mathscr{A}$ and all $\mu \in S$. Thus by Lemma 2, the mapping $f\colon \mathscr{A} \to \mathscr{A}$ is $\mathbb{C}$-linear.

We now consider the cases according to whether $\kappa = 1$ or $\kappa = -1$. First suppose $\kappa = 1$. Using (7), we can state

$$\lim_{m\to\infty} 2^{-nm} \left\| f\left(\prod_{i=1}^{n} 2^{nm} x_i\right) - \sum_{i=1}^{n} \prod_{\iota=1}^{i-1} 2^m x_\iota f(2^m x_i) \prod_{\iota=i+1}^{n} 2^m x_\iota \right\| \leqslant \lim_{m\to\infty} 2^{-nm}\varepsilon = 0.$$

Hence, f satisfies (1). Therefore, $f\colon \mathscr{A} \to \mathscr{A}$ is an n-derivation, and thus f is continuous. Now assume $\kappa = -1$. Here, we deduce that f is an approximate n-derivation. Since $\mathscr{A}$ is semisimple factorizable, Theorem 6 guarantees that f is continuous.

As a consequence, we have the following result.

COROLLARY 5. *Let $\mathscr{A}$ be a semisimple factorizable Banach algebra. If a mapping $f\colon \mathscr{A} \to \mathscr{A}$ satisfies the inequality* (7) *and*

$$\left\|\mu f\left(x_1 + \frac{\sum_{j=2}^{n} x_j}{n-1}\right) + \sum_{i=2}^{n} f\left(\mu x_i + \frac{\sum_{j=1, j\neq i}^{n} \mu x_j}{n-1}\right)\right\| \leqslant \left\|2f\left(\sum_{i=1}^{n} \mu x_i\right)\right\|$$

for all $x_1, x_2, \ldots, x_n \in \mathscr{A}$ and all $\mu \in S$, then f is continuous.

From Theorem 7, we can deduce the following result.

COROLLARY 6. *If, under the conditions of Theorem 9 (or Corollary 5), we assume in addition $\mathscr{A}$ is commutative, then f is identically zero.*

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