Mathematical
Inequalities
& Applications
Volume 24, Number 1 (2021), 1–11 doi:10.7153/mia-2021-24-01

PROOFS OF CONJECTURES OF ELEZOVIĆ AND VUKŠIĆ CONCERNING THE INEQUALITIES FOR MEANS

XUE-FENG HAN* AND CHAO-PING CHEN

(*Communicated by N. Elezović*)

Abstract. By using the asymptotic expansion method, Elezović and Vukšić conjectured certain inequalities related to Neuman-Sándor mean. The aim of this paper is to offer a proof of these inequalities.

1. Introduction

For $x,y>0$ with $x \neq y$, the Neuman-Sándor mean $M(x,y)$ was introduced in [12, 13] by

$$M(x,y) = \frac{x-y}{2\operatorname{arcsinh}(\frac{x-y}{x+y})}.$$

Let

$$H = \frac{2xy}{x+y}, \quad G = \sqrt{xy}, \quad L = \frac{x-y}{\ln x - \ln y}, \quad A = \frac{x+y}{2},$$
$$C = \frac{2}{3} \cdot \frac{x^2+xy+y^2}{x+y}, \quad Q = \sqrt{\frac{x^2+y^2}{2}}, \quad N = \frac{x^2+y^2}{x+y}$$

be the harmonic, geometric, logarithmic, arithmetic, centroidal, root-square, and contraharmonic means of two unequal and positive numbers x and y, respectively. It is known that

$$H < G < L < A < M < C < Q < N.$$

There is a large number of papers studying inequalities between Neuman-Sándor mean and convex combinations of other means. For example, Neuman [11] proved that the double inequalities

$$\xi_1 Q + (1-\xi_1)A < M < \eta_1 Q + (1-\eta_1)A \tag{1.1}$$

and

$$\xi_2 N + (1-\xi_2)A < M < \eta_2 N + (1-\eta_2)A \tag{1.2}$$

Mathematics subject classification (2010): 26E60.
Keywords and phrases: Neuman-Sándor mean, inequality, optimal convex combination.
* Corresponding author.

Paper MIA-24-01

hold if and only if

$$\xi_1 \leqslant \frac{1-\ln(1+\sqrt{2})}{(\sqrt{2}-1)\ln(1+\sqrt{2})}, \quad \eta_1 \geqslant \frac{1}{3}, \quad \xi_2 \leqslant \frac{1-\ln(1+\sqrt{2})}{\ln(1+\sqrt{2})}, \quad \eta_2 \geqslant \frac{1}{6}.$$

Zhao et al. [21] proved that the double inequalities

$$\mu_1 H + (1-\mu_1)Q < M < \nu_1 H + (1-\nu_1)Q, \tag{1.3}$$

$$\mu_2 G + (1-\mu_2)Q < M < \nu_2 G + (1-\nu_2)Q, \tag{1.4}$$

$$\mu_3 H + (1-\mu_3)N < M < \nu_3 H + (1-\nu_3)N \tag{1.5}$$

hold if and only if

$$\mu_1 \geqslant \frac{2}{9}, \quad \nu_1 \leqslant 1 - \frac{1}{\sqrt{2}\ln(1+\sqrt{2})},$$

$$\mu_2 \geqslant \frac{1}{3}, \quad \nu_2 \leqslant 1 - \frac{1}{\sqrt{2}\ln(1+\sqrt{2})},$$

$$\mu_3 \geqslant 1 - \frac{1}{2\ln(1+\sqrt{2})}, \quad \nu_3 \leqslant \frac{5}{12}.$$

Xia and Chu [18] proved that the double inequality

$$\alpha_1 C + (1-\alpha_1)H < M < \beta_1 C + (1-\beta_1)H \tag{1.6}$$

holds if and only if

$$\alpha_1 \leqslant \frac{3}{4\ln(1+\sqrt{5})} \quad \text{and} \quad \beta_1 \geqslant \frac{7}{8}.$$

Qian and Chu [15] proved that the double inequality

$$\alpha_2 C + (1-\alpha_2)A < M < \beta_2 C + (1-\beta_2)A \tag{1.7}$$

holds if and only if

$$\alpha_2 \leqslant \frac{3-3\ln(1+\sqrt{2})}{\ln(1+\sqrt{2})} \quad \text{and} \quad \beta_2 \geqslant \frac{1}{2}.$$

For other similar results see [4, 5, 10, 14, 16, 19, 20, 22].

Recently, Elezović and Vukšić [7], by using the asymptotic expansion method, gave a systematic study of inequalities of the form

$$(1-\mu)M_1 + \mu M_3 < M_2 < (1-\nu)M_1 + \nu M_3$$

which apart from Neuman-Sándor mean also contains two classical means from the list given at the beginnig of this section. For example, Elezović and Vukšić [7] proved the double inequality

$$(1-\mu)M + \mu N < C < (1-\nu)M + \nu N \tag{1.8}$$

holds if and only if

$$\mu \leqslant \frac{1}{5} \quad \text{and} \quad \nu \geqslant \frac{4\sigma - 3}{6\sigma - 3},$$

where

$$\sigma = \operatorname{arcsinh}(1) = \ln(1+\sqrt{2}). \tag{1.9}$$

In what follows, σ denotes the constant given in (1.9). See [2, 6, 8, 9, 17] for more details about comparison of means using asymptotic methods.

The following inequalities related to Neuman-Sándor mean $M(x,y)$, with the best possible constants, have been conjectured by Elezović and Vukšić [7]:

$$H < G < \frac{4}{7}H + \frac{3}{7}M, \tag{1.10}$$

$$H < L < \frac{3}{7}H + \frac{4}{7}M, \tag{1.11}$$

$$\frac{1}{4}G + \frac{3}{4}M < A < (1-\sigma)G + \sigma M, \tag{1.12}$$

$$\frac{1}{3}L + \frac{2}{3}M < A < (1-\sigma)L + \sigma M, \tag{1.13}$$

$$\frac{2}{5}L + \frac{3}{5}Q < M < \frac{\sqrt{2}\sigma - 1}{\sqrt{2}\sigma}L + \frac{1}{\sqrt{2}\sigma}Q, \tag{1.14}$$

$$\frac{5}{8}L + \frac{3}{8}N < M < \frac{2\sigma - 1}{2\sigma}L + \frac{1}{2\sigma}N, \tag{1.15}$$

$$\frac{1}{2}M + \frac{1}{2}Q < C < \frac{(3\sqrt{2}-4)\sigma}{3\sqrt{2}\sigma - 3}M + \frac{3-4\sigma}{3-3\sqrt{2}\sigma}Q. \tag{1.16}$$

In fact, (1.14) and (1.15) have been proved in [3]. The aim of this paper is to offer a proof of inequalities (1.10)–(1.13), and (1.16).

REMARK 1.1. Let $(x-y)/(x+y) = z$, and suppose $x > y$. Then $z \in (0,1)$, and the following identities hold true:

$$\frac{H(x,y)}{A(x,y)} = 1 - z^2, \quad \frac{G(x,y)}{A(x,y)} = \sqrt{1-z^2}, \quad \frac{L(x,y)}{A(x,y)} = \frac{2z}{\ln\frac{1+z}{1-z}}, \quad \frac{M(x,y)}{A(x,y)} = \frac{z}{\operatorname{arcsinh} z},$$

$$\frac{C(x,y)}{A(x,y)} = 1 + \frac{1}{3}z^2, \quad \frac{Q(x,y)}{A(x,y)} = \sqrt{1+z^2}, \quad \frac{N(x,y)}{A(x,y)} = 1 + z^2.$$

The following inequalities are required in our present investigation.

$$\ln\frac{1+z}{1-z} > 2\sum_{j=1}^{n} \frac{z^{2j-1}}{2j-1} \tag{1.17}$$

and

$$\sum_{j=0}^{2m-1}(-1)^j\frac{(2j-1)!!}{(2j)!!}\frac{z^{2j}}{2j+1}<\frac{\operatorname{arcsinh}z}{z}<\sum_{j=0}^{2m}(-1)^j\frac{(2j-1)!!}{(2j)!!}\frac{z^{2j}}{2j+1} \tag{1.18}$$

for $0<z<1$ and $m\in\mathbb{N}:=\{1,2,\ldots\}$. Here, we employ the special double factorial notation as follows:

$$\begin{aligned}
&(2n)!!=2\cdot4\cdot6\cdots(2n)=2^n n!=2^n\Gamma(n+1),\\
&(2n-1)!!=1\cdot3\cdot5\cdots(2n-1)=\pi^{-1/2}2^n\Gamma\left(n+\frac{1}{2}\right),\\
&0!!=1, \qquad (-1)!!=1
\end{aligned}$$

(see [1, p. 258]).

The numerical values given in this paper have been calculated via the computer program MAPLE 13.

2. Proofs of inequalities (1.10)–(1.13), and (1.16)

First of all, we give a proof of (1.18). It is known (see [1, p. 88]) that

$$\frac{\operatorname{arcsinh}z}{z}=\sum_{n=0}^{\infty}(-1)^n u_n(z), \quad 0<z<1,$$

where

$$u_n(z)=\frac{(2n-1)!!}{(2n)!!}\frac{z^{2n}}{2n+1}=\frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)}\frac{z^{2n}}{2n+1}.$$

Elementary calculations reveal that for $0<z<1$ and $n\geqslant 1$,

$$\frac{u_{n+1}(z)}{u_n(z)}=\frac{(2n+1)^2z^2}{(2n+2)(2n+3)}<\frac{(2n+1)^2}{(2n+2)(2n+3)}<1.$$

Hence, for every $z\in(0,1)$, the sequence $(u_n(z))_{n\geqslant1}$ is strictly decreasing. We then obtain

$$\sum_{j=0}^{2m-1}(-1)^j u_j(z)<\frac{\operatorname{arcsinh}z}{z}<\sum_{j=0}^{2m}(-1)^j u_j(z)$$

for $0<z<1$ and $m\in\mathbb{N}:=\{1,2,\ldots\}$. This proves (1.18).

We now prove inequalities (1.10)–(1.13), and (1.16).

THEOREM 2.1. *The inequalities*

$$(1-\lambda_1)H+\lambda_1 M<G<(1-\omega_1)H+\omega_1 M \tag{2.1}$$

hold if and only if

$$\lambda_1\leqslant 0 \quad \text{and} \quad \omega_1\geqslant\frac{3}{7}. \tag{2.2}$$

Proof. Clearly, the left-hand inequality of (2.1) holds for $\lambda_1 = 0$. We now prove the right-hand inequality of (2.1) with $\omega_1 = \frac{3}{7}$,

$$G < \frac{4}{7}H + \frac{3}{7}M, \tag{2.3}$$

which may be rewritten by Remark 1.1 as

$$\sqrt{1-z^2} < \frac{4}{7}(1-z^2) + \frac{3}{7}\frac{z}{\operatorname{arcsinh} z}, \qquad 0 < z < 1.$$

Using the right-hand inequality of (1.18) with $m = 1$ and

$$\sqrt{1-z^2} < 1 - \frac{1}{2}z^2 - \frac{1}{8}z^4, \qquad 0 < z < 1,$$

we find that for $0 < z < 1$,

$$\begin{aligned} 4(1-z^2) + \frac{3z}{\operatorname{arcsinh} z} - 7\sqrt{1-z^2} &> 4(1-z^2) + \frac{3}{1 - \frac{1}{6}z^2 + \frac{3}{40}z^4} - 7\left(1 - \frac{1}{2}z^2 - \frac{1}{8}z^4\right) \\ &= \frac{z^4(704 - 176z^2 + 63z^4)}{8(120 - 20z^2 + 9z^4)} > 0. \end{aligned}$$

Hence, (2.3) holds.

Conversely, if (2.1) is valid for some λ_1 and ω_1, then

$$\lambda_1 < \frac{\sqrt{1-z^2} - (1-z^2)}{\frac{z}{\operatorname{arcsinh} z} - (1-z^2)} < \omega_1.$$

The limit relations

$$\lim_{z\to 0^+} \frac{\sqrt{1-z^2} - (1-z^2)}{\frac{z}{\operatorname{arcsinh} z} - (1-z^2)} = \frac{3}{7} \quad \text{and} \quad \lim_{z\to 1^-} \frac{\sqrt{1-z^2} - (1-z^2)}{\frac{z}{\operatorname{arcsinh} z} - (1-z^2)} = 0$$

yield

$$\lambda_1 \leqslant 0 \quad \text{and} \quad \omega_1 \geqslant \frac{3}{7}.$$

The proof is complete.

THEOREM 2.2. *The inequalities*

$$(1-\lambda_2)H + \lambda_2 M < L < (1-\omega_2)H + \omega_2 M \tag{2.4}$$

hold if and only if

$$\lambda_2 \leqslant 0 \quad \textit{and} \quad \omega_2 \geqslant \frac{4}{7}. \tag{2.5}$$

Proof. Clearly, the left-hand inequality of (2.4) holds for $\lambda_2 = 0$. We now prove the right-hand inequality of (2.4) with $\omega_2 = \frac{4}{7}$,

$$L < \frac{3}{7}H + \frac{4}{7}M, \tag{2.6}$$

which may be rewritten by Remark 1.1 as

$$\frac{2z}{\ln\frac{1+z}{1-z}} < \frac{3}{7}(1-z^2) + \frac{4}{7}\frac{z}{\operatorname{arcsinh} z}, \qquad 0 < z < 1.$$

Using the right-hand inequality of (1.18) with $m = 1$ and inequality (1.17) with $n = 4$, we find that for $0 < z < 1$,

$$\begin{aligned} &3(1-z^2) + \frac{4z}{\operatorname{arcsinh} z} - \frac{14z}{\ln\left(\frac{1+z}{1-z}\right)} \\ &\quad > 3(1-z^2) + \frac{4}{1-\frac{1}{6}z^2+\frac{3}{40}z^4} - \frac{14z}{2z+\frac{2}{3}z^3+\frac{2}{5}z^5+\frac{2}{7}z^7} \\ &\quad = \frac{3z^4\Big((1820-1806z^4)+1330z^2+x^6(246-135z^2)\Big)}{(120-20z^2+9z^4)(105+35z^2+21z^4+15z^6)} > 0. \end{aligned}$$

Hence, (2.6) holds.

Conversely, if (2.4) is valid for some λ_2 and ω_2, then

$$\lambda_2 < \frac{\frac{2z}{\ln\frac{1+z}{1-z}} - (1-z^2)}{\frac{z}{\operatorname{arcsinh} z} - (1-z^2)} < \omega_2.$$

The limit relations

$$\lim_{z\to 0^+} \frac{\frac{2z}{\ln\frac{1+z}{1-z}} - (1-z^2)}{\frac{z}{\operatorname{arcsinh} z} - (1-z^2)} = \frac{4}{7} \quad \text{and} \quad \lim_{z\to 1^-} \frac{\frac{2z}{\ln\frac{1+z}{1-z}} - (1-z^2)}{\frac{z}{\operatorname{arcsinh} z} - (1-z^2)} = 0$$

yield

$$\lambda_2 \leqslant 0 \quad \text{and} \quad \omega_2 \geqslant \frac{4}{7}.$$

The proof is complete.

THEOREM 2.3. *The inequalities*

$$(1-\lambda_3)G + \lambda_3 M < A < (1-\omega_3)G + \omega_3 M \tag{2.7}$$

hold if and only if

$$\lambda_3 \leqslant \frac{3}{4} \quad \textit{and} \quad \omega_3 \geqslant \sigma. \tag{2.8}$$

Proof. By Remark 1.1, (2.7) may be rewritten for $0<z<1$ as

$$\lambda_3 < J_1(z) < \omega_3,$$

where

$$J_1(z) = \frac{1-\sqrt{1-z^2}}{\frac{z}{\operatorname{arcsinh} z} - \sqrt{1-z^2}}.$$

Elementary calculations reveal that

$$\lim_{z\to 0^+} J_1(z) = \frac{3}{4} \quad \text{and} \quad J_1(1) = \sigma.$$

In order to prove Theorem 2.3, it suffices to show that $J_1(z)$ is strictly increasing for $0<z<1$.

Differentiation yields

$$(z-\sqrt{1-z^2}\operatorname{arcsinh} z)^2\sqrt{1-z^4}J_1'(z) = U_1(z), \tag{2.9}$$

where

$$U_1(z) = \operatorname{arcsinh} z\cdot\sqrt{1+z^2}(1-\sqrt{1-z^2}) - (\operatorname{arcsinh} z)^2 z\sqrt{1+z^2}$$
$$+z\big(\sqrt{1-z^2}-(1-z^2)\big).$$

We now prove $U_1(z)>0$ for $0<z<1$. By an elementary change of variable $z=\sinh x$ $(0<x<\sigma)$, it suffices to show that

$$U_2(x)>0, \qquad 0<x<\sigma,$$

where

$$U_2(x) = x\cosh x(1-\sqrt{1-\sinh^2 x}) - x^2\sinh x\cosh x$$
$$+\sinh x\Big(\sqrt{1-\sinh^2 x}-(1-\sinh^2 x)\Big).$$

We find, for $0<x<\sigma$,

$$\begin{aligned}
U_2(x) &= x\cosh x - (x\cosh x-\sinh x)\sqrt{1-\sinh^2 x} - \frac{1}{2}x^2\sinh(2x) - \sinh x + \sinh^3 x\\
&> x\cosh x - (x\cosh x-\sinh x)\left(1-\frac{1}{2}\sinh^2 x\right) - \frac{1}{2}x^2\sinh(2x) - \sinh x + \sinh^3 x\\
&= \frac{1}{2}\sinh x\left(\sinh^2 x - 2x^2\cosh x + \frac{x}{2}\sinh(2x)\right)\\
&= \frac{1}{2}\sinh x\sum_{n=3}^{\infty}\frac{(n+1)4^n - 8n(2n-1)}{2\cdot(2n)!}x^{2n} > 0.
\end{aligned}$$

We then obtain that for $0<z<1$,

$$U_1(z)>0 \quad \text{and} \quad J_1'(z)>0.$$

Hence, $J_1(z)$ is strictly increasing for $0<z<1$. The proof is complete.

THEOREM 2.4. *The inequalities*

$$(1-\lambda_4)L+\lambda_4 M < A < (1-\omega_4)L+\omega_4 M. \tag{2.10}$$

hold if and only if

$$\lambda_4 \leqslant \frac{2}{3} \quad \textit{and} \quad \omega_4 \geqslant \sigma. \tag{2.11}$$

Proof. We first prove (2.10) with $\lambda_4 = \frac{2}{3}$ and $\omega_4 = \sigma$,

$$\frac{1}{3}L+\frac{2}{3}M < A < (1-\sigma)L+\sigma M. \tag{2.12}$$

Clearly, the right-hand side of (2.7) (with $\omega_3 = \sigma$) is sharper than the right-hand side of (2.12).

By Remark 1.1, the left-hand inequality of (2.12) may be rewritten as

$$\frac{1}{3}\frac{2z}{\ln\frac{1+z}{1-z}}+\frac{2}{3}\frac{z}{\operatorname{arcsinh} z} < 1, \qquad 0<z<1. \tag{2.13}$$

Using inequality (1.17) with $n=3$ and the left-hand inequality of (1.18) with $m=2$, we find that for $0<z<1$,

$$\begin{aligned} 3-\frac{2z}{\ln\frac{1+z}{1-z}}-\frac{2z}{\operatorname{arcsinh} z} &> 3-\frac{2z}{2z+\frac{2}{3}z^3+\frac{2}{5}z^5}-\frac{2}{1-\frac{1}{6}z^2+\frac{3}{40}z^4-\frac{5}{112}z^6} \\ &= \frac{3z^4(1540-960z^2-225z^6+3z^4)}{(15+5z^2+3z^4)(1680-280z^2+126z^4-75z^6)} > 0. \end{aligned}$$

Thus, the inequality (2.13) is true for $0<z<1$.

We then obtain (2.10) with $\lambda_4 = \frac{2}{3}$ and $\omega_4 = \sigma$.

Conversely, if (2.10) is valid for some λ_4 and ω_4, then

$$\lambda_4 < \frac{1-\frac{2z}{\ln\frac{1+z}{1-z}}}{\frac{z}{\operatorname{arcsinh} z}-\frac{2z}{\ln\frac{1+z}{1-z}}} < \omega_4, \qquad 0<z<1.$$

The limit relations

$$\lim_{z\to 0^+}\frac{1-\frac{2z}{\ln\frac{1+z}{1-z}}}{\frac{z}{\operatorname{arcsinh} z}-\frac{2z}{\ln\frac{1+z}{1-z}}} = \frac{2}{3} \quad \text{and} \quad \lim_{z\to 1^-}\frac{1-\frac{2z}{\ln\frac{1+z}{1-z}}}{\frac{z}{\operatorname{arcsinh} z}-\frac{2z}{\ln\frac{1+z}{1-z}}} = \sigma$$

yield

$$\lambda_4 \leqslant \frac{2}{3} \quad \text{and} \quad \omega_4 \geqslant \sigma.$$

The proof is complete.

THEOREM 2.5. *The inequalities*

$$(1-\lambda_5)M+\lambda_5 Q < C < (1-\omega_5)M+\omega_5 Q \tag{2.14}$$

hold if and only if

$$\lambda_5 \leqslant \frac{1}{2} \quad and \quad \omega_5 \geqslant \frac{4\sigma-3}{3\sqrt{2}\sigma-3}. \tag{2.15}$$

Proof. By Remark 1.1, (2.14) may be rewritten as

$$\lambda_5 < \frac{1+\frac{1}{3}z^2-\frac{z}{\operatorname{arcsinh} z}}{\sqrt{1+z^2}-\frac{z}{\operatorname{arcsinh} z}} < \omega_5, \qquad 0<z<1. \tag{2.16}$$

By an elementary change of variable $z=\sinh x$ $(0<x<\sigma)$, (2.16) becomes

$$\lambda_5 < G(x) < \omega_5, \qquad 0<x<\sigma,$$

where

$$G(x)=\frac{1+\frac{1}{3}\sinh^2 x-\frac{\sinh x}{x}}{\cosh x-\frac{\sinh x}{x}}.$$

Differentiation yields

$$\begin{aligned}
&3\big(x\cosh x-\sinh x\big)^2 G'(x)\\
&=\frac{3}{2}\sinh(2x)+\frac{1}{2}\left(x^2-1\right)\sinh(2x)\cosh x-(2x^2+2)\sinh x+4x\cosh x-3x-x\cosh^3 x\\
&=\frac{1}{4}(x^2-1)\sinh(3x)+\frac{3}{2}\sinh(2x)-\frac{1}{4}(7x^2+9)\sinh x-\frac{x}{4}\cosh(3x)+\frac{13x}{4}\cosh x-3x\\
&=\sum_{n=3}^{\infty}\frac{(n^2-n-3)9^n+9\cdot 4^n-21n^2+9n+3}{3\cdot(2n+1)!}x^{2n+1}>0.
\end{aligned}$$

Hence, $G(x)$ is strictly increasing for $0<x<\sigma$, and we have

$$\frac{1}{2}=\lim_{t\to 0^+}G(t)<G(x)<\lim_{t\to\sigma^-}G(t)=\frac{4\sigma-3}{3\sqrt{2}\sigma-3}, \qquad 0<x<\sigma.$$

Hence, (2.14) holds if and only if $\lambda_5\leqslant\frac{1}{2}$ and $\omega_5\geqslant\frac{4\sigma-3}{3\sqrt{2}\sigma-3}$. The proof is complete.

REMARK 2.1. Finally, we provide an alternative proof of (1.8). By Remark 1.1, (1.8) may be rewritten as

$$\mu < \frac{1+\frac{1}{3}z^2-\frac{z}{\operatorname{arcsinh} z}}{1+z^2-\frac{z}{\operatorname{arcsinh} z}} < \nu, \qquad 0<z<1. \tag{2.17}$$

By an elementary change of variable $z = \sinh x$ $(0 < x < \sigma)$, (2.17) becomes

$$\mu < F(x) < \nu, \quad \text{where} \quad F(x) = \frac{1 + \frac{1}{3}\sinh^2 x - \frac{\sinh x}{x}}{\cosh^2 x - \frac{\sinh x}{x}}, \quad 0 < x < \sigma.$$

Differentiation yields, for $0 < x < \sigma$,

$$\begin{aligned}\frac{3\left(x\cosh x - \sinh x\right)^2}{2\sinh x}F'(x) &= \sinh^2 x - 2x^2\cosh x + \frac{1}{2}x\sinh(2x)\\ &= \sum_{n=3}^{\infty}\frac{(n+1)4^n - 8n(2n-1)}{2\cdot(2n)!}x^{2n} > 0.\end{aligned}$$

So, $F(x)$ is strictly increasing for $0 < x < \sigma$, and we have

$$\frac{1}{5} = \lim_{t\to 0^+} F(t) < F(x) < \lim_{t\to\sigma^-} F(t) = \frac{4\sigma - 3}{6\sigma - 3}, \qquad 0 < x < \sigma.$$

Hence, (1.8) holds if and only if $\mu \leqslant \frac{1}{5}$ and $\nu \geqslant \frac{4\sigma-3}{6\sigma-3}$.

Acknowledgements. The authors would thank anonymous referees for their useful remarks which improve presentation of this paper. This work was supported by Key Science Research Project in Universities of Henan (20B110007).

REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, 9th printing, Washington, 1970.

[2] T. BURIĆ AND N. ELEZOVIĆ, *Asymptotic expansion of the arithmetic-geometric mean and related inequalities*, J. Math. Inequal. **9**, 4 (2015), 1181–1190.

[3] Y. M. CHU, T. H. ZHAO AND B. Y. LIU, *Optimal bounds for Neuman-Sándor mean in terms of the convex combination of logarithmic and quadratic or contra-harmonic means*, J. Math. Inequal. **8**, 2 (2014), 201–217.

[4] Y. M. CHU, T. H. ZHAO AND Y. Q. SONG, *Sharp bounds for Neuman-Sándor mean in terms of the convex combination of quadratic and first Seiffert means*, Acta Mathematica Scientia **34B**, 3 (2014), 797–806.

[5] H. C. CUI, N. WANG AND B.-Y. LONG, *Optimal Bounds for the Neuman-Sándor Mean in terms of the Convex Combination of the First and Second Seiffert Means*, Math. Probl. Eng. **2015**, Article ID 489490, 6 pages.

[6] N. ELEZOVIĆ, *Asymptotic inequalities and comparison of classical means*, J. Math. Inequal. **9**, 1 (2015), 177–196.

[7] N. ELEZOVIĆ AND L. VUKŠIĆ, *Neuman-Sándor mean, asymptotic expansions and related inequalities*, J. Math. Inequal. **9**, 4 (2015), 1337–1348.

[8] N. ELEZOVIĆ AND L. VUKŠIĆ, *Asymptotic expansions of bivariate classical means and related inequalities*, J. Math. Inequal. **8**, 4 (2014), 707–724.

[9] N. ELEZOVIĆ AND L. VUKŠIĆ, *Asymptotic expansions and comparison of bivariate parameter means*, Math. Inequal. Appl. **17**, 4 (2014), 1225–1244.

[10] W.-M. GONG, X.-H. SHEN AND Y.-M. CHU, *Bounds for the Neuman-Sándor mean in terms of logarithmic, quadratic or contraharmonic means*, Int. Math. Forum, **8**, 30 (2013), 1467–1475.

[11] E. NEUMAN, *A note on certain bivariate mean*, J. Math. Inequal. **6**, 4 (2012), 637–643.

[12] E. NEUMAN AND J. SÁNDOR, *On the Schwab-Borchardt mean*, Math. Pannon. **14**, 2 (2003), 253–266.
[13] E. NEUMAN AND J. SÁNDOR, *On the Schwab-Borchardt mean, II*, Math. Pannon. **17**, 1 (2006), 49–59.
[14] F. QI AND W. H. LI, *A unified proof of several inequalities and some new inequalities involving Neuman-Sándor mean*, Miskolc Math. Notes, **15**, 2 (2014), 665–675.
[15] W.-M. QIAN AND Y.-M. CHU, *On certain inequalities for Neuman-Sándor mean*, Abstr. Appl. Anal. **2013**, Article ID 790783, 6 pages.
[16] H. SUN , X.-H. SHEN, T.-H. ZHAO AND Y.-M. CHU, *Optimal bounds for the Neuman-Sándor means in terms of geometric and contraharmonic means*, Appl. Math. Sci. **7**, 88 (2013), 4363–4373.
[17] L. VUKŠIĆ, *Seiffert means, asymptotic expansions and inequalities*, Rad Hrvat. Akad. Znan. Umjet. Mat. Znan. **19** (2015), 129–142.
[18] W. F. XIA AND Y. M. CHU, *Optimal inequalities between Neuman-Sándor, centroidal and harmonic means*, J. Math. Inequal. **7**, 4 (2013), 593–600.
[19] F. ZHANG, Y.-M. CHU AND W.-M. QIAN, *Bounds for the arithmetic mean in terms of the Neuman-Sándor and other bivariate means*, J. Appl. Math. **2013**, Article ID 582504, 7 pages.
[20] T.-H. ZHAO AND Y.-M. CHU, *A sharp double inequality involving identric, Neuman-Sándor, and quadratic means*, Scientia Sinica Mathematica, **43**, 6 (2013), 551–562, http://dx.doi.org/10.1360/012013-128
[21] T.-H. ZHAO, Y.-M. CHU AND B.-Y. LIU, *Optimal bounds for Neuman-Sándor mean in terms of the convex combinations of harmonic, geometric, quadratic, and contraharmonic means*, Abstr. Appl. Anal. **2012**, Article ID 302635, 9 pages.
[22] T.-H. ZHAO, Y.-M. CHU, Y.-L. JIANG AND Y.-M. LI, *Best possible bounds for Neuman-Sándor mean by the identric, quadratic and contraharmonic means*, Abstr. Appl. Anal. **2013**, Article ID 348326, 12 pages.

(Received April 30, 2017)

Xue-Feng Han
School of Mathematics and Informatics
Henan Polytechnic University
Jiaozuo City 454000, Henan Province, China
e-mail: hanxuefeng8110@sohu.com

Chao-Ping Chen
School of Mathematics and Informatics
Henan Polytechnic University
Jiaozuo City 454000, Henan Province, China
e-mail: chenchaoping@sohu.com