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COMPOSITION OPERATORS AND CLOSURES OF A CLASS OF MÖBIUS INVARIANT FUNCTION SPACES IN THE BLOCH SPACE

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Abstract. Closures of a class of Möbius invariant function spaces in the Bloch space are investigated in this paper. Moreover, the boundedness and compactness of composition operators from the Bloch space to closures of such Möbius invariant space in the Bloch space are characterized.

1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $\partial\mathbb{D}$ be the boundary of $\mathbb{D}$. Set $H(\mathbb{D})$ be the class of all functions analytic in $\mathbb{D}$. Let $\mathrm{Aut}(\mathbb{D})$ denote the group of all Möbius maps of the disk. For any $a \in \mathbb{D}$, the function

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z}, \qquad z \in \mathbb{D},$$

is a Möbius map that interchanges the points a and 0. Let $g(z,a) = \log \frac{1}{|\varphi_a(z)|}$ be the Green's function of $\mathbb{D}$ with the pole at a.

For $0 < p < \infty$, the Hardy space is denoted by H^p, which consists of all functions $f \in H(\mathbb{D})$ such that (see [7] or [10])

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

As usual, H^∞ denotes the space of all bounded analytic functions on $\mathbb{D}$.

The Bloch space $\mathscr{B}$ is the set of all functions $f \in H(\mathbb{D})$ which satisfies

$$\|f\|_\beta = \sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)| < \infty.$$

It is well known that $\mathscr{B}$ is a Banach space if it is equipped with the norm

$$\|f\|_{\mathscr{B}} = |f(0)| + \|f\|_\beta.$$

Note that $H^\infty \subset \mathscr{B}$. For instance, $h(z) = \log(1-z)$ is in $\mathscr{B}$ but not in H^∞.

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The little Bloch space, denoted by $\mathscr{B}_0$, is the subspace of $\mathscr{B}$ consisting of all functions $f \in H(\mathbb{D})$ for which

$$\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0.$$

It is well known that $\mathscr{B}_0$ is the closure of polynomials in $\mathscr{B}$.

Let $0<\alpha<\infty$. Recall that the Bloch type space is the space of functions $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{\mathscr{B}^\alpha} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f'(z)| < \infty.$$

Obviously, the space $\mathscr{B}^\alpha$ turns into the Bloch space $\mathscr{B}$ when $\alpha = 1$. Let n be a positive integer. It is well known that $\|f\|_{\mathscr{B}^\alpha}$ is equivalent to $\|f\|_{\mathscr{B}^{\alpha,n}}$ (see [45, p. 1149]), where

$$\|f\|_{\mathscr{B}^{\alpha,n}} = |f(0)| + |f'(0)| + \ldots + |f^{(n-1)}(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha+n-1}|f^{(n)}(z)|.$$

This norm characterization of Bloch type spaces $\mathscr{B}^\alpha$ served as a motivation for introducing the nth weighted-type spaces (see, for example, [32, 33, 36]).

Let $0<p,s<\infty$, $-2<q<\infty$. An $f \in H(\mathbb{D})$ is said to belong to $F(p,q,s)$ (see [42]) if

$$\|f\|_{p,q,s}^p = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q g^s(z,a)\,dA(z) < \infty,$$

where dA is the normalized area measure on $\mathbb{D}$. $F(p,q,s)$ is called general function space because it can get many function spaces if it takes special parameters of p,q,s. For example, $F(p,q,s) = \mathscr{B}^{\frac{q+2}{p}}$ for $s>1$, $F(2,0,s) = Q_s$ and $F(2,0,1) = BMOA$.

Let $0<p<\infty, -1<\alpha<\infty$, and n be a positive integer. Let $Q(n,p,\alpha)$ denote the space of functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{n,p,\alpha}^p = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(f\circ\varphi_a)^{(n)}(z)|^p (1-|z|^2)^\alpha\,dA(z) < \infty.$$

Obviously, we also see that

$$\|f\|_{n,p,\alpha}^p = \sup_{\psi\in\mathrm{Aut}(\mathbb{D})} \int_{\mathbb{D}} |(f\circ\psi)^{(n)}(z)|^p (1-|z|^2)^\alpha\,dA(z) < \infty.$$

Thus the space $Q(n,p,\alpha)$ is Möbius invariant and

$$\|f\circ\psi\|_{n,p,\alpha} = \|f\|_{n,p,\alpha}, \qquad f\in Q(n,p,\alpha), \psi\in Aut(\mathbb{D}).$$

It is clear that $\|f\| = |f(0)| + \|f\|_{n,p,\alpha}$ defines a complete norm on $Q(n,p,\alpha)$ when $p \geqslant 1$. Thus $Q(n,p,\alpha)$ is a Banach space of analytic functions when $p \geqslant 1$. When $0<p<1$, the space $Q(n,p,\alpha)$ is not necessarily a Banach space but is always a complete metric space. This space was firstly introduced by K. Zhu in [46]. From [46], we see that if $2n < \alpha+3$, then $Q(n,2,\alpha) = Q_{\alpha-2(n-1)}$. In particular, $Q(n,2,2n-1) = BMOA$. See [3] for general properties of Möbius invariant Banach spaces. When $n=1$, for simplicity we denote $Q(n,p,\alpha)$ by $Q(p,\alpha)$.

Let $S(\mathbb{D})$ be the collection of all analytic self-maps of $\mathbb{D}$. Every $\varphi \in S(\mathbb{D})$ induces a composition operator C_φ, which is defined by

$$C_\varphi(f)(z) = f(\varphi(z)), \qquad z \in \mathbb{D}.$$

Composition operators have been extensively studied in recent years. One of the main themes for studying composition operators is to relate operator theoretical problems for C_φ with the function theoretical properties of the inducing map φ. Readers can refer to [6].

It is well known that the composition operator is bounded on the Bloch space by Schwartz-Pick lemma. The compactness of the composition operator on the Bloch space was firstly studied in [22]. They proved that $C_\varphi : \mathscr{B} \to \mathscr{B}$ is compact if and only if

$$\lim_{|\varphi(z)|\to 1} \frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)| = 0.$$

See [38] and [40] for another two characterizations for the compactness of the composition operator acting on the Bloch space. The boundedness and compactness of composition and some other concrete operators is a topic of a great recent interest. For some results on operators from or to Bloch-type or related weighted-type spaces consults, for example, [8, 13, 14, 19, 20, 29, 30, 32, 33, 36, 37, 26, 15, 16, 17, 18, 12, 27, 28, 31, 34, 35, 44] and the related references therein.

Suppose $X \subset \mathscr{B}$ is an analytic function space. We use $\mathscr{C}_{\mathscr{B}}(X)$ to denote the closure of the space X in the Bloch norm for simplicity. Let n be a positive integer and

$$\Omega_{n,\varepsilon}(f) = \{z \in \mathbb{D} : (1-|z|^2)^n |f^{(n)}(z)| \geqslant \varepsilon\}.$$

We denote $\Omega_{n,\varepsilon}(f)$ by $\Omega_\varepsilon(f)$ when $n=1$.

In [2], Anderson, Clunie and Pommerenke raised a question on what is the closure of H^∞ in the Bloch norm? The problem is still open. Anderson in [1] gave a description of the closure of $BMOA$ in $\mathscr{B}$, but he did not publish this result. For example, he showed that a Bloch function f is in $\mathscr{C}_{\mathscr{B}}(BMOA)$ if and only if for every $\varepsilon > 0$,

$$\sup_{a\in\mathbb{D}} \int_{\Omega_\varepsilon(f)} (1-|\sigma_a(z)|^2)\frac{dA(z)}{(1-|z|^2)^2} < \infty.$$

Ghatage and Zheng in [11] provided a complete proof for it. In 2008, Zhao studied $\mathscr{C}_{\mathscr{B}}(F(p,p-2,s))$ when $1 \leqslant p < \infty$ and $0 < s \leqslant 1$ in [43]. In 2010, Aulaskari and Zhao studied composition operators from the Bloch space to $\mathscr{C}_{\mathscr{B}}(F(p,p-2,s))$ in [4]. Monreal Galán and Nicolau in [23] studied the closure in the Bloch norm of the space $H^p \cap \mathscr{B}$ for $1 < p < \infty$. Recently, Galanopoulos, Monreal Galán and Pau in [9] extended the above result to $0 < p < \infty$. Bao and Göğüş in [5] characterized the closure in the Bloch norm of the space $\mathscr{D}^2_\alpha \cap \mathscr{B}(-1 < \alpha \leqslant 1)$, where $\mathscr{D}^2_\alpha$ is the Dirichlet type space. In 2017, Qian and Li characterized $\mathscr{C}_{\mathscr{B}^\alpha}(\mathscr{D}_\mu \cap \mathscr{B}^\alpha)$ in [25]. Among others, they obtained the following result.

THEOREM A. *Let $\alpha > 0$ and μ be a positive Borel measure defined on $\partial\mathbb{D}$. Let n be a positive integer. Suppose $f \in \mathscr{B}^\alpha$. Then $f \in \mathscr{C}_{\mathscr{B}^\alpha}(\mathscr{D}_\mu \cap \mathscr{B}^\alpha)$ if and only if for any $\varepsilon > 0$,*

$$\int_{\Gamma_{n,\alpha,\varepsilon}(f)} \frac{\mathscr{P}_\mu(z)}{(1-|z|^2)^{2\alpha}} dA(z) < \infty,$$

where

$$\mathscr{P}_\mu(z) = \int_{\partial\mathbb{D}} \frac{1-|z|^2}{|\xi - z|^2} d\mu(\xi)$$

and

$$\Gamma_{n,\alpha,\varepsilon}(f) = \{z \in \mathbb{D} : (1-|z|^2)^{\alpha+n-1} |f^{(n)}(z)| \geqslant \varepsilon\}.$$

See [21, 24, 39, 41] for more papers on the closure of some function spaces in the Bloch space.

The purpose of this paper is to study the closure of $Q(n,p,\alpha)$ space in the Bloch space. We give a complete characterization for $\mathscr{C}_{\mathscr{B}}(Q(n,p,\alpha))$. Moreover, we study the boundedness and compactness of composition operators $C_\varphi : \mathscr{B}(\mathscr{B}_0) \to \mathscr{C}_{\mathscr{B}}(Q(p,\alpha))$ and $C_\varphi : \mathscr{C}_{\mathscr{B}}(Q(p,\alpha)) \to \mathscr{C}_{\mathscr{B}}(Q(p,\alpha))$.

Throughout this paper, we say that $f \lesssim h$ if there exists a constant C such that $f \leqslant Ch$. The symbol $f \approx h$ means that $f \lesssim h \lesssim f$.

2. Characterization of $\mathscr{C}_{\mathscr{B}}(Q(n,p,\alpha))$

To state and prove our main results in this paper, we need some lemmas. The following well-known estimate can be found in [47, Lemma 3.10].

LEMMA 1. *Suppose $s > 0$ and $t > -1$. Then there exists a positive constant C such that*

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^t}{|1-\bar{z}w|^{2+t+s}} dA(w) \leqslant \frac{C}{(1-|z|^2)^s}$$

for all $z \in \mathbb{D}$.

LEMMA 2. [46] *If $np \leqslant \alpha + 2$, then f belongs to $Q(n,p,\alpha)$ if and only if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f^{(n)}(z)|^p \frac{(1-|a|^2)^{\alpha+2-np}}{|1-\bar{a}z|^{2(\alpha+2-np)}} (1-|z|^2)^\alpha dA(z) < \infty.$$

We also need the following estimate (cf. [43, Lemma 1]).

LEMMA 3. *Let $s>-1, r,t>0$, and $r+t-s>2$. If $t<s+2<r$, then there exists a positive constant C such that*

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^s}{|1-\bar{w}z|^r|1-\bar{a}z|^t} dA(z) \leqslant \frac{C}{(1-|w|^2)^{r-s-2}|1-\bar{a}w|^t}.$$

Now we are in a position to state and prove our main results in this paper.

THEOREM 1. *Let n be a positive integer, $1\leqslant p<\infty$ and $-1<\alpha<\infty$ such that $\alpha<np\leqslant\alpha+2$ and $np<\alpha+1+n$. Suppose that $f\in\mathscr{B}$. Then $f\in\mathscr{C}_{\mathscr{B}}(Q(n,p,\alpha))$ if and only if for any $\varepsilon>0$,*

$$\sup_{a\in\mathbb{D}} \int_{\Omega_{n,\varepsilon}(f)} \frac{(1-|a|^2)^{\alpha+2-np}(1-|z|^2)^{\alpha-np}}{|1-\bar{a}z|^{2(\alpha+2-np)}} dA(z)<\infty, \tag{1}$$

or

$$\sup_{a\in\mathbb{D}} \int_{\Omega_{n,\varepsilon}(f)} (1-|\sigma_a(z)|^2)^{\alpha+2-np} \frac{dA(z)}{(1-|z|^2)^2}<\infty.$$

Proof. Take a function f in the closure in the Bloch norm of $Q(n,p,\alpha)$ and $\varepsilon>0$. Then there exists a function $g\in Q(n,p,\alpha)$ such that $\|f-g\|_{\mathscr{B}^{1,n}}<\frac{\varepsilon}{2}$. Note that for all $z\in\mathbb{D}$,

$$\begin{aligned}(1-|z|^2)^n|f^{(n)}(z)| &\leqslant \sup_{w\in\mathbb{D}}(1-|w|^2)^n|f^{(n)}(w)-g^{(n)}(w)|+(1-|z|^2)^n|g^{(n)}(z)| \\ &\leqslant \frac{\varepsilon}{2}+(1-|z|^2)^n|g^{(n)}(z)|.\end{aligned}$$

This implies that $\Omega_{n,\varepsilon}(f)\subseteq\Omega_{n,\frac{\varepsilon}{2}}(g)$. Then it follows that

$$\begin{aligned}&\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|g^{(n)}(z)|^p\frac{(1-|a|^2)^{\alpha+2-np}}{|1-\bar{a}z|^{2(\alpha+2-np)}}(1-|z|^2)^{\alpha}dA(z) \\ \geqslant{}& \sup_{a\in\mathbb{D}}\int_{\Omega_{n,\frac{\varepsilon}{2}}(g)}|g^{(n)}(z)|^p(1-|z|^2)^{np}\frac{(1-|a|^2)^{\alpha+2-np}}{|1-\bar{a}z|^{2(\alpha+2-np)}}(1-|z|^2)^{\alpha-np}dA(z) \\ \geqslant{}& (\frac{\varepsilon}{2})^p\sup_{a\in\mathbb{D}}\int_{\Omega_{n,\frac{\varepsilon}{2}}(g)}\frac{(1-|a|^2)^{\alpha+2-np}}{|1-\bar{a}z|^{2(\alpha+2-np)}}(1-|z|^2)^{\alpha-np}dA(z) \\ \geqslant{}& (\frac{\varepsilon}{2})^p\sup_{a\in\mathbb{D}}\int_{\Omega_{n,\varepsilon}(f)}\frac{(1-|a|^2)^{\alpha+2-np}}{|1-\bar{a}z|^{2(\alpha+2-np)}}(1-|z|^2)^{\alpha-np}dA(z).\end{aligned}$$

Since $g\in Q(n,p,\alpha)$, by Lemma 2, we have

$$\sup_{a\in\mathbb{D}}\int_{\Omega_{n,\varepsilon}(f)}\frac{(1-|a|^2)^{\alpha+2-np}}{|1-\bar{a}z|^{2(\alpha+2-np)}}(1-|z|^2)^{\alpha-np}dA(z)<\infty.$$

Conversely, we suppose that f is a Bloch function which satisfies (1) and take $\varepsilon > 0$. Without loss of generality, we may assume that $f(0) = f'(0) = \ldots = f^{(n-1)}(0) = 0$. For any $z \in \mathbb{D}$, using Proposition 4.27 in [47], we have

$$f(z) = C_\beta \int_{\mathbb{D}} \frac{f^{(n)}(w)(1-|w|^2)^{n+\beta}}{(1-z\overline{w})^{2+\beta}\overline{w}^n} dA(w).$$

Here

$$C_\beta = \frac{1}{(\beta+2)\ldots(\beta+n)} \quad \text{and} \quad \beta > -1.$$

According to [43], we decompose $f(z) = f_1(z) + f_2(z)$, where

$$f_1(z) = C_\beta \int_{\Omega_{n,\varepsilon}(f)} \frac{f^{(n)}(w)(1-|w|^2)^{n+\beta}}{(1-z\overline{w})^{2+\beta}\overline{w}^n} dA(w)$$

and

$$f_2(z) = C_\beta \int_{\mathbb{D}\setminus\Omega_{n,\varepsilon}(f)} \frac{f^{(n)}(w)(1-|w|^2)^{n+\beta}}{(1-z\overline{w})^{2+\beta}\overline{w}^n} dA(w).$$

Obviously,

$$f_1^{(n)}(z) = (\beta+n+1) \int_{\Omega_{n,\varepsilon}(f)} \frac{f^{(n)}(w)(1-|w|^2)^{n+\beta}}{(1-z\overline{w})^{n+2+\beta}} dA(w)$$

and

$$f_2^{(n)}(z) = (\beta+n+1) \int_{\mathbb{D}\setminus\Omega_{n,\varepsilon}(f)} \frac{f^{(n)}(w)(1-|w|^2)^{n+\beta}}{(1-z\overline{w})^{n+2+\beta}} dA(w).$$

Let $h(z) = f_1(z) - \sum_{k=1}^{n-1} \frac{f_1^{(k)}(0)}{k!} z^k$. Then $h(0) = h'(0) = \ldots = h^{(n-1)}(0) = 0$, and $(f-h)^{(n)}(z) = f_2^{(n)}(z)$. Using Lemma 1, we obtain

$$\begin{aligned}
\|f-h\|_{\mathscr{B}^{1,n}} &= \sup_{z\in\mathbb{D}} (1-|z|^2)^n |(f-h)^{(n)}(z)| = \sup_{z\in\mathbb{D}} (1-|z|^2)^n |f_2^{(n)}(z)| \\
&\lesssim \varepsilon \sup_{z\in\mathbb{D}} (1-|z|^2)^n \int_{\mathbb{D}\setminus\Omega_{n,\varepsilon}(f)} \frac{|f^{(n)}(w)|(1-|w|^2)^{n+\beta}}{|1-z\overline{w}|^{n+2+\beta}} dA(w) \\
&\lesssim \varepsilon \sup_{z\in\mathbb{D}} (1-|z|^2)^n \int_{\mathbb{D}} \frac{(1-|w|^2)^{\beta}}{|1-z\overline{w}|^{n+2+\beta}} dA(w) \lesssim \varepsilon.
\end{aligned}$$

This means that $h \in \mathscr{B}$. Then using Fubini's theorem, we have

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |h^{(n)}(z)|^p \frac{(1-|a|^2)^{\alpha+2-np}}{|1-\bar{a}z|^{2(\alpha+2-np)}} (1-|z|^2)^\alpha dA(z) \\
= &\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_1^{(n)}(z)|^p \frac{(1-|a|^2)^{\alpha+2-np}}{|1-\bar{a}z|^{2(\alpha+2-np)}} (1-|z|^2)^\alpha dA(z)
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_1^{(n)}(z)|^{p-1}(1-|z|^2)^{n(p-1)}|f_1^{(n)}(z)| \frac{(1-|a|^2)^{\alpha+2-np}}{|1-\bar{a}z|^{2(\alpha+2-np)}}(1-|z|^2)^{\alpha-np+n}dA(z) \\
&\leqslant \|f_1\|_{\mathscr{B}^{1,n}}^{p-1} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_1^{(n)}(z)| \frac{(1-|a|^2)^{\alpha+2-np}}{|1-\bar{a}z|^{2(\alpha+2-np)}}(1-|z|^2)^{\alpha-np+n}dA(z) \\
&\lesssim \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^{\alpha+2-np}(1-|z|^2)^{\alpha-np+n}}{|1-\bar{a}z|^{2(\alpha+2-np)}} \left(\int_{\Omega_{n,\varepsilon}(f)} \frac{f^{(n)}(w)(1-|w|^2)^{n+\beta}}{|1-z\bar{w}|^{n+2+\beta}}dA(w)\right)dA(z) \\
&\lesssim \|f\|_{\mathscr{B}^{1,n}} \sup_{a\in\mathbb{D}} \int_{\Omega_{n,\varepsilon}(f)} (1-|w|^2)^{\beta} \left(\int_{\mathbb{D}} \frac{(1-|a|^2)^{\alpha+2-np}(1-|z|^2)^{\alpha-np+n}}{|1-\bar{a}z|^{2(\alpha+2-np)}|1-z\bar{w}|^{n+2+\beta}}dA(z)\right)dA(w).
\end{aligned}
$$

Observe that $\alpha+2-np \geqslant 0$. If we choose $\beta > \alpha+2-np-n$, then by Lemma 3, we obtain

$$
\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |h^{(n)}(z)|^p \frac{(1-|a|^2)^{\alpha+2-np}}{|1-\bar{a}z|^{2(\alpha+2-np)}}(1-|z|^2)^{\alpha}dA(z) \\
&\lesssim \sup_{a\in\mathbb{D}} \int_{\Omega_{n,\varepsilon}(f)} \frac{(1-|a|^2)^{\alpha+2-np}(1-|w|^2)^{\alpha-np}}{|1-\bar{a}w|^{2(\alpha+2-np)}}dA(w) < \infty.
\end{aligned}
$$

Hence, $h \in \mathscr{Q}(n,p,\alpha)$. Thus for any $\varepsilon > 0$, there exists a function $h \in \mathscr{Q}(n,p,\alpha)$ such that $\|f-h\|_{\mathscr{B}^{1,n}} \lesssim \varepsilon$, which means that $f \in \mathscr{C}_{\mathscr{B}}(Q(n,p,\alpha))$. The proof is complete.

From [46], we see $Q(n,p,\alpha) = \mathscr{B}$ when $np < \alpha+1$. According to Theorem 1, we get another characterization of the Bloch function as follows.

COROLLARY 1. *Let n be a positive integer, $1 \leqslant p < \infty$ and $-1 < \alpha < \infty$ such that $\alpha < np < \alpha+1$. Then $f \in \mathscr{B}$ if and only if for any $\varepsilon > 0$,*

$$
\sup_{a\in\mathbb{D}} \int_{\Omega_{n,\varepsilon}(f)} \frac{(1-|a|^2)^{\alpha+2-np}(1-|z|^2)^{\alpha-np}}{|1-\bar{a}z|^{2(\alpha+2-np)}}dA(z) < \infty.
$$

From [46], when $np = \alpha+2$, then $Q(n,p,\alpha) = B_p$, the Besov space. Since all Besov spaces contain the polynomials and are contained in $\mathscr{B}_0$, we have

$$
\mathscr{C}_{\mathscr{B}}(B_p) = \mathscr{B}_0, \quad 1 \leqslant p < \infty.
$$

From [46], when $p = 2$ and $\alpha = 2n-1$, $Q(n,2,2n-1) = BMOA$. We immediately get another characterization of $\mathscr{C}_{\mathscr{B}}(BMOA)$ as follows.

COROLLARY 2. *Let n be a positive integer. Suppose that $f \in \mathscr{B}$. Then $f \in \mathscr{C}_{\mathscr{B}}(BMOA)$ if and only if for any $\varepsilon > 0$,*

$$
\sup_{a\in\mathbb{D}} \int_{\Omega_{n,\varepsilon}(f)} \frac{(1-|a|^2)(1-|z|^2)^{-1}}{|1-\bar{a}z|^2}dA(z) < \infty,
$$

or

$$
\sup_{a\in\mathbb{D}} \int_{\Omega_{n,\varepsilon}(f)} (1-|\sigma_a(z)|^2)\frac{dA(z)}{(1-|z|^2)^2} < \infty.
$$

3. Composition operator on $\mathscr{C}_{\mathscr{B}}(Q(p,\alpha))$

Next, we characterize the boundedness and compactness of composition operators from the Bloch space $\mathscr{B}$ and the little Bloch space $\mathscr{B}_0$ to $\mathscr{C}_{\mathscr{B}}(Q(p,\alpha))$. Suppose that $\varphi \in S(\mathbb{D})$. We define

$$\varphi^{\sharp}(z) = \frac{1-|z|^2}{1-|\varphi(z)|^2}\varphi'(z), \qquad z \in \mathbb{D}.$$

THEOREM 2. *Suppose $1 \leqslant p < \infty, -1 < \alpha < \infty$ such that $\alpha < p < \alpha+2$ and $\varphi \in S(\mathbb{D})$. Then $C_\varphi : \mathscr{B} \to \mathscr{C}_{\mathscr{B}}(Q(p,\alpha))$ is bounded if and only if for any $\varepsilon > 0$,*

$$\sup_{a\in\mathbb{D}} \int_{\Lambda_\varepsilon(\varphi)} \frac{(1-|a|^2)^{\alpha+2-p}(1-|z|^2)^{\alpha-p}}{|1-\bar{a}z|^{2(\alpha+2-p)}} dA(z) < \infty, \tag{2}$$

where

$$\Lambda_\varepsilon(\varphi) = \{z \in \mathbb{D} : |\varphi^{\sharp}(z)| \geqslant \varepsilon\}.$$

Proof. For the sufficiency we assume that (2) holds for any $\varepsilon > 0$. Let $f \in \mathscr{B}$. Then

$$\begin{aligned}|(f\circ\varphi)'(z)|(1-|z|^2) &= |f'(\varphi(z))||\varphi'(z)|(1-|z|^2)\\ &= |f'(\varphi(z))|(1-|\varphi(z)|^2)\frac{|\varphi'(z)|(1-|z|^2)}{1-|\varphi(z)|^2} \leqslant \|f\|_{\mathscr{B}}|\varphi^{\sharp}(z)|.\end{aligned}$$

Thus, for any fixed $\delta > 0$, if $|(f\circ\varphi)'(z)|(1-|z|^2) > \delta$, then we have $|\varphi^{\sharp}(z)| \geqslant \frac{\delta}{\|f\|_{\mathscr{B}}} = \varepsilon$. Therefore,

$$\begin{aligned}\infty &> \sup_{a\in\mathbb{D}} \int_{\Lambda_\varepsilon(\varphi)} \frac{(1-|a|^2)^{\alpha+2-p}(1-|z|^2)^{\alpha-p}}{|1-\bar{a}z|^{2(\alpha+2-p)}} dA(z)\\ &\gtrsim \sup_{a\in\mathbb{D}} \int_{\Omega_\delta(f\circ\varphi)} \frac{(1-|a|^2)^{\alpha+2-p}(1-|z|^2)^{\alpha-p}}{|1-\bar{a}z|^{2(\alpha+2-p)}} dA(z).\end{aligned}$$

According to Theorem 1, we get that

$$f\circ\varphi \in \mathscr{C}_{\mathscr{B}}(Q(p,\alpha)).$$

This means that $C_\varphi : \mathscr{B} \to \mathscr{C}_{\mathscr{B}}(Q(p,\alpha))$ is bounded.

In order to prove the necessity, we suppose that $C_\varphi : \mathscr{B} \to \mathscr{C}_{\mathscr{B}}(Q(p,\alpha))$ is bounded. It is well known that there exists two functions $f_1,\ f_2 \in \mathscr{B}$ such that (see [20])

$$|f_1'(z)| + |f_2'(z)| \geqslant \frac{1}{1-|z|^2}.$$

Due to our assumption, we get that

$$f_1\circ\varphi,\ f_2\circ\varphi \in \mathscr{C}_{\mathscr{B}}(Q(p,\alpha)).$$

Thus for any $\varepsilon > 0$, we have

$$\sup_{a\in\mathbb{D}}\int_{\Omega_{\frac{\varepsilon}{2}}(f_1\circ\varphi)}\frac{(1-|a|^2)^{\alpha+2-p}(1-|z|^2)^{\alpha-p}}{|1-\bar{a}z|^{2(\alpha+2-p)}}dA(z)<\infty$$

and

$$\sup_{a\in\mathbb{D}}\int_{\Omega_{\frac{\varepsilon}{2}}(f_2\circ\varphi)}\frac{(1-|a|^2)^{\alpha+2-p}(1-|z|^2)^{\alpha-p}}{|1-\bar{a}z|^{2(\alpha+2-p)}}dA(z)<\infty.$$

If $z\in\Lambda_\varepsilon(\varphi)$, then we have

$$\begin{aligned}(|(f_1\circ\varphi)'(z)|+|(f_2\circ\varphi)'(z)|)(1-|z|^2)=&(|f_1'(\varphi(z))|+|f_2'(\varphi(z))|)|\varphi'(z)|(1-|z|^2)\\ \geqslant&|\varphi^\sharp(z)|\geqslant\varepsilon.\end{aligned}$$

This means that, either

$$|(f_1\circ\varphi)'(z)|(1-|z|^2)\geqslant\frac{\varepsilon}{2}$$

or

$$|(f_2\circ\varphi)'(z)|(1-|z|^2)\geqslant\frac{\varepsilon}{2}.$$

Therefore,

$$\begin{aligned}&\sup_{a\in\mathbb{D}}\int_{\Lambda_\varepsilon(\varphi)}\frac{(1-|a|^2)^{\alpha+2-p}(1-|z|^2)^{\alpha-p}}{|1-\bar{a}z|^{2(\alpha+2-p)}}dA(z)\\ \leqslant&\sup_{a\in\mathbb{D}}\int_{\Omega_{\frac{\varepsilon}{2}}(f_1\circ\varphi)\cup\Omega_{\frac{\varepsilon}{2}}(f_2\circ\varphi)}\frac{(1-|a|^2)^{\alpha+2-p}(1-|z|^2)^{\alpha-p}}{|1-\bar{a}z|^{2(\alpha+2-p)}}dA(z)\\ \leqslant&\sup_{a\in\mathbb{D}}\int_{\Omega_{\frac{\varepsilon}{2}}(f_1\circ\varphi)}\frac{(1-|a|^2)^{\alpha+2-p}(1-|z|^2)^{\alpha-p}}{|1-\bar{a}z|^{2(\alpha+2-p)}}dA(z)\\ &+\sup_{a\in\mathbb{D}}\int_{\Omega_{\frac{\varepsilon}{2}}(f_2\circ\varphi)}\frac{(1-|a|^2)^{\alpha+2-p}(1-|z|^2)^{\alpha-p}}{|1-\bar{a}z|^{2(\alpha+2-p)}}dA(z)<\infty.\end{aligned}$$

The proof is complete.

THEOREM 3. *Suppose* $1\leqslant p<\infty, -1<\alpha<\infty$ *such that* $\alpha<p<\alpha+2$ *and* $\varphi\in S(\mathbb{D})$. *Then* $C_\varphi:\mathscr{B}_0\to\mathscr{C}_{\mathscr{B}}(Q(p,\alpha))$ *is bounded if and only if*

$$\varphi\in\mathscr{C}_{\mathscr{B}}(Q(p,\alpha)).$$

Proof. The necessity of the conditions can be proved immediately. In fact, we suppose that $C_\varphi:\mathscr{B}_0\to\mathscr{C}_{\mathscr{B}}(Q(p,\alpha))$ is bounded. Notice that $f(z)=z\in\mathscr{B}_0$, then we have

$$\varphi=C_\varphi f\in\mathscr{C}_{\mathscr{B}}(Q(p,\alpha)).$$

To prove the sufficiency, we assume that $\varphi \in \mathscr{C}_{\mathscr{B}}(Q(p,\alpha))$. Let $f \in \mathscr{B}_0$. For any $\varepsilon > 0$, there is a constant r $(0 < r < 1)$ such that

$$|f'(z)|(1-|z|^2) < \frac{\varepsilon}{2}$$

whenever $|z| > r$. Let $z \in \Omega_\varepsilon(f \circ \varphi)$. Then

$$|f'(\varphi(z))|(1-|\varphi(z)|^2) \geqslant |f'(\varphi(z))||\varphi'(z)|(1-|z|^2) \geqslant \varepsilon.$$

That means $|\varphi(z)| \leqslant r$. Thus,

$$\frac{\|f\|_{\mathscr{B}}}{1-r^2}|\varphi'(z)|(1-|z|^2) \geqslant \|f\|_{\mathscr{B}} \frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)| \geqslant |f'(\varphi(z))||\varphi'(z)|(1-|z|^2) \geqslant \varepsilon.$$

Let $\delta = \frac{\varepsilon(1-r^2)}{\|f\|_{\mathscr{B}}}$. Then $|\varphi'(z)|(1-|z|^2) \geqslant \delta$. Hence, $\Omega_\varepsilon(f \circ \varphi) \subseteq \Omega_\delta(\varphi)$. Due to $\varphi \in \mathscr{C}_{\mathscr{B}}(Q(p,\alpha))$, we obtain

$$\begin{aligned}
\infty &> \sup_{a \in \mathbb{D}} \int_{\Omega_\delta(\varphi)} \frac{(1-|a|^2)^{\alpha+2-p}(1-|z|^2)^{\alpha-p}}{|1-\bar{a}z|^{2(\alpha+2-p)}} dA(z) \\
&\gtrsim \sup_{a \in \mathbb{D}} \int_{\Omega_\varepsilon(f \circ \varphi)} \frac{(1-|a|^2)^{\alpha+2-p}(1-|z|^2)^{\alpha-p}}{|1-\bar{a}z|^{2(\alpha+2-p)}} dA(z).
\end{aligned}$$

According to Theorem 1, we get that $f \circ \varphi \in \mathscr{C}_{\mathscr{B}}(Q(p,\alpha))$. Therefore, $C_\varphi : \mathscr{B}_0 \to \mathscr{C}_{\mathscr{B}}(Q(p,\alpha))$ is bounded. The proof is complete.

LEMMA 4. [22] *Assume that $\varphi \in S(\mathbb{D})$. Then the following statements are equivalent.*

(i) $C_\varphi : \mathscr{B} \to \mathscr{B}$ *is compact;*

(ii) $C_\varphi : \mathscr{B}_0 \to \mathscr{B}$ *is compact;*

(iii)

$$\lim_{r \to 1} \sup_{z \in \Omega_r(\varphi)} \frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)| = 0,$$

where $\Omega_r(\varphi) = \{z \in \mathbb{D} : |\varphi(z)| \geqslant r\}$.

THEOREM 4. *Suppose $1 \leqslant p < \infty, -1 < \alpha < \infty$ such that $\alpha < p < \alpha + 2$ and $\varphi \in S(\mathbb{D})$. Then the following statements are equivalent.*

(i) $C_\varphi : \mathscr{B} \to \mathscr{C}_{\mathscr{B}}(Q(p,\alpha))$ *is compact;*

(ii) $C_\varphi : \mathscr{C}_{\mathscr{B}}(Q(p,\alpha)) \to \mathscr{C}_{\mathscr{B}}(Q(p,\alpha))$ *is compact;*

(iii) $C_\varphi : \mathscr{B}_0 \to \mathscr{C}_{\mathscr{B}}(Q(p,\alpha))$ *is compact;*

(iv) $\varphi \in \mathscr{C}_{\mathscr{B}}(Q(p,\alpha))$ *and*

$$\lim_{r\to 1}\sup_{z\in\Omega_r(\varphi)} \frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)| = 0,$$

where $\Omega_r(\varphi) = \{z \in \mathbb{D} : |\varphi(z)| \geqslant r\}$.

Proof. $(i) \Rightarrow (ii)$. The implication is obvious because of $\mathscr{C}_{\mathscr{B}}(Q(p,\alpha)) \subseteq \mathscr{B}$.

$(ii) \Rightarrow (iii)$. The implication can be obtained since the space $\mathscr{C}_{\mathscr{B}}(Q(p,\alpha))$ contains all polynomials and $\mathscr{B}_0$ is the closure of all polynomials in $\mathscr{B}$.

$(iii) \Rightarrow (iv)$. Assume that $C_\varphi : \mathscr{B}_0 \to \mathscr{C}_{\mathscr{B}}(Q(p,\alpha))$ is compact. Obviously, $C_\varphi : \mathscr{B}_0 \to \mathscr{C}_{\mathscr{B}}(Q(p,\alpha))$ is bounded. Since $z \in \mathscr{B}_0$, we obtain $\varphi \in \mathscr{C}_{\mathscr{B}}(Q(p,\alpha))$. On the other hand, it is obvious that $\mathscr{C}_{\mathscr{B}}(Q(p,\alpha)) \subseteq \mathscr{B}$. This clearly implies that

$$\lim_{r\to 1}\sup_{z\in\Omega_r(\varphi)} \frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)| = 0,$$

where $\Omega_r(\varphi) = \{z \in \mathbb{D} : |\varphi(z)| \geqslant r\}$ by Lemma 4.

$(iv) \Rightarrow (i)$. According to the assumed condition, we see that there exists a constant r $(0 < r < 1)$, such that

$$\frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)| < \frac{\varepsilon}{2}, \quad \text{whenever} \quad |\varphi(z)| > r.$$

Let $z \in \mathbb{D}$ such that $\frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)| \geqslant \varepsilon$. Then $|\varphi(z)| \leqslant r$. Therefore,

$$\frac{1-|z|^2}{1-r^2}|\varphi'(z)| \geqslant \frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)| \geqslant \varepsilon.$$

Thus

$$(1-|z|^2)|\varphi'(z)| \geqslant (1-r^2)\varepsilon.$$

Set $\delta = (1-r^2)\varepsilon$. Then we obtain $z \in \Omega_\delta(\varphi)$. Since $\varphi \in \mathscr{C}_{\mathscr{B}}(Q(p,\alpha))$, we get

$$\begin{aligned}
\infty &> \sup_{a\in\mathbb{D}} \int_{\Omega_\delta(\varphi)} \frac{(1-|a|^2)^{\alpha+2-p}(1-|z|^2)^{\alpha-p}}{|1-\bar{a}z|^{2(\alpha+2-p)}} dA(z) \\
&\gtrsim \sup_{a\in\mathbb{D}} \int_{\Lambda_\varepsilon(\varphi)} \frac{(1-|a|^2)^{\alpha+2-p}(1-|z|^2)^{\alpha-p}}{|1-\bar{a}z|^{2(\alpha+2-p)}} dA(z).
\end{aligned}$$

According to Theorem 2, we get that $C_\varphi : \mathscr{B} \to \mathscr{C}_{\mathscr{B}}(Q(p,\alpha))$ is bounded. We know that $C_\varphi : \mathscr{B} \to \mathscr{B}$ is compact by Lemma 4. Therefore, $C_\varphi : \mathscr{B} \to \mathscr{C}_{\mathscr{B}}(Q(p,\alpha))$ is compact. The proof is complete.

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