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PROPERTIES OF SOME SUBSEQUENCES OF THE WALSH–KACZMARZ–DIRICHLET KERNELS

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Abstract. We study some properties of a family of subsequences of the Walsh-Kaczmarz-Dirichlet kernels. We prove properties related to the L^1 norm of the weighted maximal function and to the Fejér means of partial sums of Fourier series obtained by convolution with integrable functions.

1. Introduction

Let $\mathbb{Z}_2$ denote the discrete cyclic group $\mathbb{Z}_2 = \{0,1\}$, where the group operation is addition modulo 2. If $\mu(E)$ denotes the measure of the subset $E \subset \mathbb{Z}_2$, then $\mu(\{0\}) = \mu(\{1\}) = \frac{1}{2}$.

The dyadic group G is obtained by $G = \prod_{i=0}^{\infty} \mathbb{Z}_2$ (see [8]), where topology and measure are obtained from the product. The notation $\mu(E)$ is used for the probability measure for subsets E of the dyadic group G.

Let $x = (x_n)_{n\geqslant 0} \in G$. The sets $I_n(x) := \{y \in G : y_0 = x_0, \dots, y_{n-1} = x_{n-1}\}$, $n \geqslant 1$ and $I_0(x) := G$ are dyadic intervals of G. Let $I_n = I_n(0)$, and $e_n := (\delta_{in})_i$. It is easily seen that $(I_n)_n$ is a decreasing sequence of subgroups. Moreover, for every $x \in G$ and every nonnegative integer n, it can be seen that $\mu(I_n(x)) = \frac{1}{2^n}$.

Since every nonnegative integer i can be written in the form $i = \sum_{k=0}^{\infty} i_k 2^k$, where $i_k \in \{0,1\}$, we define the sequence $(z_i)_{i\geqslant 0}$ of elements from G by

$$z_i = \sum_{k=0}^{\infty} i_k e_k.$$

It is easily seen that for each positive integer n, the set $\{z_i, i < 2^n\}$ is a set of representatives of I_n-cosets.

The Walsh-Paley system is defined as the set of Walsh-Paley functions:

$$\omega_n(x) = \prod_{k=0}^{\infty} (r_k(x))^{n_k},\ n \in \mathbb{N},\ x \in G,$$

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where $n = \sum_{k=0}^{\infty} n_k 2^k$, $n_k \in \{0,1\}$ and $r_k(x) = (-1)^{x_k}$. The n-th Walsh-Kaczmarz function is

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k},$$

where $|n| = \max\{k : n_k \neq 0\}$.

For every positive integer A, define the transformation $\tau_A : G \to G$ by the formula

$$\tau_A(x) = (x_{A-1}, x_{A-2}, \ldots, x_1, x_0, x_A, x_{A+1}, \ldots), \; \forall x = (x_n)_{n \geqslant 0}.$$

It is clear that τ satisfies the property $\tau_A(\tau_A(x)) = x$, for every $x \in G$. Moreover, it can be easily seen that κ_n and ω_n are tied by the relation

$$\kappa_n(x) = r_{|n|}(x)\omega_n(\tau_{|n|}(x)), \quad \forall n \in \mathbb{N}.$$

The Dirichlet kernel functions with respect to the Walsh and Kaczmarz systems are respectively defined by the formulae

$$D_n(x) = \sum_{k=0}^{n-1} \omega_k(x), \quad D_n^{\kappa}(x) = \sum_{k=0}^{n-1} \kappa_k(x), \quad \forall n \in \mathbb{N}, \; x \in G.$$

We use the notation $S_n f$, $n \geqslant 1$, for partial sums of any function $f \in L^1(G)$ relative to the Kaczmarz system. Namely,

$$S_n f(y) = D_n^{\kappa} * f(y) = \int D_n^{\kappa}(y-x) f(x)dx, \; \forall y \in G.$$

It is known that for every nonnegative integer n

$$D_{2^n}(x) = D_{2^n}^{\kappa}(x) = \begin{cases} 2^n, \; x \in I_n; \\ 0, \quad x \in I \setminus I_n. \end{cases} \tag{1}$$

Hence, it can be seen that for every nonnegative integer n,

$$S_{2^n} f(y) = \int_{I_n(y)} f(x)dx, \; \forall y \in G.$$

Namely, $S_{2^n} f$, $n \in \mathbb{N}$ represent the mean values of f. In [7] it was noticed that the Dirichlet kernel function with respect to the Kaczmarz system can be written in the form

$$D_n^{\kappa}(x) = D_{2^{|n|}}(x) + r_{|n|}(x) D_{n-2^{|n|}}(\tau_{|n|}(x)). \tag{2}$$

In the first part of this paper Proposition 1 describes the weighted maximal function of the subsequence of Dirichlet kernels with respect to the Kaczmarz system, generated by the sequence $(\beta(n))_n$ of positive integers. This subsequence keeps the properties of the whole sequence of Dirichlet kernels concerning the L^1 norm of the weighted maximal function proved in [6, Proposition 3]. Its specificity is the fact that the sequence $(\beta(2n)+\beta(2n+1))_n$ generates a bounded maximal function of Dirichlet kernels, which is a contrast considering the fact that the integrability of the weighted maximal function $\sup\limits_n \frac{|D_n^\kappa(x)|}{\alpha(|n|)}$ depends on the convergence of the series $\sum\limits_{A=0}^{\infty} \frac{A}{\alpha(A)}$.

[4, Theorem 1] provides sufficient conditions on subsequences of Dirichlet kernels whose convolution with any integrable function converges almost everywhere. The second part of this work deals with the same question in Theorem 1, with respect to the Kaczmarz system. Our additional condition concerning the Kaczmarz system is that the sequence of positive numbers $(\alpha(n))_n$ generating the subsequence of Dirichlet kernels $(D_{\alpha(n)}^\kappa)_n$, does not grow too fast. Both of Proposition 1 and Theorem 1 are based on Lemma 1. Formula (3) is given in Lemma 1 and provides a new expression of the Dirichlet kernel with respect to the Kaczmarz system.

Throughout the paper the notation C denotes an absolute positive constant which may vary in different contexts. It is used to express the boundedness of some estimated quantities.

2. On the integrability of some weighted maximal functions

The first lemma gives a new representation of the Dirichlet kernel with respect to the Kaczmarz system. Namely, D_n^κ can be written as a linear combination of characteristic functions of the form given in formula (1). This property will enable us to estimate in an easy way the operators defined in Theorem 1, which is the main result in this paper.

LEMMA 1. *Let n be a positive integer having the dyadic representation $n = 2^{N_1} + \ldots + 2^{N_t}$, where $t \geqslant 2$, $N_1 < N_2 < \ldots < N_t$ and $N_t = |n|$. Then, $D_n^\kappa(x)$ can be written in the form*

$$D_n^\kappa(x) = D_{2^{N_t}}(x) + \sum_{j=0}^{2^{N_t}-1} A_{n,j}\left(D_{2^{N_t+1}}(x+z_j) - D_{2^{N_t}}(x+z_j)\right), \tag{3}$$

where for every $1 \leqslant j < 2^{N_t-N_1}$,

$$A_{n,j} := \begin{cases} \sum_{s=1}^{i} 2^{N_s-N_t} r_{N_t-N_{t-1}-1}(z_j)\ldots r_{N_t-N_{s+1}-1}(z_j), & i \in \{1,\ldots,t-2\}; \\ \sum_{s=1}^{t-2} 2^{N_s-N_t} r_{N_t-N_{t-1}-1}(z_j)\ldots r_{N_t-N_{s+1}-1}(z_j) + 2^{N_{t-1}-N_t}, & i = t-1,\ t \geqslant 3; \\ 2^{N_1-N_2}, & i = t-1,\ t = 2. \end{cases} \tag{4}$$

Where $i \in \{1,\ldots,t-1\}$ is the unique integer satisfying $j \in \{2^{N_t-N_{i+1}},\ldots,2^{N_t-N_i}-1\}$. For $j=0$, then $A_{n,0} := \sum_{s=1}^{t-1} 2^{N_s-N_t}$ and for $2^{N_t-N_1} \leqslant j < 2^{N_t}$, $A_{n,j} = 0$. Moreover, it can be easily seen that

$$|A_{n,j}| \leqslant 2^{N_i-N_t+1} \leqslant \frac{2}{j+1}. \tag{5}$$

Proof. First notice that for all $0 < m < s$, we have

$$(D_{2^m} \circ \tau_s)(x) = 2^{m-s} \sum_{j=0}^{2^{s-m}-1} D_{2^s}(x+z_j). \tag{6}$$

Besides, it was proved in [8] that

$$D_{n-2^{N_t}} = \sum_{i=1}^{t-1} r_{N_{t-1}} \dots r_{N_{i+1}} D_{2^{N_i}}. \tag{7}$$

Hence, combining (2), (7) then (6) we get

$$\begin{aligned}
D_n^{\kappa}(x) &= D_{2^{N_t}}(x) + r_{N_t}(x) \sum_{i=1}^{t-1} (r_{N_{t-1}} \circ \tau_{N_t})(x) \dots (r_{N_{i+1}} \circ \tau_{N_t})(x) \cdot (D_{2^{N_i}} \circ \tau_{N_t})(x) \\
&= D_{2^{N_t}}(x) + r_{N_t}(x) \sum_{i=1}^{t-1} 2^{N_i - N_t} r_{N_t - N_{t-1} - 1}(x) \dots r_{N_t - N_{i+1} - 1}(x) \sum_{j=0}^{2^{N_t - N_i} - 1} D_{2^{N_t}}(x+z_j) \\
&= D_{2^{N_t}}(x) + \sum_{i=1}^{t-1} 2^{N_i - N_t} r_{N_t - N_{t-1} - 1}(x) \dots r_{N_t - N_{i+1} - 1}(x) \\
&\quad \cdot \sum_{j=0}^{2^{N_t - N_i} - 1} \left(D_{2^{N_t+1}}(x+z_j) - D_{2^{N_t}}(x+z_j)\right) \\
&= D_{2^{N_t}}(x) + \sum_{i=1}^{t-1} \sum_{j=2^{N_t - N_{i+1}}}^{2^{N_t - N_i} - 1} \sum_{s=1}^{i} 2^{N_s - N_t} r_{N_t - N_{t-1} - 1}(z_j) \dots r_{N_t - N_{s+1} - 1}(z_j) \\
&\quad \cdot \left(D_{2^{N_t+1}}(x+z_j) - D_{2^{N_t}}(x+z_j)\right) + \sum_{s=1}^{t-1} 2^{N_s - N_t} \left(D_{2^{N_t+1}}(x) - D_{2^{N_t}}(x)\right),
\end{aligned}$$

where the second equality holds because $r_{N_t}(z_j) = 1$, for all $j < 2^{N_t} - 1$, and the third equality is obtained since for all $j < 2^{N_t} - 1$ and $s < N_t$, $r_s(x) = r_s(z_j)$ if $D_{2^{N_t}}(x+z_j)$ does not vanish. The result follows by applying the definition of $A_{n,j}$, $j \leqslant 2^{N_t} - 1$.

The following example gives an illustration of formula (3). It expresses the Dirichlet kernel D_n^{κ} as a linear combination of locally constant functions having mutually disjoint supports.

EXAMPLE 1. Let $n = 2 + 2^3 + 2^4$. Following the notations of Lemma 1, we can see that $N_1 = 1$, $N_2 = 3$ and $N_t = N_3 = 4$, where $t = 3$.

For $i = 2 = t-1$, the set $\left\{1, \dots, 2^t - 2^{t-1} - 1\right\}$ only contains $j = 1$. Therefore, (4) has the expression

$$\begin{aligned}
A_{n,1} &:= \sum_{s=1}^{1} 2^{N_s - N_t} r_{N_t - N_{t-1} - 1}(z_j) \dots r_{N_t - N_{s+1} - 1}(z_j) + 2^{N_2 - N_3} \\
&= 2^{N_1 - N_3} r_{N_3 - N_2 - 1}(z_1) + 2^{N_2 - N_3} = 2^{-3} r_0(z_1) + 2^{-1}.
\end{aligned}$$

Similarly, for $i=1$, we have $j\in\{2,\dots,2^3-1\}$, because

$$N_t-N_{i+1}=N_3-N_2=1,\ N_t-N_i=N_3-N_1=3.$$

Therefore, for every $j\in\{2,\dots,2^3-1\}$, (4) takes the form

$$A_{n,j}=\sum_{s=1}^{1}2^{N_s-N_t}r_{N_t-N_{t-1}-1}(z_j)\dots r_{N_t-N_{s+1}-1}(z_j)=2^{N_1-N_3}r_{N_3-N_2-1}(z_j)=2^{-3}r_0(z_j).$$

Besides,

$$A_{n,0}=2^{N_1-N_3}+2^{N_2-N_3}=2^{-3}+2^{-1}.$$

Since, $z_0=0$, it follows that

$$\begin{aligned}D_n^{\kappa}(x)&=D_{2^4}(x)+(2^{-3}+2^{-1})\,(D_{2^5}(x)-D_{2^4}(x))+(2^{-3}r_0(z_1)+2^{-1})\\&\quad\cdot(D_{2^5}(x+z_1)-D_{2^4}(x+z_1))+\sum_{j=2}^{2^3-1}2^{-3}r_0(z_j)\,(D_{2^5}(x+z_j)-D_{2^4}(x+z_j)).\end{aligned}$$

Notice that the locally constant functions $D_{2^4}(x)+(2^{-3}+2^{-1})\,(D_{2^5}(x)-D_{2^4}(x))$, $D_{2^5}(x+z_j)-D_{2^4}(x+z_j)$, $j\in\{1,\dots,2^3-1\}$, have mutually disjoint supports.

As said in the introduction, the following proposition discusses integrability of the functions $\sup_n\frac{|D^{\kappa}_{\beta(2n)}(x)|}{\alpha(2n)}$, $\sup_n\frac{|D^{\kappa}_{\beta(2n+1)}(x)|}{\alpha(2n+1)}$, and $\sup_n\frac{|D^{\kappa}_{\beta(2n)+\beta(2n+1)}(x)|}{\alpha(2n+1)}$. It proves that the integrability of the two first functions, which are generated from the sequences $\beta(2n)$ and $\beta(2n+1)$, depend on the convergence of the series $\sum_{A=0}^{\infty}\frac{A}{\alpha(A)}$. The third function, which is generated from the sum of the two previous sequences $\beta(2n)+\beta(2n+1)$, is always bounded and its integrability does not depend on the series $\sum_{A=0}^{\infty}\frac{A}{\alpha(A)}$.

PROPOSITION 1. *Define the sequence $(\beta(n))_n$ by $\beta(2n)=\sum_{k=0}^{n}2^{2k}$ and $\beta(2n+1)=\sum_{k=0}^{n}2^{2k+1}$, for all $n\geqslant 0$. Let $(\alpha(n))_n$ be any increasing sequence of positive integers. Then, we have*

1. $\sup_n\max\left\{\frac{|D^{\kappa}_{\beta(2n)}(x)|}{\alpha(2n)},\frac{|D^{\kappa}_{\beta(2n+1)}(x)|}{\alpha(2n+1)}\right\}\in L^1(G)$ *if and only if the series* $\sum_{A=0}^{\infty}\frac{A}{\alpha(A)}$ *converges.*

2. *The function* $\sup_n\frac{|D^{\kappa}_{\beta(2n)+\beta(2n+1)}(x)|}{\alpha(2n+1)}$ *is bounded.*

Proof. It was proved in [6, Proposition 3] that $\sup_n \frac{|D_n^\kappa(x)|}{\alpha(|n|)} \in L^1(G)$ if and only if the series $\sum_{A=0}^{\infty} \frac{A}{\alpha(A)}$ converges. More precisely, according to formula (2), the sequence of functions $(L_{\alpha,N})_N$ introduced in the proof of [6, Proposition 3] can be expressed by

$$L_{\alpha,N} = \max\left\{\sup_{n\leqslant N}\max_{0\leqslant t<n} \frac{|D^\kappa_{2^n+2^t}(x) - D_{2^n}|}{\alpha(n)}, \frac{|D_{2^n}|}{\alpha(n)}\right\}.$$

Therefore, applying [6, Proposition 1], we can see from the proof of [6, Proposition 3] that the L^1 norm of the maximal function $\sup_n \max_{0\leqslant t<n} \frac{|D^\kappa_{2^n+2^t}(x)-D_{2^n}|}{\alpha(n)}$ is finite if and only if the series $\sum_{A=0}^{\infty} \frac{A}{\alpha(A)}$ converges.

Following the same techniques, in order to prove assertion (1), it suffices to prove that for all $x \in G$ and $n \geqslant 1$

$$\max_{0\leqslant t<2n} |D^\kappa_{2^{2n}+2^t}(x) - D_{2^{2n}}(x)| \leqslant 2|D^\kappa_{\beta(2n)}(x) - D_{2^{2n}}|, \tag{8}$$

and

$$\max_{0\leqslant t<2n+1} |D^\kappa_{2^{2n+1}+2^t}(x) - D_{2^{2n+1}}(x)| \leqslant 2|D^\kappa_{\beta(2n+1)}(x) - D_{2^{2n+1}}(x)|. \tag{9}$$

According to formula (3), where we can see that $D_n^\kappa(x) - D_{2^{|n|}}(x)$ is expressed as a linear combination of characteristic functions of mutually disjoint dyadic intervals, it suffices to prove that

$$\max_{0\leqslant t<2n} | A_{2^{2n}+2^t,j} | \leqslant 2 \,| A_{\beta(2n),j} |, \forall j < 2^{2n},$$

and

$$\max_{0\leqslant t<2n+1} | A_{2^{2n+1}+2^t,j} | \leqslant 2 \,| A_{\beta(2n+1),j} |, \forall j < 2^{2n+1}.$$

Let $x \in G$ and $j < 2^{2n}$ be such that $x \in I_{2n}(z_j)$. First assume that $2^{2n-2k-1} \leqslant j < 2^{2n-2k}$, for some $k : 1 \leqslant k \leqslant n-1$. Applying formula (4) we can see that for all $t < 2n$ such that $2^{2n-t} > j$, $A_{2^{2n}+2^t,j} = 2^{t-2n}$. Besides, according to the definition of $A_{n,j}$ introduced in Lemma 1, $A_{2^{2n}+2^t,j} = 0$, for all $t < 2n$ such that $j \geqslant 2^{2n-t}$. Hence,

$$\max_{0\leqslant t<2n} | A_{2^{2n}+2^t,j} | = 2^{2k-2n}.$$

Using the notations of Lemma 1 we can see that $\beta(2n) = \sum_{s=1}^{t} 2^{N_s}$, where $N_s = 2(s-1)$ and $t = n+1$. In this case, the unique integer i satisfying $2^{N_t-N_{i+1}} \leqslant j < 2^{N_t-N_i}$ is $i = k+1$. Hence, (4) takes the form

$$A_{\beta(2n),j} = \sum_{s=1}^{k+1} 2^{2(s-1)-2n} r_1(z_j) r_3(z_j) \ldots r_{2n-2s-1}(z_j).$$

For $s<k$ it is clear that $r_{2n-2k+1}(z_j)=\ldots=r_{2n-2s-1}(z_j)=1$, hence

$$\begin{aligned}
&|A_{\beta(2n),j}| \\
&=\left|\sum_{s=1}^{k}2^{2(s-1)-2n}r_1(z_j)r_3(z_j)\ldots r_{2n-2k-1}(z_j)+2^{2k-2n}r_1(z_j)r_3(z_j)\ldots r_{2n-2k-3}(z_j)\right| \\
&=\left|r_{2n-2k-1}(z_j)\sum_{s=1}^{k}2^{2(s-1)-2n}+2^{2k-2n}\right| \\
&=-\sum_{s=1}^{k}2^{2(s-1)-2n}+2^{2k-2n}>-2^{2k-1-2n}+2^{2k-2n} \\
&=2^{2k-1-2n}=\frac{1}{2}\max_{0\leqslant t<2n}|A_{2^{2n}+2^t,j}|.
\end{aligned}$$

In this way we have proved inequality (8) for $2^{2n-2k-1}\leqslant j<2^{2n-2k}$. In a similar way we can see that for all $t<2n+1$ such that $2^{2n-t}>j$, $A_{2^{2n+1}+2^t,j}=2^{t-2n-1}$. Hence,

$$\max_{0\leqslant t<2n+1}|A_{2^{2n+1}+2^t,j}|=2^{2k-2n}.$$

Proceeding as in the estimation of $|A_{\beta(2n),j}|$ we can see that $\beta(2n+1)=\sum_{s=1}^{t}2^{N_s}$, where $N_s=2s-1$ and $t=n+1$. In this case, the unique integer i satisfying $2^{N_t-N_{i+1}}\leqslant j<2^{N_t-N_i}$ is $i=k+1$. We get

$$\begin{aligned}
|A_{\beta(2n+1),j}|=\Bigg|&\sum_{s=1}^{k}2^{2s-1-2n-1}r_1(z_j)r_3(z_j)\ldots r_{2n-2k-1}(z_j) \\
&+2^{2k-2n}r_1(z_j)r_3(z_j)\ldots r_{2n-2k-3}(z_j)\Bigg|=|A_{\beta(2n),j}|.
\end{aligned}$$

Hence, (9) is obtained for $2^{2n-2k-1}\leqslant j<2^{2n-2k}$.

Now, we estimate $|A_{\beta(2n),j}|$ for $2^{2n-2k}\leqslant j<2^{2n-2k+1}$, for some $k:1\leqslant k\leqslant n-1$. First we have

$$\max_{0\leqslant t<2n}|A_{2^{2n}+2^t,j}|=2^{2k-1-2n}.$$

As seen above, since $j<2^{2n-2k+1}<2^{2n-2(k-1)}$, we have that $2^{N_t-N_{i+1}}\leqslant j<2^{N_t-N_i}$ for $i=k$. Therefore,

$$\begin{aligned}
&|A_{\beta(2n),j}| \\
&=\left|\sum_{s=1}^{k-1}2^{2(s-1)-2n}r_1(z_j)r_3(z_j)\ldots r_{2n-2k+1}(z_j)+2^{2k-2-2n}r_1(z_j)r_3(z_j)\ldots r_{2n-2k-1}(z_j)\right| \\
&=\left|r_{2n-2k+1}(z_j)\sum_{s=1}^{k-1}2^{2(s-1)-2n}+2^{2k-2n-2}\right|=\sum_{s=1}^{k-1}2^{2(s-1)-2n}+2^{2k-2n-2} \\
&>2^{2k-2-2n}=\frac{1}{2}\max_{0\leqslant t<2n}|A_{2^{2n}+2^t,j}|.
\end{aligned}$$

Hence, (8) is proved for $2^{2n-2k} \leqslant j < 2^{2n-2k+1}$. The verification of (9) for $2^{2n-2k} \leqslant j < 2^{2n-2k+1}$ is left to the reader.

In order to prove assertion (2), notice that $\beta(2n)+\beta(2n+1) = \sum_{s=1}^{t} 2^{N_s}$, where $N_s = s-1$ and $t = 2n+2$. Hence, (4) takes the form

$$A_{\beta(2n)+\beta(2n+1),j} = \sum_{s=1}^{i} 2^{s-1-2n-1} r_0(z_j) r_1(z_j) \ldots r_{2n+1-s-1}(z_j),$$

if $2^{2n+1-i} \leqslant j < 2^{2n+1-(i-1)}$. For all $s < i-1$, we have $r_{2n-(i-2)}(z_j) = \ldots = r_{2n-s}(z_j) = 1$. It follows

$$\begin{aligned}
&|A_{\beta(2n)+\beta(2n+1),j}| \\
&= \left| \sum_{s=1}^{i-1} 2^{s-2n-2} r_0(z_j) r_1(z_j) \ldots r_{2n+1-i}(z_j) + 2^{i-2n-2} r_0(z_j) r_1(z_j) \ldots r_{2n-i}(z_j) \right| \\
&= \left| r_{2n+1-i}(z_j) \sum_{s=1}^{i-1} 2^{s-2n-2} + 2^{i-2n-2} \right| \\
&= -\sum_{s=1}^{i-1} 2^{s-2n-2} + 2^{i-2n-2} = 2^{-2n-1}.
\end{aligned}$$

Therefore,

$$|D^{\kappa}_{\beta(2n)+\beta(2n+1)}(x)| \leqslant 1, \forall x \in G, \forall n \geqslant 0.$$

3. Fejér means of some subsequences of partial sums of Fourier series

We recall the Calderón-Zygmund decomposition lemma proved in [8], [5, Theorem 1] and [5, Lemma 2]. It is used in the proof of Theorem 1.

LEMMA 2. ([8], [5, Lemma 2]) *Let $f \in L^1(G)$ be such that $\|f\|_1 < \lambda$. Then, there exist a decomposition $f = \sum_{j=0}^{\infty} f_j$ and mutually disjoint intervals $(I_{k_j}(u_j))_j$ such that the function f_j is supported on $I_{k_j}(u_j)$ for every $j \geqslant 1$, $\mu(F_\lambda) \leqslant C\frac{\|f\|_1}{\lambda}$, where $F_\lambda := \biguplus_{j=1}^{\infty} I_{k_j}(u_j)$, $\int_{I_{k_j}(u_j)} f_j = 0$, $\forall j \geqslant 1$, $\lambda < \frac{1}{\mu(I_{k_j}(u_j))} \int_{I_{k_j}(u_j)} |f_j| \leqslant C\lambda$, $\forall j \geqslant 1$, and $\|\limsup_n S_{2^n}|f_0|\|_\infty \leqslant C\lambda$.*

REMARK 1. It can be seen in the proof of [5, Lemma 2] that the constant C, mentioned in Lemma 2, does not depend on the positive number λ. Indeed, using the notations of Lemma 2, it can be seen that $\mu(F_\lambda) \leqslant \frac{\|f\|_1}{\lambda}$, $\frac{1}{\mu(I_{k_j}(u_j))} \int_{I_{k_j}(u_j)} |f_j| \leqslant 4\lambda$ and $\|\limsup_n S_{2^n}|f_0|\|_\infty \leqslant 3\lambda$.

REMARK 2. It can be seen in the proof of [5, Lemma 2] that the mutually disjoint intervals $(I_{k_j}(u_j))_j$, introduced in Lemma 2, are such that

$$\lambda < \frac{1}{\mu(I_{k_j}(u_j))} \int_{I_{k_j}(u_j)} |f| \leqslant 2\lambda .$$

The following remark provides an additional estimate of the mean values of the function $|f|$ on the set F_λ^c. It will be used in the proof of Lemma 3.

REMARK 3. Using the notations of Lemma 2, it can be easily seen that

$$\|S_{2^n}|f|\cdot 1_{F_\lambda^c}\|_\infty \leqslant C\lambda, \ \forall n \geqslant 0, \tag{10}$$

where 1_M denotes the characteristic function of the subset $M \subseteq G$, and $M^c = G\backslash M$ denotes its complement set in G. Indeed, let n be some fixed arbitrary nonnegative integer and $x \in F_\lambda^c$. If $j \in \mathbb{N}$ is such that $I_n(x)$ and $I_{k_j}(u_j)$ are disjoint, then we obviously have that

$$S_{2^n}|f_j|(x) = 2^n \int_{I_n(x)} |f_j| = 0.$$

On the other hand if $I_n(x)$ and $I_{k_j}(u_j)$ intersect, then we must have that $I_{k_j}(u_j) \subset I_n(x)$ because $x \in F_\lambda^c \subseteq (I_{k_j}(u_j))^c$. Hence, we have

$$\begin{aligned}
S_{2^n}|f|(x) &= 2^n \int_{I_n(x)} |f| = 2^n \int_{I_n(x)} |\sum_{j=0}^{\infty} f_j| \leqslant 2^n \sum_{j=0}^{\infty} \int_{I_n(x)} |f_j| \\
&\leqslant 2^n \sum_{\substack{j\geqslant 1 \\ I_{k_j}(u_j)\subset I_n(x)}} \int_{I_n(x)} |f_j| + 2^n \int_{I_n(x)} |f_0| \\
&\leqslant 2^n \sum_{\substack{j\geqslant 1 \\ I_{k_j}(u_j)\subset I_n(x)}} \int_{I_{k_j}(u_j)} |f_j| + S_{2^n}|f_0|(x) \\
&\leqslant 2^n C\lambda \sum_{\substack{j\geqslant 1 \\ I_{k_j}(u_j)\subset I_n(x)}} \mu(I_{k_j}(u_j)) + S_{2^n}|f_0|(x) \\
&\leqslant 2^n C\lambda \mu\left(\bigcup_{\substack{j\geqslant 1 \\ I_{k_j}(u_j)\subset I_n(x)}} I_{k_j}(u_j) \right) + S_{2^n}|f_0|(x) \\
&\leqslant 2^n C\lambda \mu(I_n(x)) + S_{2^n}|f_0|(x) \leqslant C\lambda ,
\end{aligned}$$

for some conveniently chosen constant C independent on the choice of n and λ as seen in Remark 1.

In the following lemma, for every $k \geqslant 1$, we form a collection $F_{k,\lambda}$ of mutually disjoint intervals analogue to those introduced in Lemma 2.

LEMMA 3. *Let $f \in L^1(G)$ be such that $\|f\|_1 < \lambda$. For every $k \geqslant 1$, there exists a collection $F_{k,\lambda}$, of mutually disjoint intervals $(I^k_{k_j}(u^k_j))_{j\geqslant 1}$, satisfying*

$$\mu\left(\bigcup_{J\in F_{k,\lambda}} J\right) \leqslant \frac{1}{2^k\lambda}\sum_{J\in F_{k,\lambda}}\int_J |f|, \forall k \geqslant 0, \tag{11}$$

and

$$\|S_{2^n}|f|\cdot 1_{(\bigcup_{J\in F_{k,\lambda}} J)^c}\|_\infty \leqslant C2^k\lambda,\ \forall n \geqslant 0, \tag{12}$$

where the constant C is independent on the choice of λ, k and n.

Proof. For every positive integer k, define the appropriate mutually disjoint intervals $(I^k_{k_j}(u^k_j))_{j\geqslant 1}$ corresponding to the intervals introduced in Lemma 2, by replacing λ with $2^k\lambda$. Then, if we define the collection $F_{k,\lambda} = \{I^k_{k_j}(u^k_j), j \geqslant 1\}$, using Remark 2, where λ is replaced with $2^k\lambda$, we obtain (11), because

$$\mu\left(\bigcup_{J\in F_{k,\lambda}} J\right) = \sum_{J\in F_{k,\lambda}}\mu(J) \leqslant \frac{1}{2^k\lambda}\sum_{J\in F_{k,\lambda}}\int_J |f|, \forall k \geqslant 0.$$

Similarly, combining Remark 3 and Remark 1 gives (12).

In [4, Theorem 1], G.Gát proves sufficient conditions on subsequences of Dirichlet kernels, related to the Walsh system, whose convolution with any function $f \in L^1(G)$ converges almost everywhere. The following result deals with the same question concerning the Kaczmarz system. The latter structure has some specific properties with regards to the Walsh system which makes it impossible to apply Gát's method. Our techniques require an additional condition which is expressed in formula (13). It describes the growth of the sequence of positive numbers $(\alpha(n))_n$ generating the subsequence of Dirichlet kernels $(D^\kappa_{\alpha(n)})_n$.

THEOREM 1. *Let $f \in L^1(G)$ and $(\alpha(n))_n$ be an increasing sequence of positive integers satisfying $\alpha(n+1) \geqslant q\alpha(n)$, for some $q > 1$ and*

$$|\alpha(n)| - |\alpha(m)| \leqslant Cn^{\beta-1}(n-m), \tag{13}$$

for some $1 < \beta < \frac{3}{2}$ and for all $n > m \geqslant 1$. Then,

$$\frac{1}{s}\sum_{n=1}^{s} D^\kappa_{\alpha(n)} * f \to f, \qquad s \to \infty,$$

almost everywhere.

Proof. Following the techniques used in the proof of [4, Theorem 1]), it suffices to prove the result for $q \geqslant 2$.

If we prove that the operator $\sup_s \frac{1}{s}\sum_{n=1}^{s} D^{\kappa}_{\alpha(n)} * f$ is of weak type $(1,1)$, then the result can be deduced by means of some standard arguments (see for example [4, Theorem 1]).

Applying formula (3) on the elements of the sequence $(\alpha(n))_n$, we have for every positive integer s and every $x \in G$

$$\frac{1}{s}\sum_{n=1}^{s}\left(D^{\kappa}_{\alpha(n)} * f\right)(x) = \frac{1}{s}\sum_{n=1}^{s} S_{2^{|\alpha(n)|}} f(x) + \frac{1}{s}\sum_{n=1}^{s}\sum_{j=0}^{2^{|\alpha(n)|}-1} A_{\alpha(n),j}\left(S_{2^{|\alpha(n)|+1}} f(x+z_j) - S_{2^{|\alpha(n)|}} f(x+z_j)\right). \tag{14}$$

Denote by

$$T_s^{(\alpha)} f(x) := \sum_{n=1}^{s}\sum_{j=0}^{2^{|\alpha(n)|}-1} A_{\alpha(n),j}\left(S_{2^{|\alpha(n)|+1}} f(x+z_j) - S_{2^{|\alpha(n)|}} f(x+z_j)\right).$$

It is known that the operator $f^*(x) = \sup_n |S_{2^n} f(x)|$ is of weak type $(1,1)$. Therefore, this is also valid for the operator $\sup_s \frac{1}{s}|\sum_{n=1}^{s} S_{2^{|\alpha(n)|}} f(x)|$. Hence, it suffices to prove that the operator $\sup_s \frac{1}{s}|T_s^{(\alpha)} f(x)|$ is of weak type $(1,1)$.

For every nonnegative integer j set $\nu_j(x) = \inf\{i \geqslant 0 : S_{2^i}|f|(x+z_j) > 2^k\lambda\}$, where k is the least integer satisfying $j < 2^k$.

For every $y \in G$ and every nonnegative integer n, denote by $\triangle_n f(y)$ the expression

$$\triangle_n f(y) := S_{2^{n+1}} f(y) - S_{2^n} f(y).$$

Since $1_{\{\nu_j(x)>m\}}$ is constant on each I_m-coset, it is clear that for all positive integers m, n and all $j < 2^m$ and $i < 2^n$ we have

$$\int 1_{\{\nu_j(x)>m\}} \triangle_m f(x+z_j) \cdot 1_{\{\nu_i(x)>n\}} \triangle_n f(x+z_i)\,dx = 0, \tag{15}$$

whenever $m \neq n$ or $i \neq j$. Moreover, if $F = \bigcup_{k=0}^{\infty}\bigcup_{j=0}^{2^k-1}\left(z_j + \left(\bigcup_{J\in F_{k,\lambda}} J\right)\right)$, we have

$$\mu\left\{x \in F^c : \frac{1}{s}\left|T_s^{(\alpha)} f(x)\right| > \lambda\right\}$$
$$\leqslant \mu\left\{x : \frac{1}{s}\left|\sum_{n=1}^{s}\sum_{j=0}^{2^{|\alpha(n)|}} A_{\alpha(n),j} \cdot 1_{\{\nu_j(x)>|\alpha(n)|\}}(x) \cdot \triangle_{|\alpha(n)|} f(x+z_j)\right| > \lambda\right\},$$

because if $x \in F^c$, then we can see that $x \in \left(z_j + \bigcup_{k:j<2^k} \bigcup_{J \in F_{k,\lambda}} J \right)^c$ for every nonnegative integer j. Indeed, the set F can also be expressed in the form

$$F = \bigcup_{j=0}^{\infty} \left(z_j + \left(\bigcup_{k:j<2^k} \bigcup_{J \in F_{k,\lambda}} J \right) \right),$$

hence from the De Morgan's law we obtain

$$F^c = \bigcap_{j=0}^{\infty} \left(z_j + \left(\bigcup_{k:j<2^k} \bigcup_{J \in F_{k,\lambda}} J \right) \right)^c.$$

Then by means of (12) obtained in Lemma 3, we get $S_{2^i}|f|(x+z_j) \leqslant 2^k \lambda$, for all $i \geqslant 0$ and every $k \geqslant 0$ such that $j < 2^k$. This means that $\nu_j(x) = \infty$ and $1_{\{\nu_j(x) > |\alpha(n)|\}}(x) = 1$, $\forall j \geqslant 0$, $n \geqslant 1$.

Since $q \geqslant 2$, then $|\alpha(m)| \neq |\alpha(n)|$ whenever $m \neq n$, hence applying (15) we get

$$\begin{aligned}
&\mu \left\{ x \in F^c : \frac{1}{s} \left| T_s^{(\alpha)} f(x) \right| > \lambda \right\} \qquad (16)\\
&\leqslant \frac{1}{s^2 \lambda^2} \int \left| \sum_{n=1}^{s} \sum_{j=0}^{2^{|\alpha(n)|}-1} A_{\alpha(n),j} \cdot 1_{\{\nu_j(x) > |\alpha(n)|\}}(x) \cdot \triangle_{|\alpha(n)|} f(x+z_j) \right|^2 dx \\
&\leqslant \frac{1}{s^2 \lambda^2} \sum_{n=1}^{s} \sum_{j=0}^{2^{|\alpha(n)|}-1} A^2_{\alpha(n),j} \int 1_{\{\nu_j(x) > |\alpha(n)|\}}(x) \left(\triangle_{|\alpha(n)|} f(x+z_j) \right)^2 dx \\
&\leqslant \frac{1}{s^2 \lambda^2} \sum_{j=0}^{2^{|\alpha(s)|}-1} \sum_{\substack{n:2^{|\alpha(n)|}>j \\ n \leqslant s}} A^2_{\alpha(n),j} \int 1_{\{\nu_j(x) > |\alpha(n)|\}}(x) \left(\triangle_{|\alpha(n)|} f(x+z_j) \right)^2 dx \\
&\leqslant \frac{1}{s^2 \lambda^2} \sum_{j=0}^{2^{|\alpha(s)|}-1} \frac{1}{(j+1)^2} \sum_{\substack{n:2^{|\alpha(n)|}>j \\ n \leqslant s}} \int 1_{\{\nu_j(x) > |\alpha(n)|\}}(x) \left(\triangle_{|\alpha(n)|} f(x+z_j) \right)^2 dx,
\end{aligned}$$

where, the last inequality is obtained from (5). Applying [2, Lemma 2.1] we obtain

$$\mu \left\{ x \in F^c : \frac{1}{s} \left| T_s^{(\alpha)} f(x) \right| > \lambda \right\} \leqslant \frac{C}{\lambda} \frac{1}{s^2} \sum_{j=0}^{2^{|\alpha(s)|}-1} \frac{1}{j+1} \|f\|_1,$$

because according to the definition of $\nu_j(x)$, if $\nu_j(x) > |\alpha(n)|$ then

$$S_{2^{|\alpha(n)|+1}} |f|(x+z_j) \leqslant 2 S_{2^{|\alpha(n)|}} |f|(x+z_j) \leqslant 4j\lambda.$$

It follows that

$$\mu \left\{ x \in F^c : \frac{1}{s} \left| T_s^{(\alpha)} f(x) \right| > \lambda \right\} \leqslant C \frac{|\alpha(s)|}{s^2 \lambda} \|f\|_1. \qquad (17)$$

We can deduce from (13) that

$$|\alpha(s)| - |\alpha(1)| \leqslant Cs^{\beta},$$

hence, (17) implies that

$$\mu\left\{x \in F^c : \frac{1}{s}\left|T_s^{(\alpha)} f(x)\right| > \lambda\right\} \leqslant C\frac{1}{s^{2-\beta}\lambda}\|f\|_1. \tag{18}$$

Notice that $T_s^{(\alpha)} f(x)$ can be written in the form

$$T_s^{(\alpha)} f(x) = \sum_{j=0}^{2^{|\alpha(s)|}-1} M_j(x),$$

where

$$M_j(x) = \sum_{\substack{n: 2^{|\alpha(n)|} > j \\ n \leqslant s}} A_{\alpha(n),j}\left(S_{2^{|\alpha(n)|+1}} f(x+z_j) - S_{2^{|\alpha(n)|}} f(x+z_j)\right).$$

Therefore, if $k^{\theta} \leqslant s < (k+1)^{\theta}$, for some fixed θ satisfying $\frac{1}{2-\beta} < \theta < \frac{1}{\beta-1}$, then we get

$$\left|T_s^{(\alpha)} f(x)\right| \leqslant \left|\sum_{j=0}^{2^{|\alpha(k^{\theta})|}-1} M_j(x)\right| + \left|\sum_{j=2^{|\alpha(k^{\theta})|}}^{2^{|\alpha(s)|}-1} M_j(x)\right|. \tag{19}$$

Proceeding as in the proof of [4, Theorem 1], the calculations made in (16) give

$$\begin{aligned} &\mu\left\{x \in F^c : \frac{1}{s}\left|\sum_{j=2^{|\alpha(k^{\theta})|}}^{2^{|\alpha(s)|}-1} M_j(x)\right| > \lambda\right\} \qquad (20) \\ &\leqslant \frac{C}{\lambda}\frac{|\alpha((k+1)^{\theta})| - |\alpha(k^{\theta})|}{k^{2\theta}}\|f\|_1 \\ &\leqslant \frac{C}{\lambda}\frac{k^{\theta-1}k^{\theta(\beta-1)}}{k^{2\theta}}, \end{aligned}$$

where the last inequality is obtained from assumption (13). Combining (18), (19) and (20) we obtain

$$\begin{aligned} &\mu\left\{x \in F^c : \sup_s \frac{1}{s}\left|T_s^{(\alpha)} f(x)\right| > 2\lambda\right\} \\ &\leqslant \mu\left\{x \in F^c : \sup_k \frac{1}{k^{\theta}} \sup_{k^{\theta} \leqslant s < (k+1)^{\theta}} \left|T_s^{(\alpha)} f(x)\right| > 2\lambda\right\} \\ &\leqslant \mu\left\{x \in F^c : \sup_k \frac{1}{k^{\theta}}\left|\sum_{j=0}^{2^{|\alpha(k^{\theta})|}-1} M_j(x)\right| + \sup_k \frac{1}{k^{\theta}} \sup_{k^{\theta} < s < (k+1)^{\theta}} \left|\sum_{j=2^{|\alpha(k^{\theta})|}}^{2^{|\alpha(s)|}-1} M_j(x)\right| > 2\lambda\right\} \end{aligned}$$

$$\begin{aligned}
&\leqslant \mu\left\{x\in F^c : \sup_k \frac{1}{k^\theta}\left|\sum_{j=0}^{2^{|\alpha(k^\theta)|}-1} M_j(x)\right| > \lambda\right\}\\
&\quad +\mu\left\{x\in F^c : \sup_k \frac{1}{k^\theta}\sup_{k^\theta<s<(k+1)^\theta}\left|\sum_{j=2^{|\alpha(k^\theta)|}}^{2^{|\alpha(s)|}-1} M_j(x)\right| > \lambda\right\}\\
&\leqslant \sum_{k=1}^{\infty}\mu\left\{x\in F^c : \frac{1}{k^\theta}\left|\sum_{j=0}^{2^{|\alpha(k^\theta)|}-1} M_j(x)\right| > \lambda\right\}\\
&\quad +\sum_{k=1}^{\infty}\sum_{s=k^\theta+1}^{(k+1)^\theta-1}\mu\left\{x\in F^c : \frac{1}{k^\theta}\left|\sum_{j=2^{|\alpha(k^\theta)|}}^{2^{|\alpha(s)|}-1} M_j(x)\right| > \lambda\right\}\\
&\leqslant \frac{C\|f\|_1}{\lambda}\sum_{k=1}^{\infty}\frac{1}{k^{\theta(2-\beta)}} + \sum_{k=1}^{\infty}\sum_{s=k^\theta}^{(k+1)^\theta-1} C\frac{\|f\|_1}{\lambda k^{\theta(2-\beta)+1}}\\
&\leqslant \frac{C\|f\|_1}{\lambda}\left(\sum_{k=1}^{\infty}\frac{1}{k^{\theta(2-\beta)}} + \sum_{k=1}^{\infty}\frac{1}{k^{\theta(1-\beta)+2}}\right) \leqslant \frac{C\|f\|_1}{\lambda}.
\end{aligned}$$

As mentioned in the discussion made after Lemma 2, the sets $(F_{k,\lambda})_k$ have mutually disjoint elements. Hence, according to (11)

$$\begin{aligned}
\mu(F) &\leqslant \sum_{k=0}^{\infty}\sum_{j=0}^{2^k-1}\mu\left(z_j+\left(\bigcup_{J\in F_{k,\lambda}} J\right)\right)\\
&\leqslant \sum_{k=0}^{\infty}\frac{1}{2^k\lambda}\sum_{j=0}^{2^k-1}\sum_{J\in F_{k,\lambda}}\int_J |f|\\
&\leqslant \frac{1}{\lambda}\sum_{k=0}^{\infty}\sum_{J\in F_{k,\lambda}}\int_J |f| \leqslant \frac{\|f\|_1}{\lambda}.
\end{aligned}$$

It follows

$$\begin{aligned}
\mu\left\{\sup_s \frac{1}{s}\left|T_s^{(\alpha)}f(x)\right| > 2\lambda\right\} &\leqslant \mu(F)+\mu\left\{x\in F^c : \sup_s \frac{1}{s}\left|T_s^{(\alpha)}f(x)\right| > 2\lambda\right\}\\
&\leqslant \frac{C\|f\|_1}{\lambda}.
\end{aligned}$$

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