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BOUNDEDNESS OF BI–PARAMETER LITTLEWOOD–PALEY g_λ^*–FUNCTION ON HARDY SPACES

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Abstract. In this paper, we prove the $H^p - L^p$ boundedness of bi-parameter Littlewood-Paley g_λ^*-function with p less than 1, improving on the $H^1 - L^1$ boundedness due to Li and Xue. The main tools include Journé's covering lemma, vector-valued theory and the atomic decomposition of product Hardy spaces.

1. Introduction

M. P. Malliavia and P. Malliavia [15], Gundy and Stein [7] introduced and systematically studied the product Hardy spaces in the 1970s. The theory of atomic decomposition of the product Hardy spaces was established by Chang and R. Fefferman [2]. It is known that the atomic decomposition of $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ is more complicated than classical Hardy spaces $H^p(\mathbb{R}^n)$. And the atomic decomposition of the product Hardy spaces $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ is a powerful tool to verify the $H^p(\mathbb{R}^n \times \mathbb{R}^m) - L^p(\mathbb{R}^n \times \mathbb{R}^m)$ and $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ boundedness of multi-parameter singular integral operators. For example, applying the rectangle atomic decomposition of product Hardy spaces $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ and Journé's covering lemma, R. Fefferman [4] proved $H^p(\mathbb{R}^n \times \mathbb{R}^m) - L^p(\mathbb{R}^n \times \mathbb{R}^m)$ boundedness of Journé's product singular integrals. More recently, Han, Lee et al. [10] showed that the Journé's singular integral are bounded on $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ in terms of vector-valued singular integral theory, Littlewood-Paley theory together with Fefferman's rectangle atomic decomposition and Journé's covering lemma. For more work about properties of product Hardy spaces and boundedness of operators on product Hardy spaces, one may refer to [5], [6] and [11].

Now, we recall the definition of the product Hardy spaces $H^p(\mathbb{R}^n \times \mathbb{R}^m)$. Given $0 < p \leqslant 1$, let

$$C_{0,0}^\infty(\mathbb{R}^n) = \Big\{ \psi \in C^\infty(\mathbb{R}^n) : \psi \text{ has a compact support and}$$
$$\int_{\mathbb{R}^n} \psi(x) x^\alpha dx = 0 \text{ for } 0 \leqslant |\alpha| \leqslant N_{p,n} \Big\},$$

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where $N_{p,n}$ is a large positive integer depending on p and n. Let $\psi \in C^{\infty}_{0,0}(\mathbb{R}^n)$ satisfy the condition

$$\int_0^{\infty} |\hat{\psi}(t\xi)|^2 \frac{dt}{t} = 1 \tag{1}$$

for all $\xi \neq 0$. Let $n_1 = n, n_2 = m$, $\psi^i \in C^{\infty}_{0,0}$ supported in the unit ball of $\mathbb{R}^{n_i}$ and ψ^i satisfy (1) with $i = 1,2$. For $t_i > 0$ and $(x_1,x_2) \in \mathbb{R}^n \times \mathbb{R}^m$, set $\psi^i_{t_i} = t_i^{-n_i}\psi(x_i/t_i)$ and $\psi_{t_1t_2}(x_1,x_2) = \psi^1_{t_1}(x_1)\psi^2_{t_2}(x_2)$, the product Hardy space $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is defined by

$$H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) = \left\{ f \in \mathscr{S}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) : g(f) \in L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \right\}$$

with $\|f\|_{H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} := \|g(f)\|_{L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})}$, where

$$g(f)(x_1,x_2) = \left\{ \int_0^{\infty}\int_0^{\infty} |\psi_{t_1t_2} * f(x_1,x_2)|^2 \frac{dt_1}{t_1}\frac{dt_2}{t_2} \right\}^{1/2}.$$

H^p-atoms on $\mathbb{R}^n \times \mathbb{R}^m$ are defined as follows.

DEFINITION 1. [2] A function $a(x_1,x_2)$ defined on $\mathbb{R}^n \times \mathbb{R}^m$ is called an $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ atom if $a(x_1,x_2)$ is supported in an open set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ with finite measure and satisfies the following conditions:
(i) $\|a\|_{L^2} \leqslant |\Omega|^{1/2-1/p}$;
(ii) $a(x_1,x_2)$ can futher be decomposed as $a(x_1,x_2) = \sum_{R\in\mathscr{M}(\Omega)} a_R(x_1,x_2)$, where a_R are supported on the double of $R = I \times J (I$ is a dyadic cubes in $\mathbb{R}^n$ and J is a dyadic cubes in $\mathbb{R}^n$) and $\mathscr{M}(\Omega)$ is the collection of all maximal dyadic rectangle contained in Ω,

$$\left\{ \sum_{R\in\mathscr{M}(\Omega)} \|a_R\|^2_{L^2} \right\}^{1/2} \leqslant |\Omega|^{1/2-1/p};$$

(iii) for all $x_2 \in \mathbb{R}^m, 0 \leqslant |\alpha| \leqslant N_{p,n}$,

$$\int_{2I} a_R(x_1,x_2)x_1^{\alpha}dx_1 = 0;$$

and for all $x_1 \in \mathbb{R}^n, 0 \leqslant |\alpha| \leqslant N_{p,m}$,

$$\int_{2J} a_R(x_1,x_2)x_2^{\alpha}dx_2 = 0.$$

Chang and R. Fefferman [2] obtained atomic decomposition of $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ which is a key tool in this paper.

LEMMA 1. *A distribution* $f \in H^p(\mathbb{R}^n \times \mathbb{R}^m)$ *if and only if* $f = \sum_j \lambda_j a_j$, *where* a_j *are* $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ *atoms,* $\sum_j |\lambda_j|^p < \infty$, *and the series converges in the distribution sense. Moreover,* $\|f\|^p_{H^p}$ *is equivalent to* $\inf\{\sum_j |\lambda_j|^p : \textit{for all } f = \sum_j \lambda_j a_j\}$.

As we all known, Littlewood and Paley introduced the Littlewood-Paley g_λ^* -functions in the 1930s. These functions play important roles in the analysis of L^p bounds for various linear operators. With the development of real-variable method from the 1950s, Stein [17] studied the classical Littlewood-Paley g_λ^* -functions in the setting of higher dimension, and which are important tools in the harmonic analysis and other fields. See [3], [9], [13], [17] and [18], for more results about Littlewood-Paley g_λ^* -functions.

Recently, Cao and Xue [1] introduced a class of bi-parameter Littlewood-Paley g_λ^* -functions as follows:

DEFINITION 2. Suppose that $\lambda=(\lambda_1,\lambda_2)$ with $\lambda_1,\lambda_2>1$, $t_1,t_2>0$ and $(x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^m$, the bi-parameter Littlewood-Paley g_λ^* -function is defined by

$$g_\lambda^*(f)(x_1,x_2)=\left(\int\int_{\mathbb{R}_+^{m+1}}\left(\frac{t_2}{t_2+|x_2-y_2|}\right)^{m\lambda_2}\int\int_{\mathbb{R}_+^{n+1}}\left(\frac{t_1}{t_1+|x_1-y_1|}\right)^{n\lambda_1}\right.$$
$$\left.\times|\theta_{t_1,t_2}(f)(y_1,y_2)|^2\frac{dt_1dy_1}{t_1^{n+1}}\frac{dt_2dy_2}{t_2^{m+1}}\right)^{1/2},$$

where

$$\theta_{t_1,t_2}(f)(y_1,y_2)=\int_{\mathbb{R}^n\times\mathbb{R}^m}K_{t_1,t_2}(y_1,y_2,z_1,z_2)f(z_1,z_2)dz_1dz_2.$$

Using the random dyadic grids and martingale difference decomposition, Cao and Xue [1] proved the L^2 boundedness of the above Littlewood-Paley g_λ^* -functions. Moreover, Li and Xue [12] showed that bi-parameter Littlewood-Paley g_λ^* -functions are bounded from $H^1(\mathbb{R}^n\times\mathbb{R}^m)$ to $L^1(\mathbb{R}^n\times\mathbb{R}^m)$ by Fefferman's rectangle atomic decomposition and Journé's covering lemma.

The main purpose of the paper is to present that bi-parameter Littlewood-Paley g_λ^* -function are bounded from $H^p(\mathbb{R}^n\times\mathbb{R}^m)$ to $L^p(\mathbb{R}^n\times\mathbb{R}^m)$ with the index p less than 1. To achieve our goal, the key tools are vector-valued theory, the atomic decomposition of product Hardy spaces and Journé's covering lemma.

In order to present our main result, we show assumptions for kernels $K_{t,s}$ of bi-parameter Littlewood-Paley g_λ^* -functions. In this paper, we always assume that indices $\alpha,\beta>0$.

ASSUMPTION 1.1. [14] (Full standard estimates) The kernel $K_{t,s}:\mathbb{R}^{n+m}\times\mathbb{R}^{n+m}\to\mathbb{C}$ is assumed to satisfy the size estimate

$$|K_{t_1,t_2}(x_1,x_2,y_1,y_2)|\leqslant C\frac{t_1^\alpha}{(t_1+|x_1-y_1|)^{n+\alpha}}\frac{t_2^\beta}{(t_2+|x_2-y_2|)^{m+\beta}}.$$

We also assume the two mixed Hölder and size estimate:

$$|K_{t_1,t_2}(x_1,x_2,y_1,y_2)-K_{t_1,t_2}(x_1,x_2,z_1,y_2)|\leqslant C\frac{|y_1-z_1|^\alpha}{(t_1+|x_1-y_1|)^{n+\alpha}}\frac{t_2^\beta}{(t_2+|x_2-y_2|)^{m+\beta}},$$

when $|y_1-z_1|\leqslant t_1/2$, and

$$|K_{t_1,t_2}(x_1,x_2,y_1,y_2)-K_{t_1,t_2}(x_1,x_2,y_1,z_2)|\leqslant C\frac{t_1^\alpha}{(t_1+|x_1-y_1|)^{n+\alpha}}\frac{|y_2-z_2|^\beta}{(t_2+|x_2-y_2|)^{m+\beta}},$$

when $|y_2-z_2|\leqslant t_2/2$. Finally, we assume the Hölder estimate

$$\begin{aligned}&|K_{t_1,t_2}(x_1,x_2,y_1,y_2)-K_{t_1,t_2}(x_1,x_2,z_1,y_2)-K_{t_1,t_2}(x_1,x_2,y_1,z_2)+K_{t_1,t_2}(x_1,x_2,z_1,z_2)|\\ \leqslant&C\frac{|y_1-z_1|^\alpha}{(t_1+|x_1-y_1|)^{n+\alpha}}\frac{|y_2-z_2|^\beta}{(t_2+|x_2-y_2|)^{m+\beta}},\end{aligned}$$

when $|y_1-z_1|\leqslant t_1/2$ and $|y_2-z_2|\leqslant t_2/2$.

ASSUMPTION 1.2. [12] (Strong Carleson condition $\times$ standard estimates) Let I be a cube with side length $l(I)$ in $\mathbb{R}^n$, we define the associated Carleson box $\widehat{I}=I\times(0,l(I))$. We assume the following combinations of Carleson and size conditions: For every cube $I\subset\mathbb{R}^n$ and $J\subset\mathbb{R}^m$ there holds that

$$\begin{aligned}&\left(\int\int_{\widehat{I}}\int_{\mathbb{R}^n}\int_I|K_{t_1,t_2}(y_1,y_2,z_1,z_2)|^2\left(\frac{t_1}{t_1+|x_1-y_1|}\right)^{n\lambda_1}dz_1\frac{dy_1dt_1dx_1}{t_1^{n+1}}\right)^{1/2}\\ \leqslant&C\frac{t_2^\beta}{(t_2+|y_2-z_2|)^{m+\beta}}\end{aligned}$$

and

$$\begin{aligned}&\left(\int\int_{\widehat{J}}\int_{\mathbb{R}^m}\int_J|K_{t_1,t_2}(y_1,y_2,z_1,z_2)|^2\left(\frac{t_2}{t_2+|x_2-y_2|}\right)^{m\lambda_2}dz_2\frac{dy_2dt_2dx_2}{t_2^{m+1}}\right)^{1/2}\\ \leqslant&C\frac{t_1^\alpha}{(t_1+|y_1-z_1|)^{n+\alpha}}.\end{aligned}$$

We also assume the following two mixed Carleson and Hölder conditions: For every cube $I\subset\mathbb{R}^n$ and $J\subset\mathbb{R}^m$ there holds that:

$$\begin{aligned}&\left(\int\int_{\widehat{I}}\int_{\mathbb{R}^n}\int_I|K_{t_1,t_2}(y_1,y_2,z_1,z_2)-K_{t_1,t_2}(y_1,y_2,z_1,z_2')|^2\right.\\ &\qquad\left.\times\left(\frac{t_1}{t_1+|x_1-y_1|}\right)^{n\lambda_1}dz_1\frac{dy_1dt_1dx_1}{t_1^{n+1}}\right)^{1/2}\leqslant C\frac{|z_2-z_2'|^\beta}{(t_2+|y_2-z_2|)^{m+\beta}}\end{aligned}$$

when $|z_2-z_2'|\leqslant t_2/2$, and

$$\begin{aligned}&\left(\int\int_{\widehat{J}}\int_{\mathbb{R}^m}\int_J|K_{t_1,t_2}(y_1,y_2,z_1,z_2)-K_{t_1,t_2}(y_1,y_2,z_1',z_2)|^2\right.\\ &\qquad\left.\times\left(\frac{t_2}{t_2+|x_2-y_2|}\right)^{m\lambda_2}dz_2\frac{dy_2dt_2dx_2}{t_2^{m+1}}\right)^{1/2}\leqslant C\frac{|z_1-z_1'|^\alpha}{(t_1+|y_1-z_1|)^{n+\alpha}},\end{aligned}$$

when $|z_1-z_1'|\leqslant t_2/2$.

ASSUMPTION 1.3. [14] (Bi-parameter Carleson condition) Let $\mathscr{D} = \mathscr{D}_n \times \mathscr{D}_m$, where $\mathscr{D}_n$ is a dyadic grid in $\mathbb{R}^n$ and $\mathscr{D}_m$ is a dyadic grid in $\mathbb{R}^m$. For $I \in \mathscr{D}_n$, let $W_I = I \times (l(I)/2, l(I))$ be the associated Whitney region. For $J \in \mathscr{D}_m$, let $W_J = J \times (l(J)/2, l(J))$ be the associated Whitney region. Denote $n_1 = n, n_2 = m$ and

$$\mathscr{C}_{IJ}^{\mathscr{D}} = \int\int_{W_J}\int\int_{W_I}\int_{\mathbb{R}^{n+m}} |\theta_{t,s}\chi(y_1,y_2)|^2 \prod_{i=1}^{2}\left(\frac{t_i}{t_i+|x_i-y_i|}\right)^{n_i\lambda_i} dy_1dy_2 \frac{dx_1dt_1}{t_1^{n_1+1}}\frac{dx_2dt_2}{t_2^{n_2+1}}.$$

We assume the following bi-parameter Carleson condition: for every $\mathscr{D} = \mathscr{D}_n \times \mathscr{D}_m$, there holds that

$$\sum_{I\times J\in\mathscr{D}, I\times J\subset\Omega} \mathscr{C}_{IJ}^{\mathscr{D}} \leqslant C|\Omega|$$

for all sets $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ such that $|\Omega| < \infty$ and such that for every $(x_1,x_2) \in \Omega$ there exists $I \times J \in \mathscr{D}$ so that $(x_1,x_2) \in I \times J \subset \Omega$.

Now we can formulate our main result in this paper.

THEOREM 1. *Let $\lambda = (\lambda_1,\lambda_2)$ with $\lambda_1,\lambda_2 > 2$, $0 < \alpha \leqslant n(\lambda_1-2)/2$ and $0 < \beta \leqslant m(\lambda_2-2)/2$. Suppose that kernels of bi-parameter Littlewood-Paley g_λ^*-functions satisfy Assmuptions 1.1, 1.2 and 1.3 with $\alpha,\beta > 0$. Then, we have*

$$\|g_\lambda^*(f)\|_{L^p} \leqslant C\|f\|_{H^p},$$

where $\max\left\{\frac{n}{n+\alpha}, \frac{m}{m+\beta}\right\} < p \leqslant 1$.

We prove the Theorem 1 by the following two steps.

Step 1. Reduce the $H^p(\mathbb{R}^n \times \mathbb{R}^m) - L^p(\mathbb{R}^n \times \mathbb{R}^m)$ boundedness of bi-parameter Littlewood-Paley g_λ^*-function to the $H^p(\mathbb{R}^n \times \mathbb{R}^m) - L^p_{\mathscr{H}}(\mathbb{R}^{n+m})$ boundedness of linear operator $\{\theta_{t_1,t_2}(f)\}_{t_1,t_2>0}$, where $L^p_{\mathscr{H}}$ is the $\mathscr{H}$-valued L^p spaces and $\mathscr{H}$ is the Hilbert space defined by

$$\mathscr{H} = \left\{ \{h_{t_1,t_2}\}_{t_1,t_2>0} : \|\{h_{t_1,t_2}\}\|_{\mathscr{H}} = \left(\int_{\mathbb{R}_+^{m+1}}\int_{\mathbb{R}_+^{n+1}}\left(\frac{t_1}{t_1+|x_1-y_1|}\right)^{n\lambda_1}\right.\right.$$
$$\left.\left.\times\left(\frac{t_2}{t_2+|x_2-y_2|}\right)^{m\lambda_2} |\, h_{t_1,t_2}(y_1,y_2)\,|^2 \frac{dy_1dt_1}{t_1^{n+1}}\frac{dy_2dt_2}{t_2^{m+1}}\right)^{\frac{1}{2}} < \infty \right\},$$

and g_λ^*-function can be written by

$$g_\lambda^*(f) = \|\{\theta_{t_1,t_2}(f)\}_{t_1,t_2>0}\|_{\mathscr{H}} := \|T(f)\|_{\mathscr{H}}.$$

Step 2. Verify $H^p(\mathbb{R}^n \times \mathbb{R}^m) - L^p_{\mathscr{H}}(\mathbb{R}^{n+m})$ boundedness of T by H^p-atoms.

In order to carry out Step 2, we need the following criterion of the boundedness of $\mathscr{H}$-valued operators T from $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ and $L^p_{\mathscr{H}}(\mathbb{R}^{n+m})$.

LEMMA 2. [10] *Let $\mathscr{L}$ be a bounded operator from $L^2(\mathbb{R}^{n+m})$ to $L^2_{\mathscr{H}}(\mathbb{R}^{n+m})$. Then, for $0<p\leqslant 1$, $\mathscr{L}$ extends to be a bounded operator from $H^p(\mathbb{R}^n\times\mathbb{R}^m)$ to $L^p_{\mathscr{H}}(\mathbb{R}^{n+m})$ if and only if $\|\mathscr{L}(a)\|_{L^p_{\mathscr{H}}(\mathbb{R}^{n+m})}\leqslant C$ for all $H^p(\mathbb{R}^n\times\mathbb{R}^m)$ atom a, where the constant C is independence of a.*

Together Lemma 2 with the fact that g^*_λ is bounded from $L^2(\mathbb{R}^{n+m})$ to $L^2_{\mathscr{H}}(\mathbb{R}^{n+m})$, it's suffices to prove the following theorem.

THEOREM 2. *For each $H^p(\mathbb{R}^n\times\mathbb{R}^m)$-atom a,*

$$\big\|T(a)\big\|_{L^p_{\mathscr{H}}(\mathbb{R}^{n+m})}\leqslant C,$$

where the constant C is independence of a.

2. Proof of Theorem 2

In this section, we give the

Proof of Theorem 2. The proof of Theorem 2 follows Fefferman's idea [4]. Suppose that a is an $H^p(\mathbb{R}^n\times\mathbb{R}^n)$ atom which supported on an open set $\Omega\subset\mathbb{R}^n\times\mathbb{R}^m$ with finite measure. Moreover, a can be decomposed as $a=\sum_{R\in\mathscr{M}(\Omega)}a_R$, where $\mathscr{M}(\Omega)$ is the collection of all maximal dyadic subrectangles contained in Ω with supp $a_R\subset 2R=2I\times 2J$, $\int_{2I}a_R(x_1,x_2)dx_1=0$ for all $x_2\in 2J$, and $\int_{2J}a_R(x_1,x_2)dx_2=0$ for all $x_1\in 2I$, where I and J are dyadic cubes in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Furthermore,

$$\|a\|_{L^2}\leqslant|\Omega|^{1/2-1/p}\ \text{ and }\ \sum_{R\in\mathscr{M}(\Omega)}\|a_R\|^2_{L^2}\leqslant|\Omega|^{1-2/p}.$$

Let

$$\Omega_1=\big\{(x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^m:M_S(\chi_\Omega)(x_1,x_2)>4^{-n-m}n^{-\frac{n}{2}}m^{-\frac{m}{2}}\big\},$$

where M_S is the strong maximal function defined by

$$M_S(f)(x_1,x_2)=\sup_{(x_1,x_2)\in R}\frac{1}{|R|}\int_R|f(y_1,y_2)|dy_1dy_2,$$

where the supremum is taken over all rectangles $R=I\times J$ with sides parallel to the axis and containing the point (x_1,x_2). It follows from the $L^p(1<p<\infty)$ boundedness of the strong maximal operator M_s that $|\Omega_1|\leqslant C|\Omega|$.

Denote

$$\Omega_2=\Big\{(x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^m:M_S(\chi_{\Omega_1})(x_1,x_2)>4^{-n-m}n^{-\frac{n}{2}}m^{-\frac{m}{2}}\Big\},$$
$$\Omega_3=\Big\{(x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^m:M_S(\chi_{\Omega_2})(x_1,x_2)>4^{-n-m}n^{-\frac{n}{2}}m^{-\frac{m}{2}}\Big\}.$$

Note that $\mathscr{M}(\Omega)\subset\mathscr{M}_i(\Omega)$, where $\mathscr{M}_i(\Omega)$ is collection of all dyadic rectangle $R\subset\Omega$ that are maximal in x_i direction, $i=1,2$. For each $R=I\times J\in\mathscr{M}(\Omega)$, we choose $\widehat{I}$ is the largest dyadic cube containing I such that $\widehat{R}=\widehat{I}\times J\subset\Omega_1$. Similarly, we can choose $\widehat{J}$ is the largest dyadic cube containing J such that $\widehat{\widehat{R}}=\widehat{I}\times\widehat{J}\subset\Omega_2$. Note that $4\sqrt{n}\widehat{I}\times 4\sqrt{m}\widehat{J}\subset\Omega_3$.

Then, we have

$$\begin{aligned}&\int_{\mathbb{R}^n\times\mathbb{R}^m}\|T(a)(x_1,x_2)\|_{\mathscr{H}}^p dx_1dx_2\\ =&\int_{\Omega_3}\|T(a)(x_1,x_2)\|_{\mathscr{H}}^p dx_1dx_2+\int_{(\Omega_3)^c}\|T(a)(x_1,x_2)\|_{\mathscr{H}}^p dx_1dx_2.\end{aligned}$$

Hölder's inequality, the $L^2-L^2_{\mathscr{H}}$ boundedness of T and size condition of $H^p(\mathbb{R}^n\times\mathbb{R}^m)$ atom a yield that

$$\begin{aligned}\int_{\Omega_3}\|T(a)(x_1,x_2)\|_{\mathscr{H}}^p dx_1dx_2&\leqslant\Big(\int_{\Omega_3}\|T(a)(x_1,x_2)\|_{\mathscr{H}}^2 dx_1dx_2\Big)^{p/2}|\Omega_3|^{1-p/2}\\ &\leqslant C\|a\|_{L^2}^p|\Omega|^{1-p/2}\\ &\leqslant C.\end{aligned}$$

Thus, we need to deal with

$$\int_{(\Omega_3)^c}\|T(a)(x_1,x_2)\|_{\mathscr{H}}^p dx_1dx_2\leqslant C\sum_{R\in\mathscr{M}(\Omega)}\int_{(\Omega_3)^c}\|T(a)(x_1,x_2)\|_{\mathscr{H}}^p dx_1dx_2,$$

where we use the inequality $(a+b)^p\leqslant a^p+b^p$ for $p\leqslant 1$.

Then

$$\begin{aligned}&\int_{(\Omega_3)^c}\|T(a)(x_1,x_2)\|_{\mathscr{H}}^p dx_1dx_2\\ \leqslant&\int_{(4\sqrt{n}\widehat{I})^c\times\mathbb{R}^m}\|T(a)(x_1,x_2)\|_{\mathscr{H}}^p dx_1dx_2+\int_{\mathbb{R}^n\times(4\sqrt{m}\widehat{J})^c}\|T(a)(x_1,x_2)\|_{\mathscr{H}}^p dx_1dx_2\\ :=&U(R)+V(R).\end{aligned}$$

We write $\gamma_1(R)=\gamma_1(R,\Omega)=\frac{l(\widehat{I})}{l(I)}$ and $\gamma_2(\widehat{R})=\gamma_2(\widehat{R},\Omega_1)=\frac{l(\widehat{J})}{l(J)}$. In order to estimate $U(R)$, we have

$$\begin{aligned}&U(R)\\ &=\int_{(4\sqrt{n}\widehat{I})^c\times 4\sqrt{m}J}\|T(a)(x_1,x_2)\|_{\mathscr{H}}^p dx_1dx_2+\int_{(4\sqrt{n}\widehat{I})^c\times(4\sqrt{m}J)^c}\|T(a)(x_1,x_2)\|_{\mathscr{H}}^p dx_1dx_2\\ &:=U_1(R)+U_2(R).\end{aligned}$$

To estimate $U_1(R)$, we split the domain of t_1 and t_2 as follows

$$\begin{aligned}(t_1,t_2)\in\mathbb{R}^+\times\mathbb{R}^+\subset&\big((0,2|y_1-x_I|]\times(0,l(\widetilde{J})]\big)\cup\big([2|y_1-x_I|,|x_1-x_I|]\times(0,l(\widetilde{J})]\big)\\ &\cup\big([|x_1-x_I|,\infty)\times(0,l(\widetilde{J})]\big)\cup\big((0,2|y_1-x_I|]\times[l(\widetilde{J}),\infty)\big)\end{aligned}$$

$$
\begin{aligned}
&\cup\left([2|y_1-x_I|,|x_1-x_I|]\times[l(\widetilde{J}),\infty)\right)\cup\left([|x_1-x_I|,\infty)\times[l(\widetilde{J}),\infty)\right)\\
:=&A_1\cup A_2\cup A_3\cup A_4\cup A_5\cup A_6,
\end{aligned}
$$

where x_I and x_J are the centers of dyadic cubes I and J respectively, and denote $l(\widetilde{J})=2\sqrt{m}l(J)$.

Thus, we have

$$
\begin{aligned}
&U_1(R_1)\\
=&\int\int_{(4\sqrt{n}\widehat{I})^c\times 4\sqrt{m}J}\Bigg(\int_0^\infty\int_{\mathbb{R}^m}\left(\frac{t_2}{t_2+|x_2-y_2|}\right)^{\lambda_2 m}\int_0^\infty\int_{\mathbb{R}^n}\left(\frac{t_1}{t_1+|x_1-y_1|}\right)^{\lambda_1 n}\\
&\times\sum_{i=1}^{6}\mathscr{X}_{A_i}(t_1,t_2)\left|\int\int_{2I\times 2J}K_{t_1,t_2}(a_R)(z_1,z_2)dz_1dz_2\right|^2dy_1\frac{dt_1}{t_1^{n+1}}dy_2\frac{dt_2}{t_2^{m+1}}\Bigg)^{\frac{p}{2}}dx_1dx_2\\
=&C\sum_{i=1}^{6}\int\int_{(4\sqrt{n}\widehat{I})^c\times 4\sqrt{m}J}\Bigg(\int_0^\infty\int_{\mathbb{R}^m}\left(\frac{t_2}{t_2+|x_2-y_2|}\right)^{\lambda_2 m}\int_0^\infty\int_{\mathbb{R}^n}\left(\frac{t_1}{t_1+|x_1-y_1|}\right)^{\lambda_1 n}\\
&\times\mathscr{X}_{A_i}(t_1t_2)\left|\int\int_{2I\times 2J}K_{t_1,t_2}(a_R)(z_1,z_2)dz_1dz_2\right|^2dy_1\frac{dt_1}{t_1^{n+1}}dy_2\frac{dt_2}{t_2^{m+1}}\Bigg)^{\frac{p}{2}}dx_1dx_2\\
:=&C\sum_{i=1}^{6}B_i.
\end{aligned}
$$

Let ε_1 satisfying that $2n+\alpha\leqslant 2n+2\varepsilon_1\leqslant n\lambda_1$. Since $y_1\in 2I, y_2\in 2J$ and $(x_1,x_2)\in(4\sqrt{n}\widehat{I})^c\times 4\sqrt{m}\widehat{J}$, then we have $|x_1-y_1|\approx|x_1-x_I|$ and $|y_1-x_I|\leqslant\sqrt{n}l(I)$. Now, let us begin with the estimate of B_1. By Hölder's inequality, Carleson and size condition of $K_{t,s}$, then

$$
\begin{aligned}
B_1=&\int_{(4\sqrt{n}\widehat{I})^c\times 4\sqrt{m}J}\Bigg(\int_0^{2l(J)}\int_{\mathbb{R}^m}\left(\frac{t_2}{t_2+|x_2-y_2|}\right)^{m\lambda_2}\int_0^{2|y_1-x_I|}\int_{\mathbb{R}^n}\left(\frac{t_1}{t_1+|x_1-y_1|}\right)^{n\lambda_1}\\
&\times\left|\int_{2I\times 2J}K_{t_1,t_2}(y_1,y_2,z_1,z_2)a_R(z_1,z_2)dz_1dz_2\right|^2\frac{dt_1}{t_1^{n+1}}dy_1\frac{dt_2}{t_2^{m+1}}dy_2\Bigg)^{\frac{p}{2}}dx_1dx_2\\
\leqslant&C\|a_R\|_{L^2}^{p}|J|^{1-\frac{p}{2}}\\
&\times\int_{(4\sqrt{n}\widehat{I})^c}\Bigg(\int_0^{2l(J)}\int_{\mathbb{R}^m}\left(\frac{t_2}{t_2+|x_2-y_2|}\right)^{m\lambda_2}\int_0^{2|y_1-x_I|}\int_{\mathbb{R}^n}\left(\frac{t_1}{t_1+|x_1-y_1|}\right)^{n\lambda_1}\\
&\times\int_{4\sqrt{m}J}\int_{2I}\int_{2J}|K_{t_1,t_2}(y_1,y_2,z_1,z_2)|^2dz_1dz_2dx_2\frac{dt_1}{t_1^{n+1}}dy_1\frac{dt_2}{t_2^{m+1}}dy_2\Bigg)^{\frac{p}{2}}dx_1
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant C|J|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p\int_{(4\sqrt{n}\widehat{I})^c}\left(\int_0^{2|y_1-x_I|}\int_{\mathbb{R}^n}\int_{2I}\left(\frac{t_1}{t_1+|z_1-y_1|}\right)^{n\lambda_1}\right.\\
&\quad\left.\times\frac{t_1^{2\alpha}}{(t_1+|z_1-y_1|)^{2n+2\alpha}}dz_1dy_1\frac{dt_1}{t_1^{n+1}}\right)^{\frac{p}{2}}dx_1\\
&\leqslant C|J|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p\int_{(4\sqrt{n}\widehat{I})^c}\left(\int_0^{2l(I)}\int_{2I}\frac{t_1^{2\alpha}}{(t_1+|x_1-z_1|)^{2n+2\alpha}}dz_1\frac{dt_1}{t_1^{n+1}}\right)^{\frac{p}{2}}dx_1\\
&\leqslant C|J|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p|I|^{\frac{p}{2}}\int_{(4\sqrt{n}\widehat{I})^c}\frac{1}{|x_1-x_2|^{(n+\alpha)p}}dx_1\left(\int_0^{2l(I)}t_1^{2\alpha-1}dt_1\right)^{\frac{p}{2}}\\
&\leqslant C|J|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p|I|^{\frac{p}{2}}l(I)^{\alpha p}l(\widehat{I})^{n-(n+\alpha)p}\\
&\leqslant C\gamma_1(R)^{n-(n+\alpha)p}|R|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p,
\end{aligned}
$$

where we use the fact

LEMMA 3. [19] *Let $\lambda_1,\lambda_2>0, A\leqslant B$, then*

$$
\int_{\mathbb{R}^n}\frac{1}{(A+|y|)^{\lambda_1}(B+|x-y|)^{\lambda_2}}\leqslant CA^{n-\lambda_3}(|x|+B)^{-\lambda_4}, \tag{2}
$$

where $\lambda_3=\max\{\lambda_1,\lambda_2\}$ and $\lambda_4=\min\{\lambda_1,\lambda_2\}$.

Now, we consider the estimate of B_2. Choose an $0<\alpha'<\alpha$. Since $t_1\geqslant 2|y_1-x_I|$, then applying Hölder's inequality, cancellation condition of $H^p(\mathbb{R}^n\times\mathbb{R}^m)$ atom a, Carleson and Hölder condition of $K_{t,s}$, we have

$$
\begin{aligned}
B_2&\leqslant C\|a_R\|_{L^2}^p|J|^{1-\frac{p}{2}}\int_{(4\sqrt{n}\widehat{I})^c}\left(\int_0^{2l(J)}\int_{\mathbb{R}^m}\left(\frac{t_2}{t_2+|x_2-y_2|}\right)^{m\lambda_2}\right.\\
&\quad\times\int_{2|y_1-x_I|}^{|x_1-x_I|}\int_{\mathbb{R}^n}\left(\frac{t_1}{t_1+|x_1-y_1|}\right)^{n\lambda_1}\int_{4\sqrt{m}J}\int_{2I}\int_{2J}|K_{t_1,t_2}(y_1,y_2,z_1,z_2)\\
&\quad\left.-K_{t_1,t_2}(y_1,y_2,x_I,z_2)|^2dz_1dz_2dx_2\frac{dt_1}{t_1^{n+1}}dy_1\frac{dt_2}{t_2^{m+1}}dy_2\right)^{\frac{p}{2}}dx_1\\
&\leqslant C|J|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p\int_{(4\sqrt{n}\widehat{I})^c}\left(\int_{2|y_1-x_I|}^{|x_1-x_I|}\int_{\mathbb{R}^n}\int_{2I}\left(\frac{t_1}{t_1+|z_1-y_1|}\right)^{n\lambda_1}\right.\\
&\quad\left.\times\frac{|z_1-x_I|^{2\alpha}}{(t_1+|z_1-y_1|)^{2n+2\alpha}}dz_1dy_1\frac{dt_1}{t_1^{n+1}}\right)^{\frac{p}{2}}dx_1\\
&\leqslant C|J|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p\int_{(4\sqrt{n}\widehat{I})^c}\left(\int_{2|y_1-x_I|}^{|x_1-x_I|}\int_{2I}\frac{|z_1-x_I|^{2\alpha}}{(t_1+|x_1-z_1|)^{2n+2\alpha}}dz_1\frac{dt_1}{t_1}\right)^{\frac{p}{2}}dx_1
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant C|J|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p\int_{(4\sqrt{n}\widehat{I})^c}\left(\int_{2I}\int_{2|y_1-x_I|}^{|x_1-x_I|}\frac{|z_1-x_I|^{2(\alpha-\alpha')}}{t_1^{2\alpha'}|x_1-z_1|^{2n+2\alpha'}}\frac{dt_1}{t_1}dz_1\right)^{\frac{p}{2}}dx_1\\
&\leqslant C|J|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p|I|^{\frac{p}{2}}l(I)^{\alpha' p}\int_{(4\sqrt{n}\widehat{I})^c}\frac{1}{|x_1-x_I|^{(n+\alpha')p}}dx_1\\
&\leqslant C|J|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p|I|^{\frac{p}{2}}l(I)^{\alpha' p}\widehat{I}^{n-(n+\alpha')p}\\
&\leqslant C\gamma_1(R)^{n-(n+\alpha')p}|R|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p.
\end{aligned}
$$

Estimate for B_3. Note that $t\geqslant|x_1-x_I|\geqslant 2|y_1-x_I|$. Thanks to cancellation condition of $H^p(\mathbb{R}^n\times\mathbb{R}^m)$ atom a, Hölder's inequality and Carleson and Hölder condition of $K_{t,s}$, then

$$
\begin{aligned}
B_3\leqslant& C\|a_R\|_{L^2}^p|J|^{1-\frac{p}{2}}\int_{(4\sqrt{n}\widehat{I})^c}\left(\int_0^{2l(J)}\int_{\mathbb{R}^m}\left(\frac{t_2}{t_2+|x_2-y_2|}\right)^{m\lambda_2}\right.\\
&\times\int_{|x_1-x_I|}^{\infty}\int_{\mathbb{R}^n}\left(\frac{t_1}{t_1+|x_1-y_1|}\right)^{n\lambda_1}\int_{4\sqrt{m}J}\int_{2I}\int_{2J}|K_{t_1,t_2}(y_1,y_2,z_1,z_2)\\
&\left.-K_{t_1,t_2}(y_1,y_2,x_I,z_2)|^2dz_1dz_2dx_2\frac{dt_1}{t_1^{n+1}}dy_1\frac{dt_2}{t_2^{m+1}}dy_2\right)^{\frac{p}{2}}dx_1\\
\leqslant& C|J|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p\int_{(4\sqrt{n}\widehat{I})^c}\left(\int_{|x_1-x_I|}^{\infty}\int_{\mathbb{R}^n}\int_{2I}\left(\frac{t_1}{t_1+|z_1-y_1|}\right)^{n\lambda_1}\right.\\
&\left.\times\frac{|z_1-x_I|^{2\alpha}}{(t_1+|z_1-y_1|)^{2n+2\alpha}}dz_1dy_1\frac{dt_1}{t_1^{n+1}}\right)^{\frac{p}{2}}dx_1\\
\leqslant& C|J|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p\int_{(4\sqrt{n}\widehat{I})^c}\int_{|x_1-x_I|}^{\infty}\int_{\mathbb{R}^n}\int_{2I}\left(\frac{|z_1-x_I|^{2\alpha}}{(t_1+|z_1-y_1|)^{2n+2\alpha+1}}dz_1\frac{dt_1}{t_1}\right)^{\frac{p}{2}}dx_1\\
\leqslant& C|J|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p l(I)^{\alpha p}|I|^{\frac{p}{2}}\int_{(4\sqrt{n}\widehat{I})^c}\frac{1}{|x_1-x_I|^{(n+\alpha)p}}dx_1\\
\leqslant& C|J|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p|I|^{\frac{p}{2}}l(I)^{\alpha p}l(\widehat{I})^{n-(n+\alpha)p}\\
\leqslant& C\gamma_1(R)^{n-(n+\alpha)p}|R|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p.
\end{aligned}
$$

Since $s\geqslant l(\widetilde{J})\geqslant 2|y_2-x_J|$. Cancellation condition of $H^p(\mathbb{R}^n\times\mathbb{R}^m)$ atom a, Hölder's inequality and mixed Hölder and size condition of $K_{t,s}$ imply that

$$
\begin{aligned}
B_4\leqslant& C|J|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p\int_{(4\sqrt{n}\widehat{I})^c}\left(\int_{2l(J)}^{\infty}\int_{\mathbb{R}^m}\left(\frac{t_2}{t_2+|x_2-y_2|}\right)^{m\lambda_2}\right.\\
&\times\int_0^{2|y_1-x_I|}\int_{\mathbb{R}^n}\left(\frac{t_1}{t_1+|x_1-y_1|}\right)^{n\lambda_1}\int_{4\sqrt{m}J}\int_{2I}\int_{2J}\frac{t_1^{2\alpha}}{(t_1+|y_1-z_1|)^{2n+2\alpha}}\\
&\left.\times\frac{|z_2-x_J|^{2\beta}}{(t_2+|y_2-z_2|)^{2m+2\beta}}dz_1dz_2dx_2\frac{dt_1}{t_1^{n+1}}dy_1\frac{dt_2}{t_2^{m+1}}dy_2\right)^{\frac{p}{2}}dx_1
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant C|J|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p \int_{(4\sqrt{n}\widehat{I})^c}\left(\int_{2l(J)}^{\infty}\int_0^{2|y_1-x_I|}\int_{4\sqrt{m}J}\int_{2I}\int_{2J}\frac{t_1^{2\alpha}}{(t_1+|x_1-z_1)^{2n+2\alpha}}\right.\\
&\quad\left.\times\frac{|z_2-x_J|^{2\beta}}{(t_2+|x_2-z_2|)^{2m+2\beta}}dz_1dz_2dx_2\frac{dt_1}{t_1^{n+1}}\frac{dt_2}{t_2^{m+1}}\right)^{\frac{p}{2}}dx_1\\
&\leqslant C|J|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p|J|^p l(J)^{\beta p}\frac{1}{l(J)^{(m+\beta)p}}\\
&\quad\times\int_{(4\sqrt{n}\widehat{I})^c}\left(\int_0^{2|y_1-x_I|}\int_{2I}\frac{t_1^{2\alpha}}{(t_1+|x_1-z_1|)^{2n+2\alpha}}dz_1\frac{dt_1}{t_1}\right)^{\frac{p}{2}}dx_1\\
&\leqslant C|J|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p|I|^{\frac{p}{2}}l(I)^{\alpha p}l(\widehat{I})^{n-(n+\alpha)p}\\
&\leqslant C\gamma_1(R)^{n-(n+\alpha)p}|R|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p.
\end{aligned}
$$

Now, we turn to estimate B_5. Since $t\geqslant 2|y_1-x_I|$ and $s\geqslant l(\widetilde{J})\geqslant|y_2-x_J|$. Choose an $0<\alpha'<\alpha$, applying cancellation condition of $H^p(\mathbb{R}^n\times\mathbb{R}^m)$ atom a, Hölder's inequality and Hölder condition of $K_{t,s}$, then

$$
\begin{aligned}
B_5 &\leqslant C|J|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p\int_{(4\sqrt{n}\widehat{I})}\left(\int_{2l(J)}^{\infty}\int_{\mathbb{R}^m}\left(\frac{t_2}{t_2+|x_2-y_2|}\right)^{m\lambda_2}\right.\\
&\quad\times\int_{2|y_1-x_I|}^{|x_1-x_I|}\int_{\mathbb{R}^n}\left(\frac{t_1}{t_1+|x_1-y_1|}\right)^{n\lambda_1}\int_{4\sqrt{m}J}\int_{2I}\int_{2J}\frac{|\,z_1-x_I\,|^{2\alpha}}{(t_1+|y_1-z_1|)^{2n+2\alpha}}\\
&\quad\left.\times\frac{\left|z_2-x_J\right|^{2\beta}}{(t_2+|y_2-z_2|)^{2m+2\beta}}dz_1dz_2dx_2\frac{dt_1}{t_1^{n+1}}dy_1\frac{dt_2}{t_2^{m+1}}dy_2\right)^{\frac{p}{2}}dx_1\\
&\leqslant C|J|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p\int_{(4\sqrt{n}\widehat{I})^c}\left(\int_{2l(J)}^{\infty}\int_{2|y_1-x_I|}^{|x_1-x_I|}\int_{4\sqrt{m}J}\int_{2I}\int_{2J}\frac{|z_1-x_I|^{2\alpha}}{(t_1+|y_1-z_1|)^{2n+2\alpha}}\right.\\
&\quad\left.\times\frac{|z_2-x_I|^{2\beta}}{(t_2+|x_2-z_2|)^{2m+2\beta}}dz_1dz_2dx_2\frac{dt_1}{t_2}\frac{dt_2}{t_2}\right)^{\frac{p}{2}}dx_1\\
&\leqslant C|J|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p|J|^p lJ^{\beta p}\frac{1}{l(J)^{(m+\beta)p}}\\
&\quad\times\int_{(4\sqrt{n}\widehat{I})^c}\left(\int_{2|y_1-x_I|}^{|x_1-x_I|}\int_{2I}\frac{|y_1-x_I|^{2\alpha}}{(t_1+|x_1-z_1|)^{2n+2\alpha}}dz_1\frac{dt_1}{t_1}\right)^{\frac{p}{2}}dx_1\\
&\leqslant C|J|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p|I|^{\frac{p}{2}}l(I)^{\alpha' p}l(\widehat{I})^{n-(n+\alpha')p}\\
&\leqslant C\gamma_1(R)^{n-(n+\alpha')p}|R|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p.
\end{aligned}
$$

Estimate for B_6. Note that $t\geqslant 2|y_1-x_I|$ and $s\geqslant l(\widetilde{J})\geqslant 2|y_2-x_J|$. Using cancellation condition of $H^p(\mathbb{R}^n\times\mathbb{R}^m)$ atom a and Hölder condition of $K_{t,s}$, then

$$\begin{aligned}
B_6 \leqslant & C|J|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p \int_{(4\sqrt{n}\widehat{I})^c}\left(\int_{2l(J)}^{\infty}\int_{\mathbb{R}^m}\left(\frac{t_2}{t_2+|x_2-y_2|}\right)^{m\lambda_2}\right. \\
& \times \int_{|x_1-x_I|}^{\infty}\int_{\mathbb{R}^n}\left(\frac{t_1}{t_1+|x_1-y_1|}\right)^{n\lambda_1}\int_{4\sqrt{m}J}\int_{2I}\int_{2J}\frac{|z_1-x_I|}{(t_1+|y_1-z_1|)^{2n+2\alpha}} \\
& \left.\times\frac{|z_2-x_J|^{2\beta}}{(t_2+|y_2-z_2|)^{2m+2\beta}}dz_1dz_2dx_2\frac{dt_1}{t_1^{n+1}}dy_1\frac{dt_2}{t_2^{m+1}}dy_2\right)^{\frac{p}{2}}dx_1 \\
\leqslant & C|J|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p \int_{(4\sqrt{n}\widehat{I})^c}\left(\int_{2l(J)}^{\infty}\int_{|x_1-x_I|}^{\infty}\int_{4\sqrt{m}J}\int_{2I}\int_{2J}\frac{|z_1-x_I|^{2\alpha}}{(t_1+|y_1-z_1|)^{2n+2\alpha}}\right. \\
& \left.\times\frac{|z_2-x_J|^{2\beta}}{(t_2+|x_2-z_2|)^{2m+2\beta}}dz_1dz_2dx_2\frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)^{\frac{p}{2}}dx_1 \\
\leqslant & C|J|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p l(I)^{\alpha p}l(J)^{\beta p}|I|^{\frac{p}{2}}|J|^p \\
& \times\int_{(4\sqrt{n}\widehat{I})^c}\left(\int_{|x_1-x_I|}^{\infty}\int_{2l(J)}^{\infty}\frac{1}{t_1^{2n+2\alpha+1}}\frac{1}{t_2^{2m+2\beta+1}}dt_1dt_2\right)^{\frac{p}{2}}dx_1 \\
\leqslant & C|J|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p l(I)^{2\alpha}|I|^{\frac{p}{2}}\int_{(4\sqrt{n}\widehat{I})^c}\frac{1}{|x_1-x_I|}dx_1 \\
\leqslant & C|J|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p|I|^{\frac{p}{2}}l(I)^{\alpha p}l(\widehat{I})^{n-(n+\alpha)p} \\
\leqslant & C\gamma_1(R)^{n-(n+\alpha')p}|R|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p.
\end{aligned}$$

Combining the estimate for $B_i (i=1,2,\cdots,6)$, we obtain

$$U(R_1)\leqslant C\gamma_1(R)^{n-(n+\alpha')p}|R|^{1-p/2}\|a_R\|_{L^2}^p. \tag{3}$$

Now we turn to estimate $U(R_2)$. We divided the domain of t and s as follows:

$$\begin{aligned}
(t,s)\in\mathbb{R}^+\times\mathbb{R}^+\subset & \big((0,2|y_1-x_I|]\times(0,2|y_2-x_J|]\big)\cup\big((0,2|y_1-x_I|]\times[2|y_2-x_J|,\infty)\big) \\
& \cup\big([2|y_1-x_I|,\infty)\times(0,2|y_2-x_J|]\big)\cup\big([2|y_1-x_I|,\infty)\times[2|y_2-x_J|,\infty)\big) \\
:= & C_1\cup C_2\cup C_3\cup C_4.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\left(\int_0^{\infty}\int_{\mathbb{R}^m}\left(\frac{t_2}{t_2+|x_2-y_2|}\right)^{\lambda_2 m}\int_0^{\infty}\int_{\mathbb{R}^n}\left(\frac{t_1}{t_1+|x_1-y_1|}\right)^{\lambda_1 n}\sum_{i=1}^{4}\mathscr{X}_{C_i}(t,s)\right. \\
&\qquad\left.\times\left|\int\int_{2I\times 2J}K_{t,s}(y_1,y_2z_1,z_2)a_R(z_1,z_2)dz_1dz_2\right|^2dy_1\frac{dt_1}{t_1^{n+1}}dy_2\frac{dt_2}{t_2^{m+1}}\right)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\sum_{i=1}^{4}\left(\int_0^\infty\int_{\mathbb{R}^m}\left(\frac{t_2}{t_2+|x_2-y_2|}\right)^{\lambda_2 m}\int_0^\infty\int_{\mathbb{R}^n}\left(\frac{t_1}{t_1+|x_1-y_1|}\right)^{\lambda_1 n}\mathscr{X}_{C_i}(t,s)\right.\\
&\qquad\left.\times\left|\int\int_{2I\times 2J}K_{t,s}(y_1,y_2z_1,z_2)a_R(z_1,z_2)dz_1dz_2\right|^2 dy_1\frac{dt_1}{t_1^{n+1}}dy_2\frac{dt_2}{t_2^{m+1}}\right)^{\frac{1}{2}}\\
&:= C\sum_{i=1}^{4}D_i.
\end{aligned}
$$

Let's begin with estimate for D_1. Using Minkowski's inequality, size condition of $K_{t,s}$, (2.1) and Hölder's inequality, then

$$
\begin{aligned}
D_1 &= \left(\int_0^{2|y_2-x_J|}\int_{\mathbb{R}^m}\left(\frac{t_2}{t_2+|x_2-y_2|}\right)^{\lambda_2 m}\int_0^{2|y_1-x_I|}\int_{\mathbb{R}^n}\left(\frac{t_1}{t_1+|x_1-y_1|}\right)^{\lambda_1 n}\right.\\
&\quad\left.\times\left|\int\int_{2I\times 2J}K_{t,s}(y_1,y_2,z_1,z_2)a_R(z_1,z_2)dz_1dz_2\right|^2 dy_1\frac{dt_1}{t_1^{n+1}}dy_2\frac{dt_2}{t_2^{m+1}}\right)^{\frac{1}{2}}\\
&\leqslant \int\int_{2I\times 2J}\left(\int_0^{2|y_2-x_J|}\int_{\mathbb{R}^m}\left(\frac{t_2}{t_2+|x_2-y_2|}\right)^{\lambda_2 m}\int_0^{2|y_1-x_I|}\int_{\mathbb{R}^n}\left(\frac{t_1}{t_1+|x_1-y_1|}\right)^{\lambda_1 n}\right.\\
&\quad\left.\times\left|K_{t,s}(y_1,y_2,z_1,z_2)\right|^2 dy_1\frac{dt_1}{t_1^{n+1}}dy_2\frac{dt_2}{t_2^{m+1}}\right)^{\frac{1}{2}}|a_R(z_1,z_2)|dz_1dz_2\\
&\leqslant \int\int_{2I\times 2J}\left(\int_0^{2|y_2-x_J|}\int_{\mathbb{R}^m}\left(\frac{t_2}{t_2+|x_2-y_2|}\right)^{\lambda_2 m}\int_0^{2|y_1-x_I|}\int_{\mathbb{R}^n}\left(\frac{t_1}{t_1+|x_1-y_1|}\right)^{\lambda_1 n}\right.\\
&\quad\left.\times\left[\frac{t_1^\alpha}{(t_1+|y_1-z_1|)^{n+\alpha}}\frac{t_2^\beta}{(t_2+|y_2-z_2|)^{m+\beta}}\right]^2 dy_1\frac{dt_1}{t_1^{n+1}}dy_2\frac{dt_2}{t_2^{m+1}}\right)^{\frac{1}{2}}|a_R(z_1,z_2)|dz_1dz_2\\
&\leqslant \int\int_{2I\times 2J}\left(\int_0^{2|y_2-x_J|}\int_0^{2|y_1-x_I|}\frac{t_1^{2\alpha}}{(t_1+|x_1-z_1|)^{2n+2\alpha}}\right.\\
&\quad\left.\times\frac{t_2^{2\beta}}{(t_2+|x_2-z_2|)^{2m+2\beta}}\frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)^{\frac{1}{2}}|a_R(z_1,z_2)|dz_1dz_2\\
&\leqslant \int\int_{2I\times 2J}\left(\int_0^{2l(J)}\int_0^{2l(I)}\frac{t_1^{2\alpha-1}}{|x_1-x_I|^{2n+2\alpha}}\frac{t_2^{2\beta-1}}{|x_2-x_J|^{2m+2\beta}}dt_1dt_2\right)^{\frac{1}{2}}|a_R(z_1,z_2)|dz_1dz_2\\
&\leqslant \left(\int\int_{2I\times 2J}|R|\left(\frac{l(I)^\alpha}{|x_1-x_I|^{n+\alpha}}\frac{l(J)^\beta}{|x_2-x_J|^{m+\beta}}|a_R(z_1,z_2)|\right)^2 dz_1dz_2\right)^{\frac{1}{2}}.
\end{aligned}
$$

By the fact that $s\geqslant 2|y_2-x_J|$, and using Minkowski's inequality, cancellation condition of $H^p(\mathbb{R}^n\times\mathbb{R}^m)$ atom a, size condition of $K_{t,s}$, (2) and Hölder's inequality, we have

$$D_2 \leqslant \int\int_{2I\times 2J}\left(\int_{2|y2-x_J|}^{\infty}\int_{\mathbb{R}^m}\left(\frac{t_2}{t_2+|x_2-y_2|}\right)^{\lambda_2 m}\int_0^{2|y_1-x_I|}\int_{\mathbb{R}^n}\left(\frac{t_1}{t_1+|x_1-y_1|}\right)^{\lambda_1 n}\right.$$
$$\left.\times\Big|K_{t,s}(y_1,y_2,z_1,z_2)-K_{t,s}(y_1,y_2,z_1,x_J)\Big|^2 dy_1\frac{dt_1}{t_1^{n+1}}dy_2\frac{dt_2}{t_2^{m+1}}\right)^{\frac{1}{2}}|a_R(z_1,z_2)|dz_1dz_2$$
$$\leqslant \int\int_{2I\times 2J}\left(\int_{2|y2-x_J|}^{\infty}\int_{\mathbb{R}^m}\left(\frac{t_2}{t_2+|x_2-y_2|}\right)^{\lambda_2 m}\int_0^{2|y_1-x_I|}\int_{\mathbb{R}^n}\left(\frac{t_1}{t_1+|x_1-y_1|}\right)^{\lambda_1 n}\right.$$
$$\left.\times\left[\frac{t_1^{\alpha}}{(t_1+|y_1-z_1|)^{n+\alpha}}\frac{|z_2-x_J|^{2\beta}}{(t_2+|y_2-z_2|)^{m+\beta}}\right]^2 dy_1\frac{dt_1}{t_1^{n+1}}dy_2\frac{dt_2}{t_2^{m+1}}\right)^{\frac{1}{2}}|a_R(z_1,z_2)|dz_1dz_2$$
$$\leqslant \int\int_{2I\times 2J}\left(\int_{2|y2-x_J|}^{\infty}\int_0^{2|y_1-x_I|}\frac{t_1^{2\alpha}}{(t_1+|x_1-z_1|)^{2n+2\alpha}}\right.$$
$$\left.\times\frac{|z_2-x_J|^{2\beta}}{(t_2+|x_2-z_2|)^{2m+2\beta}}\frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)^{\frac{1}{2}}|a_R(z_1,z_2)|dz_1dz_2$$
$$\leqslant C\int\int_{2I\times 2J}\frac{l(I)^{\alpha}}{|x_1-x_I|^{n+\alpha}}\frac{l(J)^{\beta'}}{|x_2-x_J|^{m+\beta'}}|a_R(z_1,z_2)|dz_1dz_2$$
$$\leqslant C\int\int_{2I\times 2J}\frac{l(I)^{\alpha}}{|x_1-x_I|^{n+\alpha}}\frac{l(J)^{\beta}}{|x_2-x_J|^{m+\beta'}}|a_R(z_1,z_2)|dz_1dz_2$$
$$\leqslant C\left(\int\int_{2I\times 2J}|R|\big(\frac{l(I)^{\alpha}}{|x_1-x_I|^{n+\alpha}}\frac{l(J)^{\beta'}}{|x_2-x_J|^{m+\beta'}}|a_R(z_1,z_2)|\big)^2dz_1dz_2\right)^{\frac{1}{2}}.$$

where $0<\beta'<\beta$.

By an analogues argument to D_2, we get

$$D_3 \leqslant C\left(\int\int_{2I\times 2J}|R|\big(\frac{l(I)^{\alpha'}}{|x_1-x_I|^{n+\alpha'}}\frac{l(J)^{\beta}}{|x_2-x_J|^{m+\beta}}|a_R(z_1,z_2)|\big)^2dz_1dz_2\right)^{\frac{1}{2}}$$

with $0<\beta'<\beta$.

Note that $t\geqslant 2|y_1-x_I|$ and $s\geqslant 2|y_2-x_J|$. Thus, Minkowski's inequality, cancellation condition of $H^p(\mathbb{R}^n\times\mathbb{R}^m)$ atom a, Hölder condition of $K_{t,s}$, (2) and Hölder's inequality imply that

$$D_4 \leqslant \int\int_{2I\times 2J}\left(\int_{2|y_2-x_J|}^{\infty}\int_{\mathbb{R}^m}\left(\frac{t_2}{t_2+|x_2-y_2|}\right)^{\lambda_2 m}\int_{2|y_1-x_I|}^{\infty}\int_{\mathbb{R}^n}\left(\frac{t_1}{t_1+|x_1-y_1|}\right)^{\lambda_1 n}\right.$$
$$\left.\times\frac{|z_1-x_I|^{\alpha}}{(t_1+|y_1-z_1|)^{n+\alpha}}\frac{|z_2-x_J|^{\beta}}{(t_2+|y_2-z_2|)^{m+\beta}}]dy_1\frac{dt_1}{t_1^{n+1}}dy_2\frac{dt_2}{t_2^{m+1}}\right)^{\frac{1}{2}}|a_R(z_1,z_2)|dz_1dz_2$$

$$
\begin{aligned}
&\leqslant \int\int_{2I\times 2J}\Bigg(\int_{2|y_2-x_J|}^{\infty}\int_{2|y_1-x_I|}^{\infty}\frac{|z_1-x_I|^{2\alpha}}{(t_1+|y_1-z_1|)^{2n+2\alpha}}\\
&\quad\times\frac{|z_2-x_J|^{2\beta}}{(t_2+|y_2-z_2|)^{2m+2\beta}}dy_1\frac{dt_1}{t_1^{n+1}}dy_2\frac{dt_2}{t_2^{m+1}}\Bigg)^{\frac{1}{2}}|a_R(z_1,z_2)|dz_1dz_2\\
&\leqslant C\int\int_{2I\times 2J}\frac{|z_1-x_I|^{\alpha'}}{(t_1+|y_1-z_1|)^{n+\alpha'}}\frac{|z_2-x_J|^{\beta'}}{(t_2+|y_2-z_2|)^{m+\beta'}}|a_R(z_1,z_2)|dz_1dz_2\\
&\leqslant C\int\int_{2I\times 2J}\frac{l(I)^{\alpha'}}{|x_1-x_I|^{n+\alpha'}}\frac{l(J)^{\beta'}}{|x_2-x_J|^{m+\beta'}}|a_R(z_1,z_2)|dz_1dz_2\\
&\leqslant C\left(\int\int_{2I\times 2J}|R|\left(\frac{l(I)^{\alpha'}}{|x_1-x_I|^{n+\alpha'}}\frac{l(J)^{\beta'}}{|x_2-x_J|^{m+\beta'}}|a_R(z_1,z_2)|\right)^2dz_1dz_2\right)^{\frac{1}{2}}.
\end{aligned}
$$

where $0<\alpha'<\alpha, 0<\beta'<\beta$.

Hence, by the Hölder's inequality, we obtain

$$
\begin{aligned}
&\int\int_{(4\sqrt{n}\widehat{I})^c\times(4\sqrt{m}J)^c}\sum_{i=1}^{4}(D_i)^p dx_1dx_2\\
&\leqslant C\int\int_{(4\sqrt{n}\widehat{I})^c\times(4\sqrt{m}J)^c}|R|^{p/2}\frac{l(I)^{\alpha' p}}{|x_1-x_I|^{(n+\alpha')p}}\frac{l(J)^{\beta' p}}{|x_2-x_J|)^{(m+\beta')p}}\|a_R\|_{L^2}^p dx_1dx_2 \qquad (4)\\
&\leqslant C|R|^{p/2}l(I)^{\alpha' p}l(J)^{\beta' p}l(I)^{n-(n+\alpha')p}l(J)^{m-(m+\beta')p}\|a_R\|_{L^2}^p\\
&\leqslant C\gamma_1(R)^{n-(n+\alpha')p}|R|^{1-p/2}\|a_R\|_{L^2}^p.
\end{aligned}
$$

From estimates (3) and (4), then we have

$$U(R)\leqslant C\gamma_1(R)^{n-(n+\alpha')p}|R|^{1-p/2}\|a_R\|_{L^2}^p.$$

By an similarly argument to $U(R)$, we obtain

$$V(R)\leqslant C\gamma_2(\widehat{R})^{m-(m+\beta')p}|R|^{1-\frac{p}{2}}\|a_R\|_{L^2}^p.$$

Therefore, summing over R gives

$$
\begin{aligned}
&\sum_{R\in\mathscr{M}(\Omega)}\int_{(\Omega_3)^c}\|T(a_R)(x_1,x_2)\|_{L^p_{\mathscr{H}}}^p dx_1dx_2\\
&\leqslant C\sum_{R\in\mathscr{M}(\Omega)}\gamma_1(R)^{n-(n+\alpha')p}|R|^{1-p/2}\|a_R\|_{L^2}^p+C\sum_{R\in\mathscr{M}(\Omega)}\gamma_2(\widehat{R})^{m-(m+\beta')p}|R|^{1-p/2}\|a_R\|_{L^2}^p\\
&\leqslant C\left\{\left(\sum_{R\in\mathscr{M}_2(\Omega)}\gamma_1(R)^{-\delta_1}|R|\right)^{1-p/2}+\left(\sum_{\widehat{R}\in\mathscr{M}_1(\Omega_1)}\gamma_1(\widehat{R})^{-\delta_2}|R|\right)^{1-p/2}\right\}\left(\sum_{R\in\mathscr{M}(\Omega)}\|a_R\|_{L^2}^2\right)^{p/2},
\end{aligned}
$$

where $\delta_1=\frac{2[n-(n+\alpha')p]}{p-2}>0$ and $\delta_1=\frac{2[m-(m+\beta')p]}{p-2}>0$. In order to estimate the last part above, we need the following Journé's covering lemma.

LEMMA 4. [8] $\sum_{R\in\mathscr{M}_2(\Omega)} |R|\gamma_1(R)^{-\delta} \leqslant C_\delta|\Omega|$ *and* $\sum_{R\in\mathscr{M}_1(\Omega)} |R|\gamma_2(R)^{-\delta} \leqslant C_\delta|\Omega|$ *for any* $\delta > 0$, *where the constant* C_δ *only depend on* δ.

Journé's covering lemma and size condition of a_R yield that

$$\sum_{R\in\mathscr{M}(\Omega)} \int_{(\Omega_3)^c} \|T(a_R)(x_1,x_2)\|^p_{\mathscr{H}} dx_1dx_2 \leqslant C|\Omega|^{1-p/2}|\Omega|^{p/2-1}.$$

This is a desired result, hence we complete the proof of Theorem 1.

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