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ON SOME PROPERTIES OF THE CARATHÉODORY FUNCTIONS

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Abstract. We find a sufficient condition for an analytic function in $|z|<1$ to be with positive real part in $|z|<1$. It follows sufficient conditions for a function to be convex, starlike, etc. These conditions describe the boundary behavior of the functions on $|z|=r<1$.

1. Introduction

Let $\mathcal{H}$ denote the class of functions analytic in the unit disk $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$. The set $\mathcal{P}$ is the set of all functions $p(z)\in\mathcal{H}$, $p(0)=1$, such that

$$\mathfrak{Re}\{p(z)\}>0,\quad (z\in\mathbb{D}).$$

If $p(z)$ is in $\mathcal{P}$, then is also called a Carathéodory function or a function with positive real part in $\mathbb{D}$. The Herglotz formula states that any function $p(z)$ in $\mathcal{P}$ can be obtained as a weighted average of the extreme points $(1+ze^{-it})/(1-ze^{-it})$ in $\mathcal{P}$

$$p(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1+ze^{-it}}{1-ze^{-it}}\mathrm{d}\mu(t),\quad (z\in\mathbb{D}), \tag{1}$$

where $\mu(t)$ is a real-valued nondecreasing function such that

$$\int_0^{2\pi}\mathrm{d}\mu(t)=2\pi.$$

In some sense the Herglotz formula (1) defines $p(z)$ by the boundary values on $|z|=1$. This formula should be considered in connection with the Dirichlet problem. The Dirichlet problem is that of finding a function harmonic in $E\subset\mathbb{C}$ and a continuous in $\overline{E}$, with prescribed boundary values on E. In the special case $E=\mathbb{D}$ the Poisson formula

$$u(re^{i\theta})=\frac{1}{2\pi}\int_0^{2\pi}\left\{P(r,t-\theta)\phi(e^{it})\right\}\mathrm{d}t$$

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produces the solution for every choice of a continuous boundary function $\phi(z)$, where

$$P(r,t) = \frac{1-r^2}{1-2r\cos t + r^2} = \mathfrak{Re}\frac{1+re^{it}}{1-re^{it}}$$

is the Poisson kernel. If $p(z)$ is in $\mathscr{P}$, then the Schwarz representation formula says that

$$p(z) = \frac{1}{2\pi}\int_0^{2\pi} \{\mathfrak{Re}(p(\rho e^{i\varphi}))\}\left\{\frac{\rho e^{i\varphi}+z}{\rho e^{i\varphi}-z}\right\}\mathrm{d}\varphi, \tag{2}$$

where $0 \leqslant |z| < \rho < 1$. In some sense the Schwarz formula (2) defines $p(z)$, $|z| < \rho$ by the boundary values on $|z| = \rho$ and (2) becomes (1) when $\rho \to 1$.

If $f(z) \in \mathscr{H}$ and $g(z) \in \mathscr{H}$, then we say that $f(z)$ is subordinate to $g(z)$, written as $f(z) \prec g(z)$ if there exists a Schwarz function $w(z)$, which is analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. If $\Phi(z) \prec \Phi_0(z)$, then [1]

$$\int_0^{2\pi} \left|\mathfrak{Re}\{\Phi(\rho e^{i\theta})\}\right| \mathrm{d}\theta \leqslant \int_0^{2\pi} \left|\mathfrak{Re}\{\Phi_0(\rho e^{i\theta})\}\right| \mathrm{d}\theta \quad \text{for } 0 < \rho < 1. \tag{3}$$

In this paper we consider a related condition which is sufficient for a function to be a Carathéodory function.

2. Main result

LEMMA 1. *Let $p(z) \in \mathscr{H}$, $p(0) = 1$, and suppose that*

$$\int_0^{2\pi} \left|\mathfrak{Re}\{p(re^{i\theta})\}\right| \mathrm{d}\theta \leqslant 2\pi \tag{4}$$

for all $r \in [0,1)$. Then we have

$$\mathfrak{Re}\{p(z)\} > 0, \quad (z \in \mathbb{D}), \tag{5}$$

or in other words $p(z)$ is a Carathéodory function.

Proof. Using the mean value formula for harmonic functions, we have

$$2\pi = \int_0^{2\pi} p(z)\mathrm{d}\theta = \int_0^{2\pi} \mathfrak{Re}\{p(z)\}\mathrm{d}\theta. \tag{6}$$

Therefore, from (4), we have

$$2\pi = \int_0^{2\pi} \mathfrak{Re}\{p(z)\}\mathrm{d}\theta \leqslant \int_0^{2\pi} |\mathfrak{Re}\{p(z)\}|\,\mathrm{d}\theta \leqslant 2\pi.$$

This shows that

$$\mathfrak{Re}\{p(z)\} \geqslant 0, \quad (z \in \mathbb{D}). \tag{7}$$

Since $\mathfrak{Re}\{p(z)\}$ is a harmonic function, it is well known that its zeros cannot be isolated, which directly implies $\mathfrak{Re}\{p(z)\} > 0$ if $\mathfrak{Re}\{p(z)\} \geqslant 0$. It completes the proof and $p(z)$ is a Carathéodory function. To show that there are functions satisfying (4) consider $p(z)$ such that

$$p(z) \prec 1+z.$$

Then by (3), we have

$$\begin{aligned}\int_0^{2\pi} \left|\mathfrak{Re}\{p(re^{i\theta})\}\right| \mathrm{d}\theta &\leqslant \int_0^{2\pi} \left|\mathfrak{Re}\{1+re^{i\theta}\}\right| \mathrm{d}\theta \\ &= 2\pi. \quad \square\end{aligned}$$

Let $\mathscr{A}$ denote the class of analytic functions in $\mathbb{D}$ and normalized, i.e. $\mathscr{A} = \{f \in \mathscr{H} : f(0) = 0, f'(0) = 1\}$. By $\mathscr{S}$ we denote the class of functions $f \in \mathscr{A}$ that are univalent in $\mathbb{D}$. A function $f \in \mathscr{A}$ is said to be starlike if it maps $\mathbb{D}$ onto a starlike domain with respect to the origin. It is known that it is equivalent to $f \in \mathscr{A}$ and

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in \mathbb{D}.$$

We denote by $\mathscr{S}^*$ the class of starlike functions.

A set E is said to be convex if and only if it is starlike with respect to each of its points, that is if and only if the linear segment joining any two points of E lies entirely in E. A function $f \in \mathscr{S}$ maps $\mathbb{D}$ onto a convex domain E if and only if

$$\mathfrak{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > 0, \quad z \in \mathbb{D}. \tag{8}$$

Such a function f is said to be convex in $\mathbb{D}$ (or briefly convex) and the class of convex functions we denote by $\mathscr{CV}$.

Applying Lemma 1, we have the following theorems.

THEOREM 1. *Let $f(z) \in \mathscr{A}$ and let $zf''(z)/f'(z) \in \mathscr{H}$. Suppose that*

$$\int_0^{2\pi} \left|1+\mathfrak{Re}\frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\right| \mathrm{d}\theta \leqslant 2\pi \tag{9}$$

for all $r \in [0,1)$. Then

$$1+\mathfrak{Re}\frac{zf''(z)}{f'(z)} > 0, \quad (z \in \mathbb{D}),$$

so $f(z)$ is a convex function.

Proof. Applying Lemma 1 for

$$p(z) = 1+\frac{zf''(z)}{f'(z)},$$

we obtain $\mathfrak{Re}\{p(z)\} > 0$, $z \in \mathbb{D}$. Therefore, by (8) $f(z)$ is a convex function. $\square$

COROLLARY 1. *Let* $f(z) \in \mathscr{A}$ *and let* $zf''(z)/f'(z) \in \mathscr{H}$. *Suppose that*

$$\int_0^{2\pi} \left| 1 + \mathfrak{Re}\frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})} \right| \mathrm{d}\theta \leqslant 2\pi \tag{10}$$

for all $r \in [0,1)$. *Then* $f(z)$ *is also a convex function.*

THEOREM 2. *Let* $f(z) \in \mathscr{A}$ *and let* $zf'(z)/f(z) \in \mathscr{H}$. *Then a sufficient condition for* $f(z)$ *to be starlike in* $\mathbb{D}$ *is*

$$\int_0^{2\pi} \left| \mathfrak{Re}\frac{re^{i\theta} f'(re^{i\theta})}{f(re^{i\theta})} \right| \mathrm{d}\theta \leqslant 2\pi \tag{11}$$

for all $r \in [0,1)$.

Proof. Applying Lemma 1 for

$$p(z) = \frac{zf'(z)}{f(z)},$$

we obtain $\mathfrak{Re}\{p(z)\} > 0$, $z \in \mathbb{D}$. Therefore, by (1) $f(z)$ is a starlike function. $\square$

For another recent sufficient condition for starlikeness we refer to [4]. If $f \in \mathscr{A}$ satisfies

$$\mathfrak{Re}\left\{ \frac{zf'(z)}{e^{i\alpha} g(z)} \right\} > 0, \quad (z \in \mathbb{D}) \tag{12}$$

for some $g(z) \in \mathscr{S}^*$ and some $\alpha \in (-\pi/2, \pi/2)$, then $f(z)$ is said to be close-to-convex (with respect to $g(z)$) in $\mathbb{D}$ and denoted by $f(z) \in \mathscr{C}$. An univalent function $f \in \mathscr{S}$ belongs to $\mathscr{C}$ if and only if the complement E of the image-region $F = \{f(z) : |z| < 1\}$ is the union of rays that are disjoint (except that the origin of one ray may lie on another one of the rays), see also [5].

THEOREM 3. *Let* $f(z) \in \mathscr{A}$. *Then a sufficient condition for* $f(z)$ *to be close-to-convex function is*

$$\int_0^{2\pi} \left| \mathfrak{Re}\frac{zf'(z)}{h(z)} \right| \mathrm{d}\theta \leqslant 2\pi \tag{13}$$

for all $r \in [0,1)$ *and for some* $h(z) \in \mathscr{S}^*$, *where* $z = re^{i\theta}$.

Proof. If $h(z) \in \mathscr{S}^*$, then by Lemma 1, we have

$$\mathfrak{Re}\frac{zf'(z)}{h(z)} > 0, \quad (z \in \mathbb{D}).$$

Therefore, by (12) $f(z)$ is a close-to-convex with respect to $h(z)$. $\square$

Recall that $f(z) \in \mathscr{A}$ called a Bazilevič function of type β if there exists a starlike function

$$g(z) = z + \sum_{n=2} b_n z^n, \quad (z \in \mathbb{D})$$

such that

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)}\right\} > 0, \quad z \in \mathbb{D},$$

where $\beta > 0$.

THEOREM 4. *If $f(z)$ is in the class $\mathscr{A}$, then a sufficient condition for $f(z)$ to be a Bazilevič function of type $\beta > 0$ is that*

$$\int_0^{2\pi} \left|\mathfrak{Re}\frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)}\right| d\theta \leqslant 2\pi \tag{14}$$

for all $r \in [0,1)$ and for some $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, where $z = re^{i\theta}$.

Let $\mathscr{A}(p)$ denote the class of all functions analytic in the unit disk $\mathbb{D}$ which have the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad (z \in \mathbb{D}), \tag{15}$$

where p is positive integer. A function $f(z)$ meromorphic in a domain $D \subset \mathbb{C}$ is said to be p-valent in D if for each w the equation $f(z) = w$ has at most p roots in D, where roots are counted in accordance with their multiplicity, and there is some v such that the equation $f(z) = v$ has exactly p roots in D. In [6] S. Ozaki proved that if $f(z)$ of the form (15) is analytic in a convex domain $D \subset \mathbb{C}$ and for some real α we have

$$\mathfrak{Re}\{\exp(i\alpha) f^{(p)}(z)\} > 0, \quad z \in D,$$

then $f(z)$ is at most p-valent in D. Ozaki's condition is a generalization of the well known Noshiro-Warschawski univalence condition, [2], [7].

THEOREM 5. *If $f(z)$ is in the class $\mathscr{A}(p)$ and suppose that*

$$\int_0^{2\pi} \left|\mathfrak{Re}\{f^{(p)}(z)/p!\right| d\theta \leqslant 2\pi \tag{16}$$

for all $r \in [0,1)$, where $z = re^{i\theta}$, then $f(z)$ is at most p-valent in $\mathbb{D}$.

Proof. Applying Lemma 1, we can get

$$\mathfrak{Re}\{f^{(p)}(z)\} > 0, \quad (z \in \mathbb{D}),$$

then by the above Ozaki's result, we have that $f(z)$ is at most p-valent in $\mathbb{D}$. $\square$

LEMMA 2. [3] *Let* $q(z) = 1 + \sum_{n=m}^{\infty} c_n z^n$, $c_m \neq 0$ *be an analytic function in* $\mathbb{D}$ *with* $q(z) \neq 0$. *If there exists a point* z_0, $|z_0| < 1$, *such that* $|\arg\{q(z)\}| < \pi\gamma/2$ *for* $|z| < |z_0|$ *and* $|\arg\{q(z_0)\}| = \pi\gamma/2$ *for some* $\gamma > 0$, *then we have*

$$\frac{z_0 q'(z_0)}{q(z_0)} = \frac{2ik \arg\{q(z_0)\}}{\pi}, \tag{17}$$

for some $k \geqslant m(a+a^{-1})/2 \geqslant (a+a^{-1})/2$, *where* $\{q(z_0)\}^{1/\gamma} = \pm ia$, *and* $a > 0$.

THEOREM 6. *Let* $p(z)$, $p(0) = 1$, *be analytic in* $|z| < 1$ *and suppose that there exists a real number* α, $0 \leqslant \alpha < 1$, *for which*

$$\left| \mathfrak{Im}\left\{ \frac{zp'(z)}{p(z)-\alpha} \right\} \right| < 1, \quad (z \in \mathbb{D}). \tag{18}$$

Then we have

$$\mathfrak{Re}\{p(z)\} > \alpha, \quad (z \in \mathbb{D}). \tag{19}$$

Proof. Suppose that there exists a point z_0, $|z_0| < 1$ such that

$$\mathfrak{Re}\{p(z)\} > \alpha \quad \text{for} \quad |z| < |z_0|$$

and

$$\mathfrak{Re}\{p(z_0)\} = \alpha,$$

where $0 \leqslant \alpha < 1$ and $p(z_0) \neq \alpha$. Putting

$$q(z) = \frac{p(z)-\alpha}{1-\alpha}, \quad q(0) = 1,$$

we have

$$\mathfrak{Re}\{q(z)\} > 0 \quad \text{for} \quad |z| < |z_0|$$

and

$$\mathfrak{Re}\{q(z_0)\} = 0, \quad q(z_0) \neq 0.$$

We apply now Lemma 2 for $q(z)$, with $m = \gamma = 1$. For the case $\arg\{p(z_0) - \alpha\} = \pi/2$, we have $\arg\{q(z_0)\} = \pi/2$ too and by (17) $z_0 q'(z_0)/q(z_0) = ik$. Therefore, we have

$$\begin{aligned}
\frac{z_0 q'(z_0)}{q(z_0)} &= \frac{z_0 p'(z_0)}{p(z_0)-\alpha} \\
&= \left[\frac{\mathrm{d}\log(p(z)-\alpha)}{\mathrm{d}\log z} \right]_{z=z_0} \\
&= \left[\frac{\mathrm{d}\arg\{p(z)-\alpha\}}{\mathrm{d}\theta} - \frac{i}{|p(z)-\alpha|} \frac{\mathrm{d}|(p(z)-\alpha|}{\mathrm{d}\theta} \right]_{z=z_0} \\
&= \left[\frac{-i}{|p(z)-\alpha|} \frac{\mathrm{d}|(p(z)-\alpha|}{\mathrm{d}\theta} \right]_{z=z_0} \\
&= ik
\end{aligned}$$

and therefore, we have

$$\frac{z_0 p'(z_0)}{p(z_0)-\alpha} = i\,\mathfrak{Im}\frac{z_0 p'(z_0)}{p(z_0)-\alpha} = ik,$$

where

$$k \geqslant \frac{a^2+1}{2a} \geqslant 1,$$

where $p(z_0)-\alpha = ia$ and $a>0$, this contradicts (18).

Similarly, for the case , when $\arg\{p(z_0)-\alpha\} = -\pi/2$, we have

$$\frac{z_0 p'(z_0)}{p(z_0)-\alpha} = -i\,\mathfrak{Im}\frac{z_0 p'(z_0)}{p(z_0)-\alpha} = -ik,$$

where k is a real number and $k \geqslant (a^2+1)/(2a) \geqslant 1$, $p(z_0)-\alpha = -ia$ and $a>0$. This contradicts (18) too. $\square$

THEOREM 7. *Let $p(z)$, $p(0)=1$, be analytic in $|z|<1$ and suppose that there exists a real number β, $\beta>1$, for which*

$$\left|\mathfrak{Im}\left\{\frac{zp'(z)}{p(z)-\beta}\right\}\right| < 1, \quad (z\in\mathbb{D}). \tag{20}$$

Then we have

$$\mathfrak{Re}\{p(z)\} < \beta, \quad (z\in\mathbb{D}). \tag{21}$$

Proof. Let us put

$$q(z) = \frac{\beta-p(z)}{\beta-1}, \quad q(0)=1.$$

If there exists a point z_0, $|z_0|<1$ such that

$$|\arg\{q(z)\}| < \frac{\pi}{2} \quad \text{for } |z|<|z_0|$$

and

$$|\arg\{q(z_0)\}| = \frac{\pi}{2}, \quad q(z_0)\neq 0,$$

then applying Lemma 2, we have the following properties:

For the case $\arg\{q(z_0)\} = \pi/2$,

$$\frac{z_0 q'(z_0)}{q(z_0)} = \frac{-z_0 p'(z_0)}{\beta-p(z_0)} = ik, \tag{22}$$

where

$$\frac{\beta-p(z_0)}{\beta-1} = q(z_0) = ia,\ a>0$$

and where

$$k \geqslant \frac{1}{2}\left(a+\frac{1}{a}\right) \geqslant 1$$

Together with (22), this contradicts (20).

For the case $\arg\{q(z_0)\} = -\pi/2$,

$$\frac{z_0 q'(z_0)}{q(z_0)} = \frac{-z_0 p'(z_0)}{\beta - p(z_0)} = -ik,$$

where

$$k \geqslant \frac{1}{2}\left(a + \frac{1}{a}\right),$$

$$\frac{\beta - p(z_0)}{\beta - 1} = q(z_0) = -ia, \; a > 0,$$

this also contradicts (20). □

From Theorems 6 and 7 it follows immediately the following corollaries.

COROLLARY 2. *Let $p(z)$, $p(0) = 1$, be analytic in $|z| < 1$ and suppose that there exist real numbers α, $0 \leqslant \alpha < 1$ and β, $\beta > 1$, for which*

$$\left|\mathfrak{Im}\left\{\frac{zp'(z)}{p(z) - \alpha}\right\}\right| < 1, \quad (z \in \mathbb{D}). \tag{23}$$

and at the same time

$$\left|\mathfrak{Im}\left\{\frac{zp'(z)}{p(z) - \beta}\right\}\right| < 1, \quad (z \in \mathbb{D}). \tag{24}$$

Then we have

$$\alpha < \mathfrak{Re}\{p(z)\} < \beta, \quad (z \in \mathbb{D}). \tag{25}$$

COROLLARY 3. *Let $p(z)$, $p(0) = 1$, be analytic in $|z| < 1$ and suppose that there exist real numbers α, $0 \leqslant \alpha < 1$ and β, $\beta > 1$, for which*

$$\left(\mathfrak{Im}\left\{\frac{zp'(z)}{p(z) - \alpha}\right\}\right)^2 + \left(\mathfrak{Im}\left\{\frac{zp'(z)}{p(z) - \beta}\right\}\right)^2 < 1, \quad (z \in \mathbb{D}). \tag{26}$$

Then we have

$$\alpha < \mathfrak{Re}\{p(z)\} < \beta, \quad (z \in \mathbb{D}). \tag{27}$$

Putting

$$p(z) = \frac{zf'(z)}{f(z)},$$

it follows that

$$\frac{zp'(z)}{p(z)-\alpha} = \frac{z\left\{\frac{(f'(z)+f''(z))f(z)-z(f'(z))^2}{(f(z))^2}\right\}}{\frac{zf'(z)}{f(z)}-\alpha}$$
$$= \frac{\frac{zf'(z)}{f(z)}+\left(\frac{zf'(z)}{f(z)}\right)\left(\frac{zf''(z)}{f'(z)}\right)-\left(\frac{zf'(z)}{f(z)}\right)^2}{\frac{zf'(z)}{f(z)}-\alpha}$$
$$= \frac{\frac{zf'(z)}{f(z)}\left\{1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right\}}{\frac{zf'(z)}{f(z)}-\alpha}$$
$$= \frac{1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}}{1-\alpha\left(\frac{zf'(z)}{f(z)}\right)^{-1}}$$
$$= \frac{1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}}{1-\alpha\frac{f(z)}{zf'(z)}}$$

Therefore, applying Corollaries 2 and 3 we obtain the following result.

COROLLARY 4. *Let* $f(z) = z+\sum_{n=2}^{\infty} a_n z^n$ *be analytic in* $\mathbb{D}$. *Suppose that there exist real numbers* α, $0 \leqslant \alpha < 1$ *and* β, $\beta > 1$, *for which*

$$\mathfrak{Im}\left|\frac{1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}}{1-\alpha\left(\frac{f(z)}{zf'(z)}\right)}\right| < 1, \quad (z\in\mathbb{D}), \tag{28}$$

$$\mathfrak{Im}\left|\frac{1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}}{1-\beta\left(\frac{f(z)}{zf'(z)}\right)}\right| < 1, \quad (z\in\mathbb{D}) \tag{29}$$

and

$$\left(\mathfrak{Im}\left\{\frac{1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}}{1-\alpha\left(\frac{f(z)}{zf'(z)}\right)}\right\}\right)^2+\left(\mathfrak{Im}\left\{\frac{1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}}{1-\beta\left(\frac{f(z)}{zf'(z)}\right)}\right\}\right)^2 < 1, \quad (z\in\mathbb{D}). \tag{30}$$

Then (28) *implies*

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad (z\in\mathbb{D}),$$

while (29) *implies*

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < \beta, \quad (z\in\mathbb{D}).$$

Furthermore, (28) *together with* (29)*, or* (30) *only, imply*

$$\alpha < \mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < \beta, \quad (z \in \mathbb{D}). \tag{31}$$

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