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GENERALIZED CSISZÁR'S f–DIVERGENCE FOR LIPSCHITZIAN FUNCTIONS

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Abstract. We started with the generalization of the Csiszár's f-divergence. We stated and proved Jensen's type inequality for L-Lipschitzian functions. The results for commonly used examples of f-divergences, such as the Kullbach-Leibler divergence, the Hellinger divergence, the Rényi divergence and χ^2-distance are derived. Further, we examined two specific averaging functions, previously known in the literature. Finally, we obtained interesting results concerning the Zipf-Mandelbrot law.

1. Introduction

For a function $f\colon \mathbb{R}_+ \to \mathbb{R}$ and $\mathbf{p} = (p_1,\ldots,p_n) \in \mathbb{R}^n_+$, $\mathbf{q} = (q_1,\ldots,q_n) \in \mathbb{R}^n_+$, I. Csiszár in [6] introduced the f-divergence functional by

$$C_f(\mathbf{q},\mathbf{p}) = \sum_{i=1}^{n} p_i f\left(\frac{q_i}{p_i}\right), \tag{1}$$

with undefined expressions interpreted as follows

$$f(0) = \lim_{t\to 0_+} f(t), \quad 0f\left(\frac{0}{0}\right) = 0$$

$$0f\left(\frac{a}{0}\right) = a \lim_{t\to\infty} \frac{f(t)}{t}, \quad a > 0.$$

I. Csiszár studied (1) under assumption that function f is convex. Independently, Morimoto [16] and Ali and Silvey [1] also introduced and studied these divergences. Still, (1) is widely known as Csiszár f-divergence. These divergences are well known in probability theory, in information theory, in statistical physics, economics, biology, etc.

In probability theory, an f-divergence is a function $Df(P \parallel Q)$ that measures the difference between two probability distributions P and Q. Intuitively, the divergence is an average, weighted by the function f, of the odds ratio given by P and Q.

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There are lots of articles on that subject, both recent and older such as [4], [5], [9], [10], [11], [14] and [15]. We are following the idea of Y. J. Cho, M. Matić, and J. Pečarić [3], but in discrete case and additionally generalized. In that way we get Jensen's type inequalities for Lipschitzian functions in terms of generalized Csiszár's functional. As usual, we go through some of the most frequent applications of f-divergences. Namely, we state Jensen's type inequality involving the Kullbach-Leibler divergence, the Hellinger divergence, the Rényi divergence and χ^2-divergence, all generalized.

2. Jensen's type inequalities for generalized f-divergence

For a function $f\colon \mathbb{R}_+ \to \mathbb{R}$ and $\mathbf{p},\mathbf{q},\mathbf{r} \in \mathbb{R}_+^n$, the generalized Csiszár f-divergence is defined by

$$C_f(\mathbf{q},\mathbf{p};\mathbf{r}) = \sum_{i=1}^n r_i p_i f\left(\frac{q_i}{p_i}\right). \tag{2}$$

We recall that a real-valued function $f\colon [a,b] \subset \mathbb{R} \to \mathbb{R}$ is called Lipschitz continuous if there exists a positive real constant L such that, for all $x_1, x_2 \in \mathbb{R}$

$$|f(x_1) - f(x_2)| \leqslant L|x_1 - x_2|$$

holds. Shortly, we call those functions L-Lipschitzian or just Lipschitzian.

We introduce notations

$$P_r = \sum_{i=1}^n r_i p_i, \tag{3}$$

$$\overline{Q}_r = \frac{1}{P_r}\sum_{i=1}^n r_i q_i \tag{4}$$

and get some new inequalities of Jensen's type. Jensen's inequality appears in many forms depending on the context. In it's finite discrete form, it is defined for real convex function f, positive $\mathbf{p} \in \mathbb{R}_+^{\mathbf{n}}$ and $\mathbf{q} \in \mathbb{R}^{\mathbf{n}}$ such as follows

$$f\left(\frac{\sum_{i=1}^n p_i q_i}{\sum_{i=1}^n p_i}\right) \leqslant \frac{\sum_{i=1}^n p_i f(q_i)}{\sum_{i=1}^n p_i}.$$

Our main result of this section is the following Jensen's type inequality for Lipschitzian function based on the idea of Y. J. Cho et al [3].

THEOREM 1. *For $i \in \mathbb{N}$ suppose p_i, q_i, r_i are positive real numbers. If $f\colon [a,b] \subset \mathbb{R} \to \mathbb{R}$ is an L-Lipschitzian function, then*

$$\left|\frac{1}{P_r}C_f(\mathbf{q},\mathbf{p};\mathbf{r}) - f\left(\overline{Q}_r\right)\right| \leqslant \frac{L}{P_r}\sum_{i=1}^n r_i p_i \left|\frac{q_i}{p_i} - \overline{Q}_r\right| \tag{5}$$

holds, where $C_f(\mathbf{q},\mathbf{p};\mathbf{r})$, P_r and $\overline{Q}_r$ are defined by (2), (3) and (4) respectively.

Proof. The inequality (5) follows by elementary techniques

$$\left|\frac{1}{P_r}\sum_{i=1}^{n} r_i p_i f\left(\frac{q_i}{p_i}\right) - f\left(\frac{1}{P_r}\sum_{i=1}^{n} r_i q_i\right)\right| = \frac{1}{P_r}\left|\sum_{i=1}^{n} r_i p_i \left[f\left(\frac{q_i}{p_i}\right) - f(\overline{Q}_r)\right]\right|$$
$$\leqslant \frac{1}{P_r}\sum_{i=1}^{n} r_i p_i \left|f\left(\frac{q_i}{p_i}\right) - f(\overline{Q}_r)\right| \leqslant \frac{L}{P_r}\sum_{i=1}^{n} r_i p_i \left|\frac{q_i}{p_i} - \overline{Q}_r\right| \quad \square$$

Further inequality for sequence $(q_1,\ldots,q_n)$ is also based on [3].

THEOREM 2. *For $i\in\mathbb{N}$ let p_i, r_i be positive real numbers such that C_f, P_r and $\overline{Q}_r$ are defined by (2), (3) and (4). Suppose that $m, M \in \mathbb{R}$ are such that $mp_i \leqslant q_i \leqslant Mp_i$, $i\in\mathbb{N}$, then*

$$\left|\frac{M-\overline{Q}_r}{M-m}f(m) + \frac{\overline{Q}_r-m}{M-m}f(M) - \frac{1}{P_r}C_f(\mathbf{q},\mathbf{p};\mathbf{r})\right|$$
$$\leqslant \frac{2L}{P_r(M-m)}\sum_{i=1}^{n} r_i p_i \left(M-\frac{q_i}{p_i}\right)\left(\frac{q_i}{p_i}-m\right) \tag{6}$$

holds, where $f\colon [m,M]\to\mathbb{R}$ is an L-Lipschitzian function.

Proof. Start from the left-hand side of (6), we get

$$\left|\frac{M-\overline{Q}_r}{M-m}f(m) + \frac{\overline{Q}_r-m}{M-m}f(M) - \frac{1}{P_r}\sum_{i=1}^{n} r_i p_i f\left(\frac{q_i}{p_i}\right)\right|$$
$$= \frac{1}{P_r}\left|\sum_{i=1}^{n} r_i p_i \left[\frac{M-\frac{q_i}{p_i}}{M-m}f(m) + \frac{\frac{q_i}{p_i}-m}{M-m}f(M) - f\left(\frac{q_i}{p_i}\right)\right]\right|$$
$$\leqslant \frac{1}{P_r}\sum_{i=1}^{n} r_i p_i \left|\frac{M-\frac{q_i}{p_i}}{M-m}f(m) + \frac{\frac{q_i}{p_i}-m}{M-m}f(M) - f\left(\frac{q_i}{p_i}\right)\right|$$
$$= \frac{1}{P_r}\sum_{i=1}^{n} r_i p_i \left|\frac{M-\frac{q_i}{p_i}}{M-m}\left(f(m) - f\left(\frac{q_i}{p_i}\right)\right) + \frac{\frac{q_i}{p_i}-m}{M-m}\left(f(M) - f\left(\frac{q_i}{p_i}\right)\right)\right|$$
$$\leqslant \frac{1}{P_r}\sum_{i=1}^{n} r_i p_i \left[\frac{M-\frac{q_i}{p_i}}{M-m}\left|f(m) - f\left(\frac{q_i}{p_i}\right)\right| + \frac{\frac{q_i}{p_i}-m}{M-m}\left|f(M) - f\left(\frac{q_i}{p_i}\right)\right|\right]$$
$$\leqslant \frac{2L}{P_r(M-m)}\sum_{i=1}^{n} r_i p_i \left(M-\frac{q_i}{p_i}\right)\left(\frac{q_i}{p_i}-m\right).$$

using the properties of the absolute value function. $\square$

Now, we will go through some of the most important examples of f-divergences. The Kullback-Leibler divergence [12], [13] for $\mathbf{p},\mathbf{q}\in\mathbb{R}_+^n$ is given by

$$KL(\mathbf{q},\mathbf{p}) = \sum_{i=1}^{n} q_i \log\left(\frac{q_i}{p_i}\right).$$

It is easy to see that is f-divergence, where $f(t) = t\log t$, $t > 0$. We can generalize KL-divergence by

$$KL(\mathbf{q},\mathbf{p};\mathbf{r}) = \sum_{i=1}^{n} r_i q_i \log \frac{q_i}{p_i}, \tag{7}$$

where $\mathbf{r} \in \mathbb{R}_+^n$.

PROPOSITION 1. *Let $\mathbf{p},\mathbf{q},\mathbf{r} \in \mathbb{R}_+^n$, P_r, $\overline{Q}_r$ and $KL(\mathbf{q},\mathbf{p};\mathbf{r})$ are defined by (3), (4) and (7). Suppose that $m, M \in \mathbb{R}_+$ are such that $mp_i \leqslant q_i \leqslant Mp_i$, $i \in \mathbb{N}$, then inequalities*

$$\begin{aligned}&\left| KL(\mathbf{q},\mathbf{p};\mathbf{r}) - \sum_{i=n}^{n} r_i q_i \log \frac{\sum_{i=n}^{n} r_i q_i}{P_r} \right| \\ &\leqslant \max\{|\log m + 1|, |\log M + 1|\} \sum_{i=n}^{n} r_i p_i \left| \frac{q_i}{p_i} - \overline{Q}_r \right|\end{aligned} \tag{8}$$

and

$$\begin{aligned}&\left| \frac{M-\overline{Q}_r}{M-m} m\log m + \frac{\overline{Q}_r - m}{M-m} M\log M - \frac{1}{P_r} KL(\mathbf{q},\mathbf{p};\mathbf{r}) \right| \\ &\leqslant \max\{|\log m + 1|, |\log M + 1|\} \frac{2}{P_r(M-m)} \sum_{i=1}^{n} r_i p_i \left(M - \frac{q_i}{p_i} \right) \left(\frac{q_i}{p_i} - m \right)\end{aligned} \tag{9}$$

hold.

Proof. The inequalities (8) and (9) are derived from (5) and (6) for $f(t) = t\log t$, $t > 0$. In this case $C_f(\mathbf{q},\mathbf{p};\mathbf{r}) = \sum_{i=1}^{n} r_i p_i \frac{q_i}{p_i} \log \frac{q_i}{p_i} = KL(\mathbf{q},\mathbf{p};\mathbf{r})$ and $L = \sup_{t\in[m,M]} |\log t + 1|$, since $f'(t) = \log t + 1$ is bounded on $[m,M]$. $\square$

The Hellinger divergence [2]

$$He(\mathbf{q},\mathbf{p}) = \sum_{i=1}^{n} (\sqrt{q_i} - \sqrt{p_i})^2,$$

for $\mathbf{p},\mathbf{q} \in \mathbb{R}_+^n$ is f-divergence for $f(t) = (1-\sqrt{t})^2$, $t > 0$. As before, we also generalize this divergence by

$$He(\mathbf{q},\mathbf{p};\mathbf{r}) = \sum_{i=1}^{n} r_i (\sqrt{q_i} - \sqrt{p_i})^2, \tag{10}$$

where $\mathbf{r} \in \mathbb{R}_+^n$. So we give the following estimation.

PROPOSITION 2. *Let* $\mathbf{p},\mathbf{q},\mathbf{r} \in \mathbb{R}_+^n$, $P_r, \overline{Q}_r$ *and* $He(\mathbf{q},\mathbf{p};\mathbf{r})$ *be defined by* (3), (4) *and* (10). *Suppose that* $m, M \in \mathbb{R}_+$ *are such that* $mp_i \leqslant q_i \leqslant Mp_i$, $i \in \mathbb{N}$, *then inequalities*

$$\left| He(\mathbf{q},\mathbf{p};\mathbf{r}) - \left(\sqrt{P_r} - \sqrt{\sum_{i=n}^{n} r_i q_i}\right)^2 \right| \leqslant \max\left\{ \frac{|m-\sqrt{m}|}{m}, \frac{|M-\sqrt{M}|}{M} \right\} \sum_{i=n}^{n} r_i p_i \left| \frac{q_i}{p_i} - \overline{Q}_r \right| \tag{11}$$

and

$$\left| \frac{M-\overline{Q}_r}{M-m}(1-\sqrt{m})^2 + \frac{\overline{Q}_r - m}{M-m}(1-\sqrt{M})^2 - \frac{1}{P_r} He(\mathbf{q},\mathbf{p};\mathbf{r}) \right| \tag{12}$$
$$\leqslant \frac{2}{P_r(M-m)} \max\left\{ \frac{|m-\sqrt{m}|}{m}, \frac{|M-\sqrt{M}|}{M} \right\} \sum_{i=1}^{n} r_i p_i \left(M - \frac{q_i}{p_i} \right)\left(\frac{q_i}{p_i} - m \right)$$

hold.

Proof. For $f(t) = (1-\sqrt{t})^2$, $t > 0$, we have

$$C_f(\mathbf{q},\mathbf{p};\mathbf{r}) = \sum_{i=1}^{n} r_i p_i \left(1 - \sqrt{\frac{q_i}{p_i}}\right)^2 = \sum_{i=1}^{n} r_i (\sqrt{p_i} - \sqrt{q_i})^2 = He(\mathbf{q},\mathbf{p};\mathbf{r}),$$

and

$$L = \sup_{t \in [m,M]} \left| 1 - \frac{1}{\sqrt{t}} \right| = \max\left\{ \left| 1 - \frac{1}{\sqrt{m}} \right|, \left| 1 - \frac{1}{\sqrt{M}} \right| \right\} = \max\left\{ \frac{|m-\sqrt{m}|}{m}, \frac{|M-\sqrt{M}|}{M} \right\}.$$

So, inequalities (11) and (12) follow from (5) and (6). □

The α-order entropy known as Rényi divergence [17] is given by

$$Re_\alpha(\mathbf{q},\mathbf{p}) = \sum_{i=1}^{n} p_i^{1-\alpha} q_i^{\alpha}, \quad \alpha \in \langle 1, +\infty \rangle. \tag{13}$$

We generalize (13) by

$$Re_\alpha(\mathbf{q},\mathbf{p};\mathbf{r}) = \sum_{i=1}^{n} r_i p_i^{1-\alpha} q_i^{\alpha}, \quad r \in \mathbb{R}_+^n. \tag{14}$$

For this generalized entropy we have the following result.

PROPOSITION 3. *Let* $\mathbf{p},\mathbf{q},\mathbf{r}\in\mathbb{R}_+^n$, P_r, $\overline{Q}_r$ *and* $Re_\alpha(\mathbf{q},\mathbf{p};\mathbf{r})$ *be defined by* (3), (4) *and* (14). *Suppose that* $m,M\in\mathbb{R}_+$ *are such that* $mp_i\leqslant q_i\leqslant Mp_i$, $i\in\mathbb{N}$, *then inequalities*

$$\left|Re_\alpha(\mathbf{q},\mathbf{p};\mathbf{r})-P_r^{1-\alpha}\left(\sum_{i=1}^n r_iq_i\right)^\alpha\right|\leqslant \alpha M^{\alpha-1}\sum_{i=1}^n r_ip_i\left|\frac{q_i}{p_i}-\overline{Q}_r\right| \tag{15}$$

and

$$\begin{aligned}&\left|\frac{M-\overline{Q}_r}{M-m}m^\alpha+\frac{\overline{Q}_r-m}{M-m}M^\alpha-\frac{1}{P_r}Re_\alpha(\mathbf{q},\mathbf{p};\mathbf{r})\right|\\&\leqslant\frac{2\alpha M^{\alpha-1}}{P_r(M-m)}\sum_{i=1}^n r_ip_i\left(M-\frac{q_i}{p_i}\right)\left(\frac{q_i}{p_i}-m\right)\end{aligned} \tag{16}$$

hold.

Proof. For $f(t)=t^\alpha$, $t>0$, $\alpha>1$, we have

$$C_f(\mathbf{q},\mathbf{p};\mathbf{r})=\sum_{i=1}^n r_ip_i\left(\frac{q_i}{p_i}\right)^\alpha=\sum_{i=1}^n r_ip_i^{1-\alpha}q_i^\alpha=Re_\alpha(\mathbf{q},\mathbf{p};\mathbf{r}),$$

and

$$L=\sup_{t\in[m,M]}\left|\alpha t^{\alpha-1}\right|=\alpha\sup_{t\in[m,M]}\left|t^{\alpha-1}\right|=\alpha M^{\alpha-1},$$

so we obtain (15) and (16) from (5) and (6). □

The next interesting result is concerned the χ^2-divergence defined by

$$D_{\chi^2}(\mathbf{q},\mathbf{p})=\sum_{i=1}^n\frac{(q_i-p_i)^2}{p_i},\quad \mathbf{p},\mathbf{q}\in\mathbb{R}_+^n.$$

For generalized χ^2-divergence

$$D_{\chi^2}(\mathbf{q},\mathbf{p};\mathbf{r})=\sum_{i=1}^n r_i\frac{(q_i-p_i)^2}{p_i},\quad \mathbf{r}\in\mathbb{R}_+^n \tag{17}$$

we give the following statement.

PROPOSITION 4. *Let* $\mathbf{p},\mathbf{q},\mathbf{r}\in\mathbb{R}_+^n$, P_r, $\overline{Q}_r$ *and* $D_{\chi^2}(\mathbf{q},\mathbf{p};\mathbf{r})$ *be defined by* (3), (4) *and* (17). *Suppose that* $m,M\in\mathbb{R}$ *are such that* $mp_i\leqslant q_i\leqslant Mp_i$, $i\in\mathbb{N}$, *then inequalities*

$$\left|D_{\chi^2}(\mathbf{q},\mathbf{p};\mathbf{r})-\frac{1}{P_r}\left(\sum_{i=1}^n r_iq_i-P_r\right)^2\right|\leqslant 2\max\{|m-1|,|M-1|\}\sum_{i=1}^n r_ip_i\left|\frac{q_i}{p_i}-\overline{Q}_r\right| \tag{18}$$

and

$$\left|\frac{M-\overline{Q}_r}{M-m}(m-1)^2+\frac{\overline{Q}_r-m}{M-m}(M-1)^2-\frac{1}{P_r}D_{\chi^2}(\mathbf{q},\mathbf{p};\mathbf{r})\right| \tag{19}$$
$$\leqslant \frac{4}{P_r(M-m)}\max\{|m-1|,|M-1|\}\sum_{i=1}^n r_ip_i\left(M-\frac{q_i}{p_i}\right)\left(\frac{q_i}{p_i}-m\right)$$

hold.

Proof. For $f(t)=(t-1)^2$, $t>0$, we have

$$C_f(\mathbf{q},\mathbf{p};\mathbf{r})=\sum_{i=1}^n r_ip_i\left(\frac{q_i}{p_i}-1\right)^2=\sum_{i=1}^n r_i\frac{(q_i-p_i)^2}{p_i}=D_{\chi^2}(\mathbf{q},\mathbf{p};\mathbf{r}).$$

Since $f'(t)=2(t-1)$, we have

$$L=2\sup_{t\in[m,M]}|t-1|=2\max\{|m-1|,|M-1|\}.$$

Inequalities (18) and (19) follow from (5) and (6). □

The Shannon entropy of a positive probability distribution $\mathbf{p}=(p_1,\ldots,p_n)$ is defined by

$$H(\mathbf{p})=-\sum_{i=1}^n p_i\log(p_i). \tag{20}$$

It is easy to see that (20) is a special case of (1) for $\mathbf{q}=(1,\ldots,1)\in\mathbb{R}_+^n$ and function $f(t)=\log t$, $t>0$. We can also generalize Shannon entropy with a weight $\mathbf{r}\in\mathbb{R}_+^n$

$$H(\mathbf{p};\mathbf{r})=-\sum_{i=1}^n r_ip_i\log(p_i). \tag{21}$$

PROPOSITION 5. *Let* $\mathbf{p},\mathbf{r}\in\mathbb{R}_+^n$, P_r *and* $H(\mathbf{p};\mathbf{r})$ *be defined by* (3) *and* (21). *Suppose that* $m,M\in\mathbb{R}$ *are such that* $m\leqslant\frac{1}{p_i}\leqslant M$, $i\in\mathbb{N}$, *then inequalities*

$$\left|H(\mathbf{p};\mathbf{r})-P_r\log(\overline{Q}_r)\right|\leqslant\frac{1}{m}\sum_{i=1}^n r_ip_i\left|\frac{1}{p_i}-\overline{Q}_r\right| \tag{22}$$

and

$$\left|\frac{M-\overline{Q}_r}{M-m}f(m)+\frac{\overline{Q}_r-m}{M-m}f(M)-\frac{1}{P_r}H(\mathbf{p};\mathbf{r})\right| \tag{23}$$
$$\leqslant\frac{2}{m(M-m)P_r}\sum_{i=1}^n r_ip_i\left(M-\frac{1}{p_i}\right)\left(\frac{1}{p_i}-m\right)$$

hold, where $\overline{Q}_r=\frac{1}{P_r}\sum_{i=1}^n r_i$.

Proof. For $f(t) = \log t$, $t > 0$ and $\mathbf{q} = (1, \dots, 1)$, we have

$$C_f(\mathbf{1}, \mathbf{p}; \mathbf{r}) = \sum_{i=1}^{n} r_i p_i \log\left(\frac{1}{p_i}\right) = -\sum_{i=1}^{n} r_i p_i \log(p_i) = H(\mathbf{p}; \mathbf{r}).$$

Since $f'(t) = \dfrac{1}{t}$, the Lipschitz constant in this case is

$$L = \sup_{t \in [m,M]} \left|\frac{1}{t}\right| = \max\left\{\left|\frac{1}{m}\right|, \left|\frac{1}{M}\right|\right\} = \max\left\{\frac{1}{m}, \frac{1}{M}\right\} = \frac{1}{m}.$$

Inequalities (22) and (23) are following from (5) and (6). □

3. The mappings of H and F

In this section, we study discrete general case of the mappings called H and F introduced in [7] and [8]. For a given function $f\colon I \subset \mathbb{R} \to \mathbb{R}$ and for $a, b \in I$, $a < b$, S. S. Dragomir consider the following two mappings $H, F\colon [0,1] \to \mathbb{R}$ defined by

$$H(t) = \frac{1}{b-a} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) dx$$

and

$$F(t) = \frac{1}{(b-a)^2} \int_a^b \int_a^b f(tx + (1-t)y) dx dy$$

for all $t \in [0,1]$. Under assumption of convexity of f, mapping H and F have been tested on convexity on $[0,1]$, monotonicity and other properties. On this lead, Cho et al [3] are also studied generalized functions of this type.

In this article, we consider f to be Lipschitzian function and dealing with discrete generalization, so our next results come naturally. We prove some of the properties of the functions F and H, such as Lipschitz property.

THEOREM 3. *Let $\mathbf{p}, \mathbf{r} \in \mathbb{R}_+^n$ and $\mathbf{q} \in \mathbb{R}^n$ and $f\colon [a,b] \subset \mathbb{R} \to \mathbb{R}$ be an L-Lipschitzian function. For a mapping $H\colon [0,1] \to \mathbb{R}$ defined by*

$$H(\lambda) = \frac{1}{P_r} \sum_{i=1}^{n} p_i r_i f\left(\lambda \frac{q_i}{p_i} + (1-\lambda)\overline{Q}_r\right) \tag{24}$$

we have the following:

1. the mapping H is L_1-Lipschitzian on $[0,1]$, where

$$L_1 = \frac{L}{P_r} \sum_{i=1}^{n} p_i r_i \left(\frac{q_i}{p_i} - \overline{Q}_r\right) \tag{25}$$

2. *the inequalities*

$$\left|H(\lambda)-\frac{1}{P_r}\sum_{i=1}^{n}r_ip_if\left(\frac{q_i}{p_i}\right)\right|\leqslant(1-\lambda)L_1, \tag{26}$$

$$\left|f\left(\frac{1}{P_r}\sum_{i=1}^{n}r_iq_i\right)-H(\lambda)\right|\leqslant\lambda L_1 \tag{27}$$

and

$$\left|H(\lambda)-\frac{\lambda}{P_r}\sum_{i=1}^{n}r_ip_if\left(\frac{q_i}{p_i}\right)-(1-\lambda)f(\overline{Q}_r)\right|\leqslant 2\lambda(1-\lambda)L_1 \tag{28}$$

hold, for all $\lambda\in[0,1]$.

Proof. For $\lambda_1,\lambda_2\in[0,1]$, we calculate

$$\begin{aligned}
&|H(\lambda_2)-H(\lambda_1)|\\
&=\frac{1}{P_r}\left|\sum_{i=1}^{n}p_ir_i\left[f\left(\lambda_2\frac{q_i}{p_i}+(1-\lambda_2)\overline{Q}_r\right)-f\left(\lambda_1\frac{q_i}{p_i}+(1-\lambda_1)\overline{Q}_r\right)\right]\right|\\
&\leqslant\frac{1}{P_r}\sum_{i=1}^{n}p_ir_i\left|f\left(\lambda_2\frac{q_i}{p_i}+(1-\lambda_2)\overline{Q}_r\right)-f\left(\lambda_1\frac{q_i}{p_i}+(1-\lambda_1)\overline{Q}_r\right)\right|\\
&\leqslant\frac{L}{P_r}\sum_{i=1}^{n}p_ir_i\left|\lambda_2\frac{q_i}{p_i}+(1-\lambda_2)\overline{Q}_r-\lambda_1\frac{q_i}{p_i}-(1-\lambda_1)\overline{Q}_r\right|\\
&=\frac{L|\lambda_2-\lambda_1|}{P_r}\sum_{i=1}^{n}p_ir_i\left|\frac{q_i}{p_i}-\overline{Q}_r\right|
\end{aligned}$$

and get $|H(\lambda_2)-H(\lambda_1)|\leqslant L_1|\lambda_2-\lambda_1|$, for L_1 defined by (25). For $\lambda_1=1$ and $\lambda_2=\lambda$, left hand side in (26) is equal to $|H(\lambda)-H(1)|$. Since we already proved H is L_1-Lipschitzian function, inequality (26) hold. Analogously, (27) follows for $\lambda_1=\lambda$ and $\lambda_2=0$. Finally, inequality (28) follows from (26) and (27),

$$\begin{aligned}
&\left|H(\lambda)-\frac{\lambda}{P_r}\sum_{i=1}^{n}r_ip_if\left(\frac{q_i}{p_i}\right)-(1-\lambda)f(\overline{Q}_r)\right|\\
&\leqslant\left|\lambda H(\lambda)-\frac{\lambda}{P_r}\sum_{i=1}^{n}r_ip_if\left(\frac{q_i}{p_i}\right)\right|+\left|-(1-\lambda)f\left(\frac{1}{P_r}\sum_{i=1}^{n}r_iq_i\right)+(1-\lambda)H(\lambda)\right|\\
&\leqslant 2\lambda(1-\lambda)L_1.\quad\square
\end{aligned}$$

THEOREM 4. *Let* $\mathbf{p},\mathbf{r}\in\mathbb{R}^n_+$, $\mathbf{q}\in\mathbb{R}^n$ *and* $f\colon[a,b]\subset\mathbb{R}\to\mathbb{R}$ *be a* L*-Lipschitzian function. For a mapping* $F\colon[0,1]\to\mathbb{R}$ *defined by*

$$F(\lambda)=\frac{1}{P_r^2}\sum_{i=1}^{n}\sum_{j=1}^{n}p_ir_ip_jr_jf\left(\lambda\frac{q_i}{p_i}+(1-\lambda)\frac{q_j}{p_j}\right) \tag{29}$$

we have the following:

1. the mapping F is symmetric, i.e. $F(\lambda) = F(1-\lambda)$, $\lambda \in [0,1]$

2. the mapping F is L_2-Lipschitzian on $[0,1]$, where

$$L_2 = \frac{L}{P_r^2}\sum_{i=1}^{n}\sum_{j=1}^{n} p_i r_i p_j r_j \left|\frac{q_i}{p_i} - \frac{q_j}{p_j}\right| \tag{30}$$

3. the inequalities

$$\left|F(\lambda) - \frac{1}{P_r^2}\sum_{i=1}^{n}\sum_{j=1}^{n} p_i r_i p_j r_j f\left[\frac{1}{2}\left(\frac{q_i}{p_i} + \frac{q_j}{p_j}\right)\right]\right| \leqslant \frac{L_2}{2}|2\lambda - 1| \tag{31}$$

and

$$\left|F(\lambda) - \frac{1}{P_r}\sum_{i=1}^{n} p_i r_i f\left(\frac{q_i}{p_i}\right)\right| \leqslant L_2 \min\{\lambda, 1-\lambda\} \tag{32}$$

holds for all $\lambda \in [0,1]$.

Proof. The first property follows immediately from the the definition (29). For proving next property, let $\lambda_1, \lambda_2 \in [0,1]$. Then we have

$$\begin{aligned}
&|F(\lambda_2) - F(\lambda_1)| \\
&= \frac{1}{P_r^2}\left|\sum_{i=1}^{n}\sum_{j=1}^{n} p_i r_i p_j r_j \left[f\left(\lambda_2\frac{q_i}{p_i} + (1-\lambda_2)\frac{q_j}{p_j}\right) - f\left(\lambda_1\frac{q_i}{p_i} + (1-\lambda_1)\frac{q_j}{p_j}\right)\right]\right| \\
&\leqslant \frac{1}{P_r^2}\sum_{i=1}^{n}\sum_{j=1}^{n} p_i r i_i p_j r_j \left|f\left(\lambda_2\frac{q_i}{p_i} + (1-\lambda_2)\frac{q_j}{p_j}\right) - f\left(\lambda_1\frac{q_i}{p_i} + (1-\lambda_1)\frac{q_j}{p_j}\right)\right| \\
&\leqslant \frac{L|\lambda_2 - \lambda_1|}{P_r^2}\sum_{i=1}^{n}\sum_{j=1}^{n} p_i r_i p_j r_j \left|\frac{q_i}{p_i} - \frac{q_j}{p_j}\right| \\
&= L_2|\lambda_2 - \lambda_1|.
\end{aligned}$$

Inequality (31) follows from Lipschitzian property of F for $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = \lambda$. So, we have $\left|F(\lambda) - F\left(\frac{1}{2}\right)\right| \leqslant L_2|\lambda - \frac{1}{2}| = \frac{L_2}{2}|\lambda - 1|$. Analogously, (32) follows for $\lambda_1 = 1$, $\lambda_2 = \lambda$ and $\lambda_1 = 1$, $\lambda_2 = 1-\lambda$. Namely, by combining

$$|F(\lambda) - F(1)| = \left|F(\lambda) - \frac{1}{P_r}\sum_{i=1}^{n} p_i r_i f\left(\frac{q_i}{p_i}\right)\right| \leqslant |\lambda - 1| = 1 - \lambda$$

and

$$|F(1-\lambda) - F(1)| = |F(\lambda) - F(1)| \leqslant |1 - \lambda - 1| = \lambda$$

we get (32). □

The next result offer us the relation between the mappings F and H, defined by (24) and (29).

THEOREM 5. *For mappings* $F\colon [0,1]\to\mathbb{R}$ *and* $H\colon [0,1]\to\mathbb{R}$ *defined by* (29) *and* (24)*, inequality*

$$|F(\lambda)-H(\lambda)| \leqslant (1-\lambda)L_1 \tag{33}$$

hold, for all $\lambda\in[0,1]$*, where* L_1 *is defined by* (25)*.*

Proof. The inequality (33) hold as follows:

$$\begin{aligned}
&|F(\lambda)-H(\lambda)|\\
&=\left|\frac{1}{P_r^2}\sum_{i=1}^{n}\sum_{j=1}^{n}p_ir_ip_jr_jf\left(\lambda\frac{q_i}{p_i}+(1-\lambda)\frac{q_j}{p_j}\right)-\frac{1}{P_r}\sum_{i=1}^{n}p_ir_if\left(\lambda\frac{q_i}{p_i}+(1-\lambda)\overline{Q}_r\right)\right|\\
&=\frac{1}{P_r^2}\left|\sum_{i=1}^{n}\sum_{j=1}^{n}p_ir_ip_jr_j\left[f\left(\lambda\frac{q_i}{p_i}+(1-\lambda)\frac{q_j}{p_j}\right)-f\left(\lambda\frac{q_i}{p_i}+(1-\lambda)\overline{Q}_r\right)\right]\right|\\
&\leqslant\frac{1}{P_r^2}\sum_{i=1}^{n}\sum_{j=1}^{n}p_ir_ip_jr_j\left|f\left(\lambda\frac{q_i}{p_i}+(1-\lambda)\frac{q_j}{p_j}\right)-f\left(\lambda\frac{q_i}{p_i}+(1-\lambda)\overline{Q}_r\right)\right|\\
&\leqslant\frac{L}{P_r^2}\sum_{i=1}^{n}\sum_{j=1}^{n}p_ir_ip_jr_j\left|\lambda\frac{q_i}{p_i}+(1-\lambda)\frac{q_j}{p_j}-\lambda\frac{q_i}{p_i}-(1-\lambda)\overline{Q}_r\right|\\
&=\frac{L(1-\lambda)}{P_r}\sum_{i=1}^{n}p_ir_i\left|\frac{q_i}{p_i}-\overline{Q}_r\right|=(1-\lambda)L_1. \qquad\square
\end{aligned}$$

4. The Zipf-Mandelbrot law

DEFINITION 1. [10] Zipf-Mandelbrot law is a discrete probability distribution, depends on three parameters $N\in\{1,2,\dots\}$, $t\in[0,\infty\rangle$ and $v>0$, and it is defined by

$$\phi\,(i;N,t,v):=\frac{1}{(i+t)^v H_{N,t,v}},\qquad i=1,\dots,N,$$

where

$$H_{N,t,v}:=\sum_{j=1}^{N}\frac{1}{(j+t)^v}.$$

When $t=0$, then Zipf–Mandelbrot law becomes Zipf's law.

Now, we can apply our results for distributions on the Zipf-Mandelbrot law.

Let $\mathbf{p},\mathbf{q}$ be two Zipf-Mandelbrot laws with parameters $N\in\{1,2,\dots\}$, t_1, $t_2\geqslant 0$ and v_1, $v_2>0$, respectively. It is

$$p_i=\phi\,(i;N,t_1,v_1):=\frac{1}{(i+t_1)^{v_1}H_{N,t_1,v_1}},\qquad i=1,\dots,N, \tag{34}$$

and

$$q_i=\phi\,(i;N,t_2,v_2):=\frac{1}{(i+t_2)^{v_2}H_{N,t_2,v_2}},\qquad i=1,\dots,N, \tag{35}$$

where

$$H_{N,t_k,v_k} := \sum_{j=1}^{N} \frac{1}{(j+t_k)^{v_k}}, \quad k = 1,2. \tag{36}$$

Then the generalized Csiszár divergence for such $\mathbf{p},\mathbf{q}$, and for $\mathbf{r} \in \mathbb{R}^n_+$ is given by

$$C_f(\mathbf{q},\mathbf{p};\mathbf{r}) = \frac{1}{H_{N,t_1,v_1}} \sum_{i=1}^{N} \frac{r_i}{(i+t_1)^{v_1}} f\left(\frac{(i+t_1)^{v_1} H_{N,t_1,v_1}}{(i+t_2)^{v_2} H_{N,t_2,v_2}}\right). \tag{37}$$

Using (34) and (35), we have the following expressions for (3) and (4)

$$P_r = \sum_{i=1}^{N} \frac{r_i}{(i+t_1)^{v_1} H_{N,t_1,v_1}} = \frac{1}{H_{N,t_1,v_1}} \sum_{i=1}^{N} \frac{r_i}{(i+t_1)^{v_1}}, \tag{38}$$

$$\overline{Q}_r = \frac{\sum_{i=1}^{N} \frac{r_i}{(i+t_2)^{v_2} H_{N,t_2,v_2}}}{\sum_{i=1}^{N} \frac{r_i}{(i+t_1)^{v_1} H_{N,t_1,v_1}}} = \frac{H_{N,t_1,v_1}}{H_{N,t_2,v_2}} \cdot \frac{\sum_{i=1}^{N} \frac{r_i}{(i+t_2)^{v_2}}}{\sum_{i=1}^{N} \frac{r_i}{(i+t_1)^{v_1}}}. \tag{39}$$

For m and M from Theorem 2 we choose

$$m = \frac{(1+t_1)^{v_1} H_{N,t_1,v_1}}{(N+t_2)^{v_2} H_{N,t_2,v_2}}$$

and

$$M = \frac{(N+t_1)^{v_1} H_{N,t_1,v_1}}{(1+t_2)^{v_2} H_{N,t_2,v_2}}.$$

Thus we have the following results.

COROLLARY 1. *Let* $\mathbf{p},\mathbf{q}$ *be two Zipf-Mandelbrot laws with parameters* $N \in \{1,2,\dots\}$, t_1, $t_2 \geqslant 0$ *and* $v_1, v_2 > 0$, *respectively, and* $\mathbf{r} \in \mathbb{R}^n_+$. *If* $C_f(\mathbf{q},\mathbf{p};\mathbf{r})$, P_r *and* $\overline{Q}_r$ *are defined by* (37), (38) *and* (39), *respectively, we have*

$$\left|C_f(\mathbf{q},\mathbf{p};\mathbf{r}) - P_r f(\overline{Q}_r)\right| \leqslant \frac{L}{H_{N,t_2,v_2}} \sum_{i=1}^{N} \frac{r_i}{(i+t_1)^{v_1}} \left| \frac{(i+t_1)^{v_1}}{(i+t_2)^{v_2}} - \frac{\sum_{i=1}^{N} \frac{r_i}{(i+t_2)^{v_2}}}{\sum_{i=1}^{N} \frac{r_i}{(i+t_1)^{v_1}}} \right|,$$

and

$$\begin{aligned} &\left| \sum_{i=1}^{N} \frac{r_i}{(i+t_1)^{v_1}} \left[\left(\frac{(N+t_1)^{v_1}}{(1+t_2)^{v_2}} - \frac{\sum_{i=1}^{N} \frac{r_i}{(i+t_2)^{v_2}}}{\sum_{i=1}^{N} \frac{r_i}{(i+t_1)^{v_1}}} \right) f\left(\frac{(1+t_1)^{v_1} H_{N,t_1,v_1}}{(N+t_2)^{v_2} H_{N,t_2,v_2}} \right) \right. \right. \\ &\quad \left. + \left(\frac{\sum_{i=1}^{N} \frac{r_i}{(i+t_2)^{v_2}}}{\sum_{i=1}^{N} \frac{r_i}{(i+t_1)^{v_1}}} - \frac{(1+t_1)^{v_1}}{(N+t_2)^{v_2}} \right) f\left(\frac{(N+t_1)^{v_1} H_{N,t_1,v_1}}{(1+t_2)^{v_2} H_{N,t_2,v_2}} \right) \right] \\ &\quad \left. - \left(\frac{(N+t_1)^{v_1}}{(1+t_2)^{v_2}} - \frac{(1+t_1)^{v_1}}{(N+t_2)^{v_2}} \right) C_f(\mathbf{q},\mathbf{p};\mathbf{r}) \right| \\ &\leqslant 2L \frac{H_{N,t_1,v_1}}{H_{N,t_2,v_2}} \sum_{i=1}^{N} \frac{r_i}{(i+t_1)^{v_1}} \left(\frac{(N+t_1)^{v_1}}{(1+t_2)^{v_2}} - \frac{(i+t_1)^{v_1}}{(i+t_2)^{v_2}} \right) \left(\frac{(i+t_1)^{v_1}}{(i+t_2)^{v_2}} - \frac{(1+t_1)^{v_1}}{(N+t_2)^{v_2}} \right). \end{aligned} \tag{40}$$

Proof. Inequality (40) can be obtained from

$$\Bigg|\Bigg[\left(\frac{(N+t_1)^{v_1}H_{N,t_1,v_1}}{(1+t_2)^{v_2}H_{N,t_2,v_2}}-\frac{H_{N,t_1,v_1}}{H_{N,t_2,v_2}}\frac{\sum_{i=1}^{N}\frac{r_i}{(i+t_2)^{v_2}}}{\sum_{i=1}^{N}\frac{r_i}{(i+t_1)^{v_1}}}\right)f\left(\frac{(1+t_1)^{v_1}H_{N,t_1,v_1}}{(N+t_2)^{v_2}H_{N,t_2,v_2}}\right)$$
$$+\left(\frac{H_{N,t_1,v_1}}{H_{N,t_2,v_2}}\frac{\sum_{i=1}^{N}\frac{r_i}{(i+t_2)^{v_2}}}{\sum_{i=1}^{N}\frac{r_i}{(i+t_1)^{v_1}}}-\frac{(1+t_1)^{v_1}H_{N,t_1,v_1}}{(N+t_2)^{v_2}H_{N,t_2,v_2}}\right)f\left(\frac{(N+t_1)^{v_1}H_{N,t_1,v_1}}{(1+t_2)^{v_2}H_{N,t_2,v_2}}\right)\Bigg]$$
$$\times\sum_{i=1}^{N}\frac{r_i}{(i+t_1)^{v_1}H_{N,t_1,v_1}}-\left(\frac{(N+t_1)^{v_1}H_{N,t_1,v_1}}{(1+t_2)^{v_2}H_{N,t_2,v_2}}-\frac{(1+t_1)^{v_1}H_{N,t_1,v_1}}{(N+t_2)^{v_2}H_{N,t_2,v_2}}\right)C_f(\mathbf{q},\mathbf{p};\mathbf{r})\Bigg|$$
$$\leqslant 2L\sum_{i=1}^{N}\frac{r_i}{(i+t_1)^{v_1}H_{N,t_1,v_1}}\left(\frac{(N+t_1)^{v_1}H_{N,t_1,v_1}}{(1+t_2)^{v_2}H_{N,t_2,v_2}}-\frac{(i+t_1)^{v_1}H_{N,t_1,v_1}}{(i+t_2)^{v_2}H_{N,t_2,v_2}}\right)$$
$$\times\left(\frac{(i+t_1)^{v_1}H_{N,t_1,v_1}}{(i+t_2)^{v_2}H_{N,t_2,v_2}}-\frac{(1+t_1)^{v_1}H_{N,t_1,v_1}}{(N+t_2)^{v_2}H_{N,t_2,v_2}}\right). \quad \square$$

Now, we will give some examples of the Zipf-Mandelbrot law for special f-divergences. If $\mathbf{p},\mathbf{q}$ are two Zipf-Mandelbrot laws with parameters $N\in\{1,2,\ldots\}$, t_1, $t_2\geqslant 0$ and v_1, $v_2>0$, respectively, and $\mathbf{r}\in\mathbb{R}^n_+$, for the generalized Kullback-Leibler divergence we have the following expression

$$KL(\mathbf{q},\mathbf{p};\mathbf{r})=\frac{1}{H_{N,t_2,v_2}}\sum_{i=1}^{N}\frac{r_i}{(i+t_2)^{v_2}}\log\left(\frac{(i+t_1)^{v_1}H_{N,t_1,v_1}}{(i+t_2)^{v_2}H_{N,t_2,v_2}}\right). \tag{41}$$

The following results hold.

COROLLARY 2. *Let $\mathbf{p},\mathbf{q}$ be two Zipf-Mandelbrot laws with parameters $N\in\{1,2,\ldots\}$, $t_1,t_2\geqslant 0$ and $v_1,v_2>0$, respectively, and $\mathbf{r}\in\mathbb{R}^n_+$. If $KL(\mathbf{q},\mathbf{p};\mathbf{r})$ is defined by (41), then inequalities*

$$\left|H_{N,t_2,v_2}KL(\mathbf{q},\mathbf{p};\mathbf{r})-\sum_{i=1}^{N}\frac{r_i}{(i+t_2)^{v_2}}\left(\log\frac{H_{N,t_1,v_1}}{H_{N,t_2,v_2}}+\log\sum_{i=1}^{N}\frac{r_i}{(i+t_2)^{v_2}}-\log\sum_{i=1}^{N}\frac{r_i}{(i+t_1)^{v_1}}\right)\right|$$
$$\leqslant\max\left\{\left|\log\frac{H_{N,t_1,v_1}}{H_{N,t_2,v_2}}+s_1\log(1+t_1)-v_2\log(N+t_2)+1\right|,\right.$$
$$\left.\left|\log\frac{H_{N,t_1,v_1}}{H_{N,t_2,v_2}}+v_1\log(N+t_1)-v_2\log(1+t_2)+1\right|\right\}$$
$$\times\sum_{i=1}^{N}\frac{r_i}{(i+t_1)^{v_1}}\left|\frac{(i+t_1)^{v_1}}{(i+t_2)^{v_2}}-\frac{\sum_{i=1}^{N}\frac{r_i}{(i+t_2)^{v_2}}}{\sum_{i=1}^{N}\frac{r_i}{(i+t_1)^{v_1}}}\right|,$$

and

$$\left|\frac{(1+t_1)^{v_1}}{(N+t_2)^{v_2}}\left(\frac{(N+t_1)^{v_1}}{(1+t_2)^{v_2}}\sum_{i=1}^{N}\frac{r_i}{(i+t_1)^{v_1}}-\sum_{i=1}^{N}\frac{r_i}{(i+t_2)^{v_2}}\right)\right.$$
$$+\frac{(N+t_1)^{v_1}}{(1+t_2)^{v_2}}\left(\sum_{i=1}^{N}\frac{r_i}{(i+t_2)^{v_2}}-\frac{(1+t_1)^{v_1}}{(N+t_2)^{v_2}}\sum_{i=1}^{N}\frac{r_i}{(i+t_1)^{v_1}}\right)$$

$$
\begin{aligned}
& - H_{N,t_2,v_2}\left(\frac{(N+t_1)^{v_1}}{(1+t_2)^{v_2}} - \frac{(1+t_1)^{v_1}}{(N+t_2)^{v_2}}\right) KL(\mathbf{q},\mathbf{p};\mathbf{r})\Bigg| \\
& \leqslant 2\max\left\{\left|\log\frac{H_{N,t_1,v_1}}{H_{N,t_2,v_2}} + v_1\log(1+t_1) - v_2\log(N+t_2)+1\right|,\right. \\
& \left.\left|\log\frac{H_{N,t_1,v_1}}{H_{N,t_2,v_2}} + v_1\log(N+t_1) - v_2\log(1+t_2)+1\right|\right\} \\
& \times \sum_{i=1}^{N}\frac{r_i}{(i+t_1)^{v_1}}\left(\frac{(N+t_1)^{v_1}}{(1+t_2)^{v_2}} - \frac{(i+t_1)^{v_1}}{(i+t_2)^{v_2}}\right)\left(\frac{(i+t_1)^{v_1}}{(i+t_2)^{v_2}} - \frac{(1+t_1)^{v_1}}{(N+t_2)^{v_2}}\right)
\end{aligned}
$$

hold.

For $\mathbf{p},\mathbf{q}$ two Zipf-Mandelbrot laws with parameters $N \in \{1,2,\dots\}$, t_1, $t_2 \geqslant 0$ and v_1, $v_2 > 0$, respectively, and $\mathbf{r} \in \mathbb{R}^n_+$, the generalized Hellinger divergence has the following representation

$$
He(\mathbf{q},\mathbf{p};\mathbf{r}) = \frac{1}{H_{N,t_1,v_1}H_{N,t_2,v_2}}\sum_{i=1}^{N} r_i \frac{\left(\sqrt{(i+t_1)^{v_1}H_{N,t_1,v_1}} - \sqrt{(i+t_2)^{v_2}H_{N,t_2,v_2}}\right)^2}{(i+t_1)^{v_1}(i+t_2)^{v_2}}. \tag{42}
$$

The following results hold true.

COROLLARY 3. *Let* $\mathbf{p},\mathbf{q}$ *be two Zipf-Mandelbrot laws with parameters* $N \in \{1,2,\dots\}$, t_1, $t_2 \geqslant 0$ *and* $v_1, v_2 > 0$, *respectively, and* $\mathbf{r} \in \mathbb{R}^n_+$. *If* $He(\mathbf{q},\mathbf{p};\mathbf{r})$ *is defined by* (42), *then inequalities*

$$
\begin{aligned}
& \left| He(\mathbf{q},\mathbf{p};\mathbf{r}) - \left(\sqrt{\sum_{i=1}^{N}\frac{r_i}{(i+t_1)^{v_1}H_{N,t_1,v_1}}} - \sqrt{\sum_{i=1}^{N}\frac{r_i}{(i+t_2)^{v_2}H_{N,t_2,v_2}}}\right)^2\right| \\
& \leqslant \max\left\{\left|1-\sqrt{\frac{(N+t_2)^{v_2}H_{N,t_2,v_2}}{(1+t_1)^{v_1}H_{N,t_1,v_1}}}\right|, \left|1-\sqrt{\frac{(1+t_2)^{v_2}H_{N,t_2,v_2}}{(N+t_1)^{v_1}H_{N,t_1,v_1}}}\right|\right\} \\
& \times \sum_{i=1}^{N}\frac{r_i}{(i+t_1)^{v_1}H_{N,t_2,v_2}}\left|\frac{(i+t_1)^{v_1}}{(i+t_2)^{v_2}} - \frac{\sum_{i=1}^{N}\frac{r_i}{(i+t_2)^{v_2}}}{\sum_{i=1}^{N}\frac{r_i}{(i+t_1)^{v_1}}}\right|,
\end{aligned}
$$

and

$$
\begin{aligned}
& \Bigg|\Bigg[\left(\frac{(N+t_1)^{v_1}}{(1+t_2)^{v_2}}\sum_{i=1}^{N}\frac{r_i}{(i+t_1)^{v_1}} - \sum_{i=1}^{N}\frac{r_i}{(i+t_2)^{v_2}}\right)\left(1-\sqrt{\frac{(1+t_1)^{v_1}H_{N,t_1,v_1}}{(N+t_2)^{v_2}H_{N,t_2,v_2}}}\right)^2 \\
& + \left(\sum_{i=1}^{N}\frac{r_i}{(i+t_2)^{v_2}} - \frac{(1+t_1)^{v_1}}{(N+t_2)^{v_2}}\sum_{i=1}^{N}\frac{r_i}{(i+t_1)^{v_1}}\right)\left(1-\sqrt{\frac{(N+t_1)^{v_1}H_{N,t_1,v_1}}{(1+t_2)^{v_2}H_{N,t_2,v_2}}}\right)^2\Bigg] \\
& - H_{N,t_1,v_1}\left(\frac{(N+t_1)^{v_1}}{(1+t_2)^{v_2}} - \frac{(1+t_1)^{v_1}}{(N+t_2)^{v_2}}\right) He(\mathbf{q},\mathbf{p};\mathbf{r})\Bigg|
\end{aligned}
$$

$$\leqslant 2\frac{H_{N,t_1,v_1}}{H_{N,t_2,v_2}}\max\left\{\left|1-\sqrt{\frac{(N+t_2)^{v_2}H_{N,t_2,v_2}}{(1+t_1)^{v_1}H_{N,t_1,v_1}}}\right|,\left|1-\sqrt{\frac{(1+t_2)^{v_2}H_{N,t_2,v_2}}{(N+t_1)^{v_1}H_{N,t_1,v_1}}}\right|\right\}$$
$$\times\sum_{i=1}^{N}\frac{r_i}{(i+t_1)^{v_1}}\left(\frac{(N+t_1)^{v_1}}{(1+t_2)^{v_2}}-\frac{(i+t_1)^{v_1}}{(i+t_2)^{v_2}}\right)\left(\frac{(i+t_1)^{v_1}}{(i+t_2)^{v_2}}-\frac{(1+t_1)^{v_1}}{(N+t_2)^{v_2}}\right)$$

hold.

Analogously, for $\mathbf{p},\mathbf{q}$ two Zipf-Mandelbrot laws with parameters $N\in\{1,2,\ldots\}$, t_1, $t_2\geqslant 0$ and v_1, $v_2>0$, respectively, and $\mathbf{r}\in\mathbb{R}_+^n$, the generalized Rényi divergence has the following expression

$$Re_\alpha(\mathbf{q},\mathbf{p};\mathbf{r})=\frac{H_{N,t_1,v_1}^{\alpha-1}}{H_{N,t_2,v_2}^{\alpha}}\sum_{i=1}^{N}r_i\frac{(i+t_1)^{(\alpha-1)v_1}}{(i+t_2)^{\alpha v_2}}\ ,\ \ \alpha\in\langle 1,+\infty\rangle. \tag{43}$$

The following results hold.

COROLLARY 4. *Let* $\mathbf{p},\mathbf{q}$ *be two Zipf-Mandelbrot laws with parameters* $N\in\{1,2,\ldots\}$, t_1, $t_2\geqslant 0$ *and* $v_1,v_2>0$, *respectively, and* $\mathbf{r}\in\mathbb{R}_+^n$. *If* $Re_\alpha(\mathbf{q},\mathbf{p};\mathbf{r})$ *is defined by* (43), *then inequalities*

$$\left|\frac{H_{N,t_2,v_2}^{\alpha}}{H_{N,t_1,v_1}^{\alpha-1}}Re_\alpha(\mathbf{q},\mathbf{p};\mathbf{r})-\left(\sum_{i=1}^{N}\frac{r_i}{(i+t_1)^{v_1}}\right)^{1-\alpha}\left(\sum_{i=1}^{N}\frac{r_i}{(i+t_2)^{v_2}}\right)^{\alpha}\right|$$
$$\leqslant\alpha\frac{(N+t_1)^{(\alpha-1)v_1}}{(1+t_2)^{(\alpha-1)v_2}}\sum_{i=1}^{N}\frac{r_i}{(i+t_1)^{v_1}}\left|\frac{(i+t_1)^{v_1}}{(i+t_2)^{v_2}}-\frac{\sum_{i=1}^{N}\frac{r_i}{(i+t_2)^{v_2}}}{\sum_{i=1}^{N}\frac{r_i}{(i+t_1)^{v_1}}}\right|,$$

and

$$\left|\frac{(1+t_1)^{\alpha v_1}}{(N+t_2)^{\alpha v_2}}\left(\frac{(N+t_1)^{v_1}}{(1+t_2)^{v_2}}\sum_{i=1}^{N}\frac{r_i}{(i+t_1)^{v_1}}-\sum_{i=1}^{N}\frac{r_i}{(i+t_2)^{v_2}}\right)\right.$$
$$+\frac{(N+t_1)^{\alpha v_1}}{(1+t_2)^{\alpha v_2}}\left(\sum_{i=1}^{N}\frac{r_i}{(i+t_2)^{v_2}}-\frac{(1+t_1)^{v_1}}{(N+t_2)^{v_2}}\sum_{i=1}^{N}\frac{r_i}{(i+t_1)^{v_1}}\right)$$
$$\left.-\frac{H_{N,t_2,v_2}^{\alpha}}{H_{N,t_1,v_1}^{\alpha-1}}\left(\frac{(N+t_1)^{v_1}}{(1+t_2)^{v_2}}-\frac{(1+t_1)^{v_1}}{(N+t_2)^{v_2}}\right)Re_\alpha(\mathbf{q},\mathbf{p};\mathbf{r})\right|$$
$$\leqslant 2\alpha\frac{(N+t_1)^{(\alpha-1)v_1}}{(1+t_2)^{(\alpha-1)v_2}}\sum_{i=1}^{N}\frac{r_i}{(i+t_1)^{v_1}}\left(\frac{(N+t_1)^{v_1}}{(1+t_2)^{v_2}}-\frac{(i+t_1)^{v_1}}{(i+t_2)^{v_2}}\right)$$
$$\times\left(\frac{(i+t_1)^{v_1}}{(i+t_2)^{v_2}}-\frac{(1+t_1)^{v_1}}{(N+t_2)^{v_2}}\right)$$

hold.

For $\mathbf{p},\mathbf{q}$ two Zipf-Mandelbrot laws with parameters $N\in\{1,2,\ldots\}$, t_1, $t_2\geqslant 0$ and v_1, $v_2>0$, respectively, and $\mathbf{r}\in\mathbb{R}^n_+$, the generalized χ^2-divergence has the following form

$$D_{\chi^2}(\mathbf{q},\mathbf{p};\mathbf{r})=H_{N,t_1,v_1}\sum_{i=1}^{N}r_i(i+t_1)^{v_1}\left(\frac{1}{(i+t_2)^{v_2}H_{N,t_2,v_2}}-\frac{1}{(i+t_1)^{v_1}H_{N,t_1,v_1}}\right)^2. \quad (44)$$

We have the following results.

COROLLARY 5. *Let* $\mathbf{p},\mathbf{q}$ *be two Zipf-Mandelbrot laws with parameters* $N\in\{1,2,\ldots\}$, t_1, $t_2\geqslant 0$ *and* $v_1,v_2>0$, *respectively, and* $\mathbf{r}\in\mathbb{R}^n_+$. *If* $D_{\chi^2}(\mathbf{q},\mathbf{p};\mathbf{r})$ *is defined by* (44), *then inequalities*

$$\left|D_{\chi^2}(\mathbf{q},\mathbf{p};\mathbf{r})-\frac{H_{N,t_1,v_1}}{\sum_{i=1}^{N}\frac{r_i}{(i+t_1)^{v_1}}}\left(\sum_{i=1}^{N}\frac{r_i}{(i+t_2)^{v_2}H_{N,t_2,v_2}}-\sum_{i=1}^{N}\frac{r_i}{(i+t_1)^{v_1}H_{N,t_1,v_1}}\right)^2\right|$$
$$\leqslant\frac{2}{H_{N,t_2,v_2}}\max\left\{\left|\frac{(1+t_1)^{v_1}H_{N,t_1,v_1}}{(N+t_2)^{v_2}H_{N,t_2,v_2}}-1\right|,\left|\frac{(N+t_1)^{v_1}H_{N,t_1,v_1}}{(1+t_2)^{v_2}H_{N,t_2,v_2}}-1\right|\right\}$$
$$\times\sum_{i=1}^{N}\frac{r_i}{(i+t_1)^{v_1}}\left|\frac{(i+t_1)^{v_1}}{(i+t_2)^{v_2}}-\frac{\sum_{i=1}^{N}\frac{r_i}{(i+t_2)^{v_2}}}{\sum_{i=1}^{N}\frac{r_i}{(i+t_1)^{v_1}}}\right|$$

and

$$\left|\sum_{i=1}^{N}\frac{r_i}{(i+t_1)^{v_1}}\left[\left(\frac{(N+t_1)^{v_1}}{(1+t_2)^{v_2}}-\frac{\sum_{i=1}^{N}\frac{r_i}{(i+t_2)^{v_2}}}{\sum_{i=1}^{N}\frac{r_i}{(i+t_1)^{v_1}}}\right)\left(\frac{(1+t_1)^{v_1}H_{N,t_1,v_1}}{(N+t_2)^{v_2}H_{N,t_2,v_2}}-1\right)^2\right.\right.$$
$$\left.+\left(\frac{\sum_{i=1}^{N}\frac{r_i}{(i+t_2)^{v_2}}}{\sum_{i=1}^{N}\frac{r_i}{(i+t_1)^{v_1}}}-\frac{(1+t_1)^{v_1}}{(N+t_2)^{v_2}}\right)\left(\frac{(N+t_1)^{v_1}H_{N,t_1,v_1}}{(1+t_2)^{v_2}H_{N,t_2,v_2}}-1\right)^2\right]$$
$$\left.-D_{\chi^2}(\mathbf{q},\mathbf{p};\mathbf{r})H_{N,t_1,v_1}\left(\frac{(N+t_1)^{v_1}}{(1+t_2)^{v_2}}-\frac{(1+t_1)^{v_1}}{(N+t_2)^{v_2}}\right)\right|$$
$$\leqslant 4\left(\frac{H_{N,t_1,v_1}}{H_{N,t_2,v_2}}\right)^2\max\left\{\left|\frac{(1+t_1)^{v_1}H_{N,t_1,v_1}}{(N+t_2)^{v_2}H_{N,t_2,v_2}}-1\right|,\left|\frac{(N+t_1)^{v_1}H_{N,t_1,v_1}}{(1+t_2)^{v_2}H_{N,t_2,v_2}}-1\right|\right\}$$
$$\times\sum_{i=1}^{N}\frac{r_i}{(i+t_1)^{v_1}}\left(\frac{(N+t_1)^{v_1}}{(1+t_2)^{v_2}}-\frac{(i+t_1)^{v_1}}{(i+t_2)^{v_2}}\right)\left(\frac{(i+t_1)^{v_1}}{(i+t_2)^{v_2}}-\frac{(1+t_1)^{v_1}}{(N+t_2)^{v_2}}\right)$$

hold.

Finally, if $\mathbf{p}$ is the Zipf-Mandelbrot law with parameters $N\in\{1,2,\ldots\}$, $t_1\geqslant 0$ and $v_1>0$, and $\mathbf{r}\in\mathbb{R}^n_+$, then the generalized Shannon entropy $H(\mathbf{p};\mathbf{r})$ has the following representation

$$H(\mathbf{p};\mathbf{r})=\frac{1}{H_{N,t_1,v_1}}\sum_{i=1}^{N}\frac{r_i}{(i+t_1)^{v_1}}\log\left[(i+t_1)^{v_1}H_{N,t_1,v_1}\right]. \quad (45)$$

We have the following results.

COROLLARY 6. *Let* $\mathbf{p}$ *be the Zipf-Mandelbrot law with parameters* $N \in \{1,2,\ldots\}$, $t_1 \geqslant 0$ *and* $v_1 > 0$, *and* $\mathbf{r} \in \mathbb{R}_+^n$. *If* $H(\mathbf{p};\mathbf{r})$ *is defined by* (45), *then inequalities*

$$\left| H_{N,t_1,v_1} H(\mathbf{p};\mathbf{r}) - \sum_{i=1}^{N} \frac{r_i}{(1+t_1)^{v_1}} \log\left(H_{N,t_1,v_1} \frac{\sum_{i=1}^{N} r_i}{\sum_{i=1}^{N} \frac{r_i}{(1+t_1)^{v_1}}} \right) \right| \leqslant \frac{1}{(1+t_1)^{v_1}} \sum_{i=1}^{N} \frac{r_i}{(1+t_1)^{v_1}} \left| (1+t_1)^{v_1} - \frac{\sum_{i=1}^{N} r_i}{\sum_{i=1}^{N} \frac{r_i}{(1+t_1)^{v_1}}} \right|$$

and

$$\left| \left[(N+t_1)^{v_1} - \frac{\sum_{i=1}^{N} r_i}{\sum_{i=1}^{N} \frac{r_i}{(i+t_1)^{v_1}}} \right] f\left((1+t_1)^{v_1} H_{N,t_1,v_1}\right) + \left[\frac{\sum_{i=1}^{n} r_i}{\sum_{i=1}^{n} \frac{r_i}{(i+t_1)^{v_1}}} - (1+t_1)^{v_1} \right] \times f\left((N+t_1)^{v_1} H_{N,t_1,v_1}\right) - \frac{H_{N,t_1,v_1}}{\sum_{i=1}^{N} \frac{r_i}{(i+t_1)^{v_1}}} \left[(N+t_1)^{v_1} - (1+t_1)^{v_1} \right] H(\mathbf{p};\mathbf{r}) \right| \leqslant \frac{2}{(1+t_1)^{v_1} \sum_{i=1}^{N} \frac{r_i}{(i+t_1)^{v_1}}} \times \sum_{i=1}^{N} \frac{r_i}{(i+t_1)^{v_1}} \left[(N+t_1)^{v_1} - (i+t_1)^{v_1} \right] \left[(i+t_1)^{v_1} - (1+t_1)^{v_1} \right]$$

hold.

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