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TRIGONOMETRIC APPROXIMATION OF FUNCTIONS IN SEMINORMED SPACES

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Abstract. In this paper, we study the approximation properties of 2π-periodic functions in a seminormed space. We use a general matrix method of summability, and the moduli of continuity in the seminormed space as a measure of approximation. Our results generalize and improve some of the previous results available in the literature.

1. Introduction

Let C and L^p $(1 \leqslant p < \infty)$ denote the spaces of 2π-periodic real valued continuous functions and $p-$th power Lebesgue integrable functions, respectively, equipped with the following norms

$$\|f\|_C := \|f(\cdot)\|_C = \max_{-\pi \leqslant t \leqslant \pi} |f(t)|$$

and

$$\|f\|_{L^p} := \|f(\cdot)\|_{L^p} = \left\{ \int_{-\pi}^{\pi} |f(t)|^p \, dt \right\}^{\frac{1}{p}}.$$

For a seminorm P, we define the following seminormed spaces:

$$(L^p, P) = \{f \in L^p : P(f) < \infty\},$$

$$(C, P) = \{f \in C : P(f) < \infty\}$$

with the property that $f(\cdot + h) \in (L^p, P)$ or $f(\cdot + h) \in (C, P)$ for any $h \in \mathbb{R}$, respectively.

In this paper we will consider the seminorm P satisfying, for all $f, g \in (L^p, P)$ (or $f, g \in (C, P)$), the following conditions:

1. for any $h \in \mathbb{R}$

$$P(f(\cdot + h)) = P(f(\cdot)), \tag{1}$$

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2. if $|f(x)| \leqslant |g(x)|$ for every $x \in [-\pi, \pi]$, then

$$P(f) \leqslant P(g). \tag{2}$$

For $f \in L^1$ we will consider the trigonometric Fourier series

$$S[f](x) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt + \sum_{\nu=1}^{\infty}\left[\frac{\cos \nu x}{\pi}\int_{-\pi}^{\pi} f(t)\cos \nu t\,dt + \frac{\sin \nu x}{\pi}\int_{-\pi}^{\pi} f(t)\sin \nu t\,dt\right]$$

with the partial sums $S_k[f]$.

We will also need some notations on methods of summability of the series $S[f]$.

If $A := (a_{n,k})_{0\leqslant n,k<\infty}$ is an infinite matrix of real numbers such that $a_{n,k} \geqslant 0$ for all $k,n = 0,1,2,\ldots,$ or $A_0 := (a_{n,k})_{0\leqslant n,k<\infty}$ is an infinite matrix of real numbers such that $a_{n,k} \geqslant 0$ for $k = 0,1,2,\ldots,n$ and $a_{n,k} = 0$ when $k = n+1, n+2, \ldots,$ with $\lim_{n\to\infty} a_{n,k} = 0$ for every k and $\sum_{k=0}^{\infty} a_{n,k} = 1$ for every n, then

$$T_{n,A}[f](x) := \sum_{k=0}^{\infty} a_{n,k} S_k[f](x) \quad (n = 0,1,2,\ldots)$$

or

$$T_{n,A_0}[f](x) := \sum_{k=0}^{n} a_{n,k} S_k[f](x) \quad (n = 0,1,2,\ldots),$$

define the respective matrix means of $S[f]$.

It is well-known that in Fourier analysis, various interesting results are established by assuming monotonicity of the entries of a matrix A. One of the methods of obtaining generalizations of these results was the use of the so-called bounded variation concept. There have been defined many classes of bounded variation sequences (see for example [5], [12]–[16]). We will also use one of them in this paper. Namely, following Leindler [6] we define the class of Head Bounded Variations Sequences

$$HBVS := \left\{ (c_k) \subset [0,\infty) : \sum_{k=0}^{m-1} |c_k - c_{k+1}| = O(c_m) \text{ for all } m \in \mathbb{N} \right\}.$$

For $f \in (L^p, P)$ (or $f \in (C, P)$), we denote the best approximation by

$$E_n(f)_P = \inf_{T_n}\{P(f - T_n)\},\ n = 0,1,2,\ldots,$$

where the infimum is taken over all trigonometric polynomials

$$T_n(x) = \sum_{k=-n}^{n} c_n e^{inx}$$

of the degree at most n.

The modulus of continuity of $f \in (L^p, P)$ (or $f \in (C, P)$) with respect to P is defined by the following formula:

$$\Omega f(\delta)_P = \sup_{0 \leqslant h \leqslant \delta} P(f(\cdot + h) - f(\cdot)).$$

With such modulus of continuity, we can define the following class of functions:

$$Lip_P(\omega) = \{f \in (L^p, P) \, (\text{or } f \in (C, P)) : \Omega f(\delta)_P = O(\omega(\delta)), \; \delta \geqslant 0\},$$

where ω is a function of modulus of continuity type on the interval $[0, 2\pi]$, i.e., a non-decreasing continuous function having the following properties: $\omega(0) = 0$, $\omega(\delta_1 + \delta_2) \leqslant \omega(\delta_1) + \omega(\delta_2)$ for any $0 \leqslant \delta_1 \leqslant \delta_2 \leqslant \delta_1 + \delta_2 \leqslant 2\pi$.

We know that the following Jackson-type inequality (see [17, Theorems 1, 2, p. 159]) holds:

$$E_n(f)_P = O\left(\Omega f\left(\frac{1}{n+1}\right)_P\right); \; n = 0, 1, 2, \ldots,$$

for $f \in (L^p, P)$ (or $f \in (C, P)$).

With these definitions, it is pertinent to study the problem of approximation in the seminormed spaces defined above. Our results presented in this paper generalize and improve the results of A. Guven [2], B. Szal [11, Theorem 5 (v)] and R. N, Mohapatra, B. Szal [10, Theorem 5 (vi)] using seminorms instead of special kind of the Orlicz norms and considering more general matrix means.

2. Main results

In the beginning we present a general result dealing with the degree of approximation in terms of the best approximation and the modulus of continuity.

THEOREM 1. *Let $f \in (L^p, P)$ with $1 \leqslant p < \infty$, where seminorm P satisfies (1) and (2) or the following condition*

$$P(f) \leqslant \|f(\cdot)\|_{L^p}. \tag{3}$$

If entries of the matrix A satisfy the conditions

$$\sum_{k=0}^{\infty} (k+1)^\beta \left| \frac{a_{n,k}}{(k+1)^\beta} - \frac{a_{n,k+1}}{(k+2)^\beta} \right| = O\left(\frac{1}{n+1}\right), \tag{4}$$

for some $\beta \geqslant 0$ and

$$\sum_{k=0}^{\infty} (k+1) a_{n,k} = O(n+1), \tag{5}$$

then

$$P(T_{n,A}[f] - f) = O\left(E_n(f)_P + \sum_{k=0}^{n} a_{n,k} E_k(f)_P\right), \tag{6}$$

for $n = 0, 1, 2, \ldots$

Let GM be a class of all sequences (c_k) of real numbers such that there exist constants $M > 0$ and $\lambda > 1$ for which

$$\sum_{k=n}^{2n-1} |c_k - c_{k+1}| \leqslant M \sum_{k=[n/\lambda]}^{[\lambda n]} \frac{|c_k|}{k}$$

for all $n \in \mathbb{N}$ (see [8]).

If $((k+1)^{-\beta} a_{n,k}) \in GM$ with respect to k for some $\beta \geqslant 0$ and

$$\sum_{k=0}^{\infty} \frac{a_{n,k}}{k+1} = O\left(\frac{1}{n+1}\right) \tag{7}$$

for all $n = 0, 1, 2, \ldots$, then for some $M_0 > 0$

$$\begin{aligned}
\sum_{k=1}^{\infty} (k+1)^{\beta} \left| \frac{a_{n,k}}{(k+1)^{\beta}} - \frac{a_{n,k+1}}{(k+2)^{\beta}} \right| &= \sum_{s=0}^{\infty} \sum_{k=2^s}^{2^{s+1}-1} (k+1)^{\beta} \left| \frac{a_{n,k}}{(k+1)^{\beta}} - \frac{a_{n,k+1}}{(k+2)^{\beta}} \right| \\
&\leqslant M \sum_{s=0}^{\infty} \left(2^{s+1}\right)^{\beta} \sum_{k=[2^s/\lambda]}^{[\lambda 2^s]} \frac{a_{n,k}}{k(k+1)^{\beta}} \\
&\leqslant MM_0 \sum_{k=0}^{\infty} \frac{a_{n,k}}{k+1} = O\left(\frac{1}{n+1}\right).
\end{aligned}$$

Thus (4) is satisfied. Using this and Theorem 1 we can formulate the following corollary:

COROLLARY 1. *Let $f \in (L^p, P)$ with $1 \leqslant p < \infty$, where the seminorm P satisfy (1) and (2) or (3). If entries of the matrix A satisfy conditions $((k+1)^{-\beta} a_{n,k}) \in GM$ with respect to k for some $\beta \geqslant 0$, (5) and (7), then (6) holds.*

In case of the triangular means defined by matrices A_0, we have the following remark:

REMARK 1. We can observe that all lower triangular matrices satisfy (5). Indeed

$$\sum_{k=0}^{\infty} (k+1) a_{n,k} = \sum_{k=0}^{n} (k+1) a_{n,k} \leqslant (n+1) \sum_{k=0}^{n} a_{n,k} = n+1.$$

Moreover, using the monotonicity of $E_n(f)_P$ with respect to n, we get

$$\sum_{k=0}^{n} a_{n,k} E_k(f)_P \geqslant E_n(f)_P \sum_{k=0}^{n} a_{n,k} = E_n(f)_P.$$

Using Theorem 1 and Remark 1 we get the following corollary:

COROLLARY 2. *Let $f \in (L^p, P)$ with $1 \leqslant p < \infty$, where the seminorm P satisfy (1) and (2) or (3). If entries of the matrix A_0 satisfy condition (4) for some $\beta \geqslant 0$, then*

$$P\left(T_{n,A_0}[f]-f\right) = O\left(\sum_{k=0}^{n} a_{n,k} E_k(f)_P\right), \tag{8}$$

for $n = 0, 1, 2, \ldots$

Suppose that A_0 and $((k+1)^{-\beta} a_{n,k}) \in HBVS$ with respect to k for some $\beta \geqslant 0$ and $(n+1)a_{n,n} = O(1)$. Then

$$\begin{aligned}
&\sum_{k=0}^{\infty} (k+1)^{\beta} \left| \frac{a_{n,k}}{(k+1)^{\beta}} - \frac{a_{n,k+1}}{(k+2)^{\beta}} \right| \\
&= \sum_{k=0}^{n-1} (k+1)^{\beta} \left| \frac{a_{n,k}}{(k+1)^{\beta}} - \frac{a_{n,k+1}}{(k+2)^{\beta}} \right| + a_{n,n} \\
&\leqslant (n+1)^{\beta} \sum_{k=0}^{n-1} \left| \frac{a_{n,k}}{(k+1)^{\beta}} - \frac{a_{n,k+1}}{(k+2)^{\beta}} \right| + a_{n,n} \\
&\leqslant (n+1)^{\beta} O\left(\frac{a_{n,n}}{(n+1)^{\beta}}\right) + a_{n,n} = O\left(a_{n,n}\right) = O\left(\frac{1}{n+1}\right).
\end{aligned}$$

Thus (4) is satisfied. Using this and Theorem 1 we can formulate the following corollary:

COROLLARY 3. *Let $f \in (L^p, P)$ with $1 \leqslant p < \infty$, where the seminorm P satisfy (1) and (2) or (3). If entries of the matrix A_0 satisfy conditions $((k+1)^{-\beta} a_{n,k}) \in HBVS$ with respect to k for some $\beta \geqslant 0$ and $(n+1)a_{n,n} = O(1)$, then (8) holds.*

Next we consider some special cases and approximate $f \in Lip_P(\omega)$.

THEOREM 2. *Let $f \in Lip_P(\omega)$, where the seminorm P satisfy (1) and (2) or (3). If entries of the matrix A satisfy conditions (4) for some $\beta \geqslant 0$ and (5), then*

$$P\left(T_{n,A}[f]-f\right) = \begin{cases} O\left(\omega\left(\frac{1}{n+1}\right)\right), \text{ when } \beta > 0, \\ O\left(\frac{1}{n+1} \sum_{k=0}^{n} \omega\left(\frac{1}{k+1}\right)\right), \text{ when } \beta = 0, \end{cases} \tag{9}$$

for $n = 0, 1, 2, \ldots$

In the similar way as in the proof of Theorem 2 using Corollary 1, 2 and 3 we obtain the following corollaries:

COROLLARY 4. *Let $f \in Lip_P(\omega)$ with $1 \leqslant p < \infty$, where the seminorm P satisfy (1) and (2) or (3). If entries of the matrix A satisfy conditions $((k+1)^{-\beta} a_{n,k}) \in GM$ with respect to k for some $\beta \geqslant 0$, (5) and (7), then*

$$P\left(T_{n,A}[f]-f\right) = O\left(\omega\left(\frac{1}{n+1}\right)\right) \tag{10}$$

for $n = 0, 1, 2, \ldots$

COROLLARY 5. *Let* $f \in Lip_P(\omega)$ *with* $1 \leqslant p < \infty$*, where the seminorm* P *satisfy* (1) *and* (2) *or* (3)*. If entries of the matrix* A_0 *satisfy condition* (4) *for some* $\beta \geqslant 0$*, then*

$$P\left(T_{n,A}[f] - f\right) = \begin{cases} O\left(\omega\left(\frac{1}{n+1}\right)\right), & \text{when } \beta > 0, \\ O\left(a_{n,n} \sum_{k=0}^{n} \omega\left(\frac{1}{k+1}\right)\right), & \text{when } \beta = 0, \end{cases}$$

for $n = 0, 1, 2, \ldots$

COROLLARY 6. *Let* $f \in Lip_P(\omega)$*, where the seminorm* P *satisfy* (1) *and* (2) *or* (3)*. If entries of the matrix* A_0 *satisfy conditions* $((k+1)^{-\beta} a_{n,k}) \in HBVS$ *for some* $\beta \geqslant 0$ *with respect to* k *and* $(n+1)a_{n,n} = O(1)$*, then* (9) *holds.*

Finally, we have the following examples and remarks.

EXAMPLE 1. One can easily verify that $a_{n,k} = e^{-n} \sum_{j=k}^{\infty} \frac{n^j}{(j+1)!}$, where $n, k = 0, 1, 2, \ldots$, satisfies the conditions (4) for any $\beta \geqslant 0$ with respect to k and (5).

EXAMPLE 2. Let $a_{n,k} = \frac{(k+1)^{\beta} - k^{\beta}}{(n+1)^{\beta}}$ for $k \leqslant n$ and $a_{n,k} = 0$ for $k > n$, where $n, k = 0, 1, 2, \ldots$. We can easily show that this sequence satisfies the conditions (4) for any $\beta > 1$ and $(n+1)a_{n,n} = O(1)$.

REMARK 2. Theorem 1 and Corollaries 1-3 will also be true for $f \in (C, P)$ because $(C, P) \subset (L^p, P)$ for $1 \leqslant p < \infty$.

REMARK 3. Let the seminorm P satisfy (2) or (3). If $f \in Lip_P(\omega)$ with $\omega(\delta) = \delta^{\alpha}$, where $\alpha \in (0, 1]$, then from Corollary 6 the results of [1], [2], [7], [10, Theorem 5 (vi)] and [11, Theorem 5 (v)] follow at once in the more general and improved forms. The similar results can also be seen in papers [3] and [4].

3. Lemmas

We need the following lemmas for the proofs of our theorems.

LEMMA 1. (see [17, Theorem 1, p. 58]) *Let* g *be a continuous function of two variables,* 2π *- periodic on* $[-\pi, \pi] \times [-\pi, \pi]$*. If* $g(\cdot, t) \in (C, P)$ *for every* $t \in [-\pi, \pi]$ *and* (1) *holds, then*

$$\int_{-\pi}^{\pi} g(\cdot, t)\, dt \in (C, P)$$

and

$$P\left(\int_{-\pi}^{\pi} g(\cdot, t)\, dt\right) \leqslant \int_{-\pi}^{\pi} P(g(\cdot, t))\, dt.$$

LEMMA 2. (cf. [17, Corollary 1, p. 59]) *Let* $f \in (L^p, P)$ *with* $1 \leqslant p < \infty$ *and* $K \in (C, P)$. *If the seminorm* P *satisfies the condition (1) and (3), then*

$$\int_{-\pi}^{\pi} f(\cdot + t) K(t)\, dt \in (L^p, P)$$

and

$$P\left(\int_{-\pi}^{\pi} f(\cdot + t) K(t)\, dt\right) \leqslant P(f) \int_{-\pi}^{\pi} |K(t)|\, dt.$$

Proof. If $f, K \in (C, P)$, then using Lemma 1 we have

$$\begin{aligned} P\left(\int_{-\pi}^{\pi} f(\cdot + t) K(t)\, dt\right) &\leqslant \int_{-\pi}^{\pi} P(f(\cdot + t) K(t))\, dt \\ &= \int_{-\pi}^{\pi} P(f(\cdot + t)) |K(t)|\, dt \\ &= \int_{-\pi}^{\pi} P(f(\cdot)) |K(t)|\, dt = P(f) \int_{-\pi}^{\pi} |K(t)|\, dt. \end{aligned} \tag{11}$$

We know that the set of continuous functions is dense in the space L^p for $1 \leqslant p < \infty$ (see [9]). Therefore, for every $f \in (L^p, P)$ with $1 \leqslant p < \infty$, there exists a sequence (f_n) of continuous functions such that $\|f_n - f\|_{L^p} \to 0$ as $n \to \infty$. Hence for every $f \in (L^p, P)$ with $1 \leqslant p < \infty$

$$g_n(x) = \int_{-\pi}^{\pi} f_n(x + t) K(t)\, dt = \int_{-\pi}^{\pi} f_n(t) K(t - x)\, dt \to \int_{-\pi}^{\pi} f(t) K(t - x)\, dt = g(x)$$

uniformly with respect to x as $n \to \infty$, since by the Hölder inequality

$$|g_n(x) - g(x)| \leqslant \max_{|s| \leqslant \pi} |K(s)| \int_{-\pi}^{\pi} |f_n(t) - f(t)|\, dt \leqslant (2\pi)^{1-1/p} \max_{|s| \leqslant \pi} |K(s)|\, \|f_n - f\|_{L^p}.$$

Next, applying (3) we get for $1 \leqslant p < \infty$

$$\lim_{n \to \infty} |P(f_n) - P(f)| \leqslant \lim_{n \to \infty} P(f_n - f) \leqslant \lim_{n \to \infty} \|f_n - f\|_{L^p} = 0.$$

Therefore, P is continuous and thus using (11), we obtain

$$\begin{aligned} P\left(\int_{-\pi}^{\pi} f(\cdot + t) K(t)\, dt\right) &= P(g) = \lim_{n \to \infty} P(g_n) \\ &= \lim_{n \to \infty} P\left(\int_{-\pi}^{\pi} f_n(\cdot + t) K(t)\, dt\right) \\ &\leqslant \lim_{n \to \infty} P(f_n) \int_{-\pi}^{\pi} |K(t)|\, dt = P(f) \int_{-\pi}^{\pi} |K(t)|\, dt. \end{aligned}$$

Using (3) and the Minkowski inequality

$$\begin{aligned}
P\left(\int_{-\pi}^{\pi} f(\cdot+t)K(t)\,dt\right) &\leqslant \left\|\int_{-\pi}^{\pi} f(\cdot+t)K(t)\,dt\right\|_{L^p} \\
&\leqslant \int_{-\pi}^{\pi} \|f(\cdot+t)\|_{L^p}\,|K(t)|\,dt \\
&= \|f(\cdot)\|_{L^p}\int_{-\pi}^{\pi} |K(t)|\,dt \leqslant 2\pi\,\|f(\cdot)\|_{L^p}\max_{|s|\leqslant\pi}|K(s)| < \infty.
\end{aligned}$$

Thus

$$\int_{-\pi}^{\pi} f(\cdot+t)K(t)\,dt \in (L^p,P)$$

and this ends our proof.

LEMMA 3. *Let $f\in(L^p,P)$ with $1\leqslant p<\infty$ and $K\in L^1$. If a seminorm P satisfies (1) and (2), then*

$$\int_{-\pi}^{\pi} f(\cdot+t)K(t)\,dt \in (L^p,P)$$

and

$$P\left(\int_{-\pi}^{\pi} f(\cdot+t)K(t)\,dt\right) \leqslant P(f)\int_{-\pi}^{\pi}|K(t)|\,dt.$$

Proof. Using (2) we can easily show that $P(f)=P(|f|)$. Therefore, applying again (2) and absolute homogeneity of the seminorm P, we obtain

$$\begin{aligned}
P\left(\int_{-\pi}^{\pi} f(\cdot+t)K(t)\,dt\right) &\leqslant P\left(\int_{-\pi}^{\pi} |f(\cdot+t)K(t)|\,dt\right) \\
&= P\left(\int_{-\pi}^{\pi}|K(t)|\,dt\int_{-\pi}^{\pi}\frac{|f(\cdot+t)K(t)|}{\int_{-\pi}^{\pi}|K(t)|\,dt}dt\right) \\
&= \int_{-\pi}^{\pi}|K(t)|\,dt P\left(\frac{1}{\int_{-\pi}^{\pi}|K(t)|\,dt}\int_{-\pi}^{\pi}|f(\cdot+t)K(t)|\,dt\right).
\end{aligned}$$

Using the Jensen inequality, we obtain

$$\begin{aligned}
P\left(\frac{1}{\int_{-\pi}^{\pi}|K(t)|\,dt}\int_{-\pi}^{\pi}|f(\cdot+t)K(t)|\,dt\right) &\leqslant \frac{1}{\int_{-\pi}^{\pi}|K(t)|\,dt}\int_{-\pi}^{\pi}P(|f(\cdot+t)|)\,|K(t)|\,dt \\
&= \frac{1}{\int_{-\pi}^{\pi}|K(t)|\,dt}\int_{-\pi}^{\pi}P(|f(\cdot)|)\,|K(t)|\,dt \\
&= P(|f(\cdot)|) = P(f)
\end{aligned}$$

and our result follows.

Next we present some estimate of the kernel.

LEMMA 4. *If $\beta \geqslant 0$ and $0 < t \leqslant \pi$, then*

$$\left|\sum_{k=0}^{n}(k+1)^{\beta}\frac{\sin\frac{(2k+1)t}{2}}{2\sin\frac{t}{2}}\right| \leqslant \pi^2\frac{(n+1)^{\beta}}{t^2}.$$

Proof. Using the Abel transformation and $\sum_{l=0}^{n}\frac{\sin\frac{(2l+1)t}{2}}{2\sin\frac{t}{2}} = \frac{\sin^2\frac{(n+1)t}{2}}{2\sin^2\frac{t}{2}}$ (see [18]), we have

$$\begin{aligned}
&\left|\sum_{k=0}^{n}(k+1)^{\beta}\frac{\sin\frac{(2k+1)t}{2}}{2\sin\frac{t}{2}}\right| \\
&\leqslant \left|\sum_{k=0}^{n-1}\left[(k+1)^{\beta}-(k+2)^{\beta}\right]\sum_{l=0}^{k}\frac{\sin\frac{(2l+1)t}{2}}{2\sin\frac{t}{2}} + (n+1)^{\beta}\sum_{l=0}^{n}\frac{\sin\frac{(2l+1)t}{2}}{2\sin\frac{t}{2}}\right| \\
&\leqslant \sum_{k=0}^{n-1}\left|(k+1)^{\beta}-(k+2)^{\beta}\right|\frac{\sin^2\frac{(k+1)t}{2}}{2\sin^2\frac{t}{2}} + (n+1)^{\beta}\frac{\sin^2\frac{(n+1)t}{2}}{2\sin^2\frac{t}{2}} \\
&\leqslant \frac{\pi^2}{2t^2}\left[\sum_{k=0}^{n-1}\left|(k+1)^{\beta}-(k+2)^{\beta}\right| + (n+1)^{\beta}\right] \\
&= \frac{\pi^2}{2t^2}\left[2(n+1)^{\beta}-1\right] \leqslant \frac{\pi^2}{t^2}(n+1)^{\beta}
\end{aligned}$$

and this ends our proof.

Application of Lemma 4 gives next more general estimate.

LEMMA 5. *If (4) for some $\beta \geqslant 0$ and (5) hold, then*

$$\int_{-\pi}^{\pi}\left|\sum_{k=0}^{\infty}a_{n,k}\left(\frac{1}{2}+\sum_{\nu=1}^{k}\cos\nu t\right)\right|dt = O(1).$$

Proof. Since $\frac{1}{2}+\sum_{\nu=1}^{k}\cos\nu t = \frac{\sin\frac{(2k+1)t}{2}}{2\sin\frac{t}{2}}$ (see [18]),

$$\int_{-\pi}^{\pi}\left|\sum_{k=0}^{\infty}a_{n,k}\left(\frac{1}{2}+\sum_{\nu=1}^{k}\cos\nu t\right)\right|dt = \int_{-\pi}^{\pi}\left|\sum_{k=0}^{\infty}a_{n,k}\frac{\sin\frac{(2k+1)t}{2}}{2\sin\frac{t}{2}}\right|dt.$$

Using the Abel transformation, Lemma 4 and (4), we have

$$\begin{aligned}
&\left|\sum_{k=0}^{\infty}\frac{a_{n,k}}{(k+1)^{\beta}}(k+1)^{\beta}\frac{\sin\frac{(2k+1)t}{2}}{2\sin\frac{t}{2}}\right| \\
&= \left|\sum_{k=0}^{\infty}\left[\frac{a_{n,k}}{(k+1)^{\beta}}-\frac{a_{n,k+1}}{(k+2)^{\beta}}\right]\sum_{l=0}^{k}(l+1)^{\beta}\frac{\sin\frac{(2l+1)t}{2}}{2\sin\frac{t}{2}}\right| \\
&\leqslant \pi^2\sum_{k=0}^{\infty}\left|\frac{a_{n,k}}{(k+1)^{\beta}}-\frac{a_{n,k+1}}{(k+2)^{\beta}}\right|\frac{(k+1)^{\beta}}{t^2} = O\left(\frac{1}{t^2(n+1)}\right).
\end{aligned}$$

Thus, by the assumption (5), we have

$$\begin{aligned}\int_0^\pi \left|\sum_{k=0}^\infty a_{n,k}\frac{\sin\frac{(2k+1)t}{2}}{2\sin\frac{t}{2}}\right| dt &= \left(\int_0^{\frac{\pi}{n+1}} + \int_{\frac{\pi}{n+1}}^\pi\right)\left|\sum_{k=0}^\infty a_{n,k}\frac{\sin\frac{(2k+1)t}{2}}{2\sin\frac{t}{2}}\right| dt \\ &\leqslant \int_0^{\frac{\pi}{n+1}} \sum_{k=0}^\infty a_{n,k}\frac{2k+1}{2} dt + \int_{\frac{\pi}{n+1}}^\pi O\left(\frac{1}{t^2(n+1)}\right) dt \\ &\leqslant \frac{\pi}{n+1}(n+1) + \frac{O(1)}{n+1}\left(\frac{\pi}{n+1}\right)^{-1} = O(1)\end{aligned}$$

and our estimate follows.

We yet need an approximation result.

LEMMA 6. *Suppose $f \in (L^p, P)$ with $1 \leqslant p < \infty$, where seminorm P satisfies (1) and (2) or (3), and let T_n be a trigonometric polynomial of the degree at most n, such that $P(f - T_n) = O(E_n(f)_P)$. If (4) for some $\beta \geqslant 0$ and (5) hold, then*

$$P\left(\sum_{k=0}^\infty a_{n,k} S_k[f - T_n]\right) = O(E_n(f)_P).$$

Proof. It is clear that

$$S_k[f - T_n](x) = \frac{1}{\pi}\int_{-\pi}^\pi [f - T_n](x+t)\left(\frac{1}{2} + \sum_{\nu=1}^k \cos\nu t\right) dt$$

and therefore,

$$\sum_{k=0}^\infty a_{n,k} S_k[f - T_n](x) = \frac{1}{\pi}\int_{-\pi}^\pi [f - T_n](x+t)\sum_{k=0}^\infty a_{n,k}\left(\frac{1}{2} + \sum_{\nu=1}^k \cos\nu t\right) dt.$$

Hence, by Lemma 2 or Lemma 3, we have

$$P\left(\sum_{k=0}^\infty a_{n,k} S_k[f - T_n]\right) \leqslant \frac{1}{\pi}\int_{-\pi}^\pi P(f - T_n)\left|\sum_{k=0}^\infty a_{n,k}\left(\frac{1}{2} + \sum_{\nu=1}^k \cos\nu t\right)\right| dt$$

and further,

$$P\left(\sum_{k=0}^\infty a_{n,k} S_k[f - T_n]\right) = O(E_n(f)_P)\frac{1}{\pi}\int_{-\pi}^\pi \left|\sum_{k=0}^\infty a_{n,k}\left(\frac{1}{2} + \sum_{\nu=1}^k \cos\nu t\right)\right| dt.$$

Since by Lemma 5

$$\frac{1}{\pi}\int_{-\pi}^\pi \left|\sum_{k=0}^\infty a_{n,k}\left(\frac{1}{2} + \sum_{\nu=1}^k \cos\nu t\right)\right| dt = O(1),$$

whence

$$P\left(\sum_{k=0}^\infty a_{n,k} S_k[f - T_n]\right) = O(E_n(f)_P)$$

and our proof is ended.

4. Proofs of main results

4.1. Proof of Theorem 1

For each $n=0,1,2,\ldots$, let T_n be a trigonometric polynomial such that $P(f-T_n)=O(E_n(f)_P)$ (which exists by [17, Theorem 1, p. 36]), then

$$\begin{aligned}
&P\left(T_{n,A}[f]-f\right)\\
&=P\left(T_{n,A}[f]-\sum_{k=0}^{n}a_{n,k}T_k-\sum_{k=n+1}^{\infty}a_{n,k}T_n+\sum_{k=0}^{n}a_{n,k}T_k+\sum_{k=n+1}^{\infty}a_{n,k}T_n-f\right)\\
&\leqslant P\left(T_{n,A}[f]-\sum_{k=0}^{n}a_{n,k}T_k-\sum_{k=n+1}^{\infty}a_{n,k}T_n\right)+P\left(\sum_{k=0}^{n}a_{n,k}T_k+\sum_{k=n+1}^{\infty}a_{n,k}T_n-f\right)\\
&=P\left(\sum_{k=0}^{n}a_{n,k}\left\{S_k[f]-T_k\right\}+\sum_{k=n+1}^{\infty}a_{n,k}\left\{S_k[f]-T_n\right\}\right)\\
&\quad+P\left(\sum_{k=0}^{n}a_{n,k}(f-T_k)+\sum_{k=n+1}^{\infty}a_{n,k}(f-T_n)\right)\\
&\leqslant P\left(\sum_{k=0}^{\infty}a_{n,k}S_k[f-T_n]\right)+P\left(\sum_{k=0}^{n}a_{n,k}(f-T_k)\right)+P\left(\sum_{k=n+1}^{\infty}a_{n,k}(f-T_n)\right)\\
&=P\left(\sum_{k=0}^{\infty}a_{n,k}S_k[f-T_n]\right)+\sum_{k=0}^{n}a_{n,k}O(E_k(f)_P)+\sum_{k=n+1}^{\infty}a_{n,k}O(E_n(f)_P),
\end{aligned}$$

since

$$S_k[f-T_n](x)=\begin{cases}S_kf(x)-T_k(x), & \text{for } k\leqslant n,\\ S_kf(x)-T_n(x), & \text{for } k\geqslant n.\end{cases}$$

We note that by Lemma 6,

$$P\left(\sum_{k=0}^{\infty}a_{n,k}S_k[f-T_n]\right)=O(E_n(f)_P)$$

and thus our result follows.

4.2. Proof of Theorem 2

The subadditivity of ω implies $\omega(n\delta)\leqslant n\omega(\delta)$, whence $\omega(\lambda\delta)\leqslant(\lambda+1)\omega(\delta)$ and therefore $\frac{\omega(\delta_2)}{\delta_2}\leqslant 2\frac{\omega(\delta_1)}{\delta_1}$ since

$$\begin{aligned}
\omega(\delta_2)&=\omega\left(\frac{\delta_1}{\delta_1}\delta_2\right)\leqslant\left(\frac{\delta_2}{\delta_1}+1\right)\omega(\delta_1)\\
&=\left(\frac{\delta_2}{\delta_1}+\frac{\delta_1}{\delta_1}\right)\omega(\delta_1)\leqslant\left(\frac{\delta_2}{\delta_1}+\frac{\delta_2}{\delta_1}\right)\omega(\delta_1)=2\frac{\delta_2}{\delta_1}\omega(\delta_1),
\end{aligned}$$

where $n \in \mathbb{N}_0$, $\lambda \geqslant 0$ and $0 < \delta_1 \leqslant \delta_2$. Hence, using the Abel transformation and (4) with $\beta > 0$,

$$\begin{aligned}
&\sum_{k=0}^{n} a_{n,k}\omega\left(\frac{1}{k+1}\right) = \sum_{k=0}^{n} \frac{a_{n,k}}{k+1} \frac{\omega(\frac{1}{k+1})}{\frac{1}{k+1}} \\
&\leqslant 2(n+1)\,\omega\left(\frac{1}{n+1}\right) \sum_{k=0}^{n} \frac{a_{n,k}}{k+1} \\
&\leqslant 2(n+1)\,\omega\left(\frac{1}{n+1}\right) \sum_{k=0}^{\infty} (k+1)^{\beta-1} \frac{a_{n,k}}{(k+1)^{\beta}} \\
&\leqslant 2(n+1)\,\omega\left(\frac{1}{n+1}\right) \sum_{k=0}^{\infty} \left| \frac{a_{n,k}}{(k+1)^{\beta}} - \frac{a_{n,k+1}}{(k+2)^{\beta}} \right| \sum_{l=0}^{k} (l+1)^{\beta-1} \\
&\leqslant 2\left(1+\frac{1}{\beta}\right)(n+1)\,\omega\left(\frac{1}{n+1}\right) \sum_{k=0}^{\infty} (k+1)^{\beta} \left| \frac{a_{n,k}}{(k+1)^{\beta}} - \frac{a_{n,k+1}}{(k+2)^{\beta}} \right| \\
&= O\left(\omega\left(\frac{1}{n+1}\right)\right)
\end{aligned}$$

but with $\beta = 0$

$$\begin{aligned}
\sum_{k=0}^{n} a_{n,k}\omega\left(\frac{1}{k+1}\right) &\leqslant \sum_{k=0}^{n-1} \left|a_{n,k} - a_{n,k+1}\right| \sum_{l=0}^{k} \omega\left(\frac{1}{l+1}\right) + a_{n,n} \sum_{l=0}^{n} \omega\left(\frac{1}{l+1}\right) \\
&\leqslant \left(\sum_{k=0}^{n-1} \left|a_{n,k} - a_{n,k+1}\right| + \sum_{k=n}^{\infty} \left(a_{n,k} - a_{n,k+1}\right) \right) \sum_{l=0}^{n} \omega\left(\frac{1}{l+1}\right) \\
&= O\left(\frac{1}{n+1} \sum_{l=0}^{n} \omega\left(\frac{1}{l+1}\right) \right).
\end{aligned}$$

Thus our result, by Theorem 1 and the Jackson-type inequality, follows.

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REFERENCES

[1] P. CHANDRA, *Trigonometric approximation of functions in L_p-norm*, J. Math. Anal. Appl., **275** (2002), 13–26.

[2] A. GUVEN, *Trigonometric approximation in reflexive Orlicz spaces*, Anal. Theory Appl., **27** (2) (2011), 125–137.

[3] A. GUVEN, D. M. ISRAFILOV, *Approximation by trigonometric polynomials in weighted Orlicz spaces*, Stud. Math., **174** (2006), 147–168.

[4] A. GUVEN, D. M. ISRAFILOV, *Improved inverse theorems in weighted Lebesgue and Smirnov spaces*, Bull. Belg. Math. Soc., **14** (2007), 681–692.

[5] L. LEINDLER, *On the uniform convergence and boundedness of a certain class of sine series*, Analysis Math., **27** (2001), 279–285.

[6] L. LEINDLER, *On the degree of approximation of continuous functions*, Acta Math. Hungar., **104** (2004), 105–113.
[7] L. LEINDLER, *Trigonometric approximation in L_p-norm*, J. Math. Anal. Appl., **302** (1) (2005), 129–136.
[8] E. LIFLYAND, S. TIKHONOV, *A concept of general monotonicity and applications*, Math. Nachr., **284** (8–9) (2011), 1083–1098.
[9] U. S. PUGACHEV, I. N. SINITSY, *Lectures on functional analysis and applications*, Word Scientific Publ. (1999).
[10] R. N, MOHAPATRA, B. SZAL, *On trigonometric approximation of functions in the L^q norm*, Demonstratio Mathematica, **51** (1) (2018), 17–26
[11] B. SZAL, *Trigonometric approximation by Nörlund type means in L^p-norm*, Comm. Math. Universitatis Carolinae, **50** (4) (2009), 575–589.
[12] S. TIKHONOV, *Trigonometric series with general monotone coefficients*, J. Math. Anal. Appl., **326** (1) (2007), 721–735.
[13] S. TIKHONOV, *On uniform convergence of trigonometric series*, Mat. Zametki, **81** (2) (2007), 304–310, Translation in Math. Notes, **81** (2) (2007), 268–274.
[14] S. TIKHONOV, *Best approximation and moduli of smoothness: Computation and equivalence theorems*, J. Approx. Theory, **153** (2008), 19–39.
[15] D. S. YU, S. P. ZHOU, *A generalization of monotonicity and applications*, Acta Math. Hungar., **115** (3) (2007), 247–267.
[16] D. S. YU, S. P. ZHOU, P. ZHOU, *Ultimate generalization to monotonicity for uniform convergence of trigonometric series*, Sci. China Math., **53** (7) (2010), 1853–1862.
[17] V. V. ZHUK, *Approximation of periodic functions*, (in Russian), Leningr. Gos. Univ. Press, Leningrad, (1982).
[18] A. ZYGMUND, *Trigonometric Series, Vol I*, Cambridge Univ. Press, 2nd edition, (1959).

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