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CONVERGENCE IN MEASURE OF FEJÉR MEANS OF TWO PARAMETER CONJUGATE WALSH TRANSFORMS

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Abstract. Weisz proved-among others – that for $f \in L\log L$ the Fejér means $\widetilde{\sigma}_{n,m}^{(t,u)}$ of conjugate transform of two-parameter Walsh-Fourier series a. e. converges to $f^{(t,u)}$. The main aim of this paper is to prove that for any Orlicz space, which is not a subspace of $L\log L$, the set of functions for which Walsh-Fejér Means of two parameter Conjugate Transforms converge in measure is of first Baire category.

1. Definitions and notations

We shall denote the set of all non-negative integers by $\mathbb{N}$, the set of all integers by $\mathbb{Z}$ and the set of dyadic rational numbers in the unit interval $\mathbb{I} := [0,1)$ by $\mathbb{Q}$. In particular, each element of $\mathbb{Q}$ has the form $\frac{p}{2^n}$ for some $p,n \in \mathbb{N}$, $0 \leqslant p \leqslant 2^n$.

Denote the dyadic expension of $n \in \mathbb{N}$ and $x \in \mathbb{I}$ by

$$n = \sum_{j=0}^{\infty} n_j 2^j, \quad n_j = 0,1$$

and

$$x = \sum_{j=0}^{\infty} \frac{x_j}{2^{j+1}}, \quad x_j = 0,1.$$

In the case of $x \in \mathbb{Q}$ chose the expension which terminates in zeros. n_i, x_i are the i-th coordinates of n, x, respectively. Define the dyadic addition $\dotplus$ as

$$x \dotplus y = \sum_{k=0}^{\infty} |x_k - y_k| \, 2^{-(k+1)}.$$

Denote by $\oplus$ the dyadic (or logical) addition. That is,

$$k \oplus n := \sum_{i=0}^{\infty} |k_i - n_i| \, 2^i,$$

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where k_i, n_i are the ith coordinate of natural numbers k, n with respect to number system based 2.

The sets $I_n(x) := \{y \in \mathbb{I} : y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\}$ for $x \in \mathbb{I}, I_n := I_n(0)$ for $0 < n \in \mathbb{N}$ and $I_0(x) := \mathbb{I}$ are the dyadic intervals of $\mathbb{I}$. For $0 < n \in \mathbb{N}$ denote by $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$, that is, $2^{|n|} \leqslant n < 2^{|n|+1}$. Set $e_j := 1/2^{j+1}$, the i th coordinate of e_i is 1, the rest sre are zeros $(i \in \mathbb{N})$.

The Rademacher system is defined by

$$r_n(x) := (-1)^{x_n} \quad (x \in \mathbb{I},\ n \in \mathbb{N}).$$

The Walsh-Paley system is defined as the sequence of the Walsh-Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = (-1)^{\sum\limits_{k=0}^{|n|} n_k x_k}, \ \ (x \in \mathbb{I},\ n \in \mathbb{N}).$$

The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that (see [12])

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in [0, 2^{-n}) \\ 0, & \text{if } x \in [2^{-n}, 1) \end{cases}, \tag{1}$$

The σ-algebra generated by the dyadic intervals $\{I_n(x) : x \in G\}$ is denoted by A^n, more precisely,

$$A^n := \sigma\{I_n(x) : x \in G\}.$$

Denote by $f = (f_n, n \in \mathbb{N})$ martingale with respect to $(A^n, n \in \mathbb{N})$ (for details see, e. g. [16, 17]). For a martingale

$$f \sim \sum_{n=0}^{\infty} (f_n - f_{n-1}), \ \ f_{-1} = 0$$

the conjugate transforms are defined by

$$\widetilde{f}^{(t)} \sim \sum_{n=0}^{\infty} r_n(t)(f_n - f_{n-1}),$$

where $t \in \mathbb{I}$ is fixed.

Note that $\widetilde{f}^{(0)} = f$. As is well known, if f is an integrable function, then conjugate transforms $\widetilde{f}^{(t)}$ do exist almost everywhere, but they are not integrable in general.

Let

$$\rho_0(t) := r_0(t), \rho_k(t) := r_n(t) \ \text{ if } \ 2^{n-1} \leqslant k < 2^n.$$

Then the nth partial sums of the conjugate transforms is given by

$$\widetilde{S}_n^{(t)}(x;f) := \sum_{k=0}^{n-1} \rho_k(t)\widehat{f}(k)w_k(x) \quad (t \in \mathbb{I},\ n \in \mathbb{P}).$$

The conjugate $(C,1)$-means of a martingale f are introduced by

$$\widetilde{\sigma}_n^{(t)}(x;f) := \frac{1}{n}\sum_{k=0}^{n-1} \widetilde{S}_k^{(t)}(x;f) \quad (t \in \mathbb{I},\ n \in \mathbb{P}).$$

Set

$$\widetilde{\sigma}_n^{(0)}(x;f) := \sigma_n(x;f) = \frac{1}{n}\sum_{k=0}^{n-1} S_k(f) \quad (n \in \mathbb{P}).$$

We consider the double system $\left\{w_{n^1}(x^1) \times w_{n^2}(x^2) : n^1, n^2 \in \mathbb{N}\right\}$ on the unit square $\mathbb{I}^2 = [0,1) \times [0,1)$.

For a set $X \neq \varnothing$ let X^2 be its Cartesian product $X \times X$ taken with itself. The Cartesian product of two dyaduc intervals is said to be a dyadic rectangle. Clearrly, the dyadic rectangle of area $2^{-n^1} \times 2^{-n^2}$ containing $\left(x^1, x^2\right) \in \mathbb{I}^2$ is given by $I_{n^1}\left(x^1\right) \times I_{n^2}\left(x^2\right)$ The σ-algebra generated by the dyadic rectangles $\left\{I_{n^1}\left(x^1\right) \times I_{n^2}\left(x^2\right) : x^1, x^2 \in \mathbb{I}\right\}$ will be denoted by $A^{n^1,n^2}\left(n^1, n^2 \in \mathbb{N}\right)$. Let $\left(f_{n^1,n^2} : n^1, n^2 \in \mathbb{N}\right)$ be two-parameter martingale with respect to $\left(A^{n^1,n^2} : n^1, n^2 \in \mathbb{N}\right)$ (for details see, e. g. [16, 17]).

We denote by $L_0(\mathbb{I}^2)$ the Lebesgue space of functions that are measurable and finite almost everywhere on $\mathbb{I}^2$. $\mu(A)$ is the Lebesgue measure of the set $A \subset \mathbb{I}^2$.

We denote by $L_p\left(\mathbb{I}^2\right)$ the class of all measurable functions f that are 1-periodic with respect to all variable and satisfy

$$\|f\|_p := \left(\int_{\mathbb{I}^2} |f|^p\right)^{1/p} < \infty.$$

Let $L_Q = L_Q(\mathbb{I}^2)$ be the Orlicz space [13] generated by Young function Q, i.e. Q is convex continuous even function such that $Q(0) = 0$ and

$$\lim_{u\to+\infty} \frac{Q(u)}{u} = +\infty, \qquad \lim_{u\to 0} \frac{Q(u)}{u} = 0.$$

This space is endowed with the norm

$$\|f\|_{L_Q(\mathbb{I}^2)} = \inf\{k > 0 : \int_{\mathbb{I}^2} Q(|f|/k) \leqslant 1\}.$$

In particular, if $Q(u) = u\log(1+u)$, $u > 0$, then the corresponding space will be denoted by $L\log L$.

For a martingale

$$f \sim \sum_{n^1,n^2=0}^{\infty} \left(f_{n^1,n^2} - f_{n^1-1,n^2} - f_{n^1,n^2-1} + f_{n^1-1,n^2-1}\right), \ f_{-1,n^2} = f_{n^1,-1} = f_{-1,-1} = 0$$

the conjugate transform is defined by the martingale

$$\widetilde{f}^{\left(t^1,t^2\right)} \sim \sum_{n^1,n^2=0}^{\infty} r_{n^1}\left(t^1\right) r_{n^2}\left(t^2\right) \left(f_{n^1,n^2} - f_{n^1-1,n^2} - f_{n^1,n^2-1} + f_{n^1-1,n^2-1}\right),$$

where $t^1, t^2 \in \mathbb{I}$ are fixed. Note that $\widetilde{f}^{(0,0)} = f$. As is well known, if $f \in L\log L\left(\mathbb{I}^2\right)$ then the conjugate transforms $\widetilde{f}^{\left(t^1,t^2\right)}$ do exists almost everywhere, but they are not integrable in general.

If $f \in L_1\left(\mathbb{I}^2\right)$, then

$$\hat{f}\left(n^1,n^2\right) = \int_{\mathbb{I}^2} f\left(y^1,y^2\right) w_{n^1}(y^1) w_{n^2}(y^2) dy^1 dy^2$$

is the $\left(n^1,n^2\right)$-th Fourier coefficient of f.

The rectangular partial sums of double Fourier series with respect to the Walsh system are defined by

$$S_{N^1,N^2}\left(x^1,x^2;f\right) = \sum_{n^1=0}^{N^1-1} \sum_{n^2=0}^{N^2-1} \hat{f}\left(n^1,n^2\right) w_{n^1}(x^1) w_{n^2}(x^2).$$

It is easy to see that the sequence $\left\{S_{2^{n^1},2^{n^2}}(f) = f_{n^1,n^2} : n^1,n^2 \in \mathbb{N}\right\}$ is two-parameter martingale.

Then the $\left(n^1,n^2\right)$ th partial sum of the conjugate transforms is given by

$$\widetilde{S}_{n^1,n^2}^{\left(t^1,t^2\right)}\left(x^1,x^2;f\right) := \sum_{v^1=0}^{n^1-1} \sum_{v^2=0}^{n^2-1} \rho_{v^1}\left(t^1\right) \rho_{v^2}\left(t^2\right) \widehat{f}\left(v^1,v^2\right) w_{v^1}\left(x^1\right) w_{v^2}\left(x^2\right).$$

The conjugate $(C,1,1)$ means of the function f are introduced by

$$\widetilde{\sigma}_{n^1,n^2}^{\left(t^1,t^2\right)}\left(x^1,x^2;f\right) := \frac{1}{n^1 n^2} \sum_{v^1=1}^{n^1} \sum_{v^2=1}^{n^2} \widetilde{S}_{v^1,v^2}^{\left(t^1,t^2\right)}\left(x^1,x^2;f\right).$$

The rectangular partial sums of the Walsh-Fourier series $S_{n^1,n^2}(f)$, of the function $f \in L_p\left(\mathbb{I}^2\right)$, $1 < p < \infty$ converge in L^p norm to the function f as $n^1,n^2 \to \infty$, [11, 19]. In the case $L_1\left(\mathbb{I}^2\right)$ this result does not hold [6, 12]. But in the one-dimensional case the operators S_n are of weak type $(1,1)$ [15], that is the analogue of the estimate of Kolmogorov for conjugate function [8]. This estimate implies the convergence of $S_n(f)$ in measure on $\mathbb{I}$ to the function $f \in L_1(\mathbb{I})$. However, for double Walsh-Fourier series this result [5, 14] fails to hold.

Classical regular summation methods often improve the convergence of Walsh-Fourier series. For instance, the Fejér means $\sigma_{n^1,n^2}(f) := \widetilde{\sigma}^{(0,0)}_{n^1,n^2}(f)$ of the Walsh-Fourier series of the function $f \in L_1\left(\mathbb{I}^2\right)$, converge in norm $L_1\left(\mathbb{I}^2\right)$ to the function f, as $n^1, n^2 \to \infty$ [9, 19, 7].

In 1992 Móricz, Schipp and Wade [10] proved with respect to the Walsh-Paley system that

$$\sigma_{n^1,n^2}(f) = \frac{1}{n^1 n^2} \sum_{\nu^1=0}^{n^1-1} \sum_{\nu^2=0}^{n^2-1} S_{\nu^1,\nu^2}(f) \to f$$

a.e. for each $f \in L\log^+ L(\mathbb{I}^2)$, when $\min\left\{n^1, n^2\right\} \to \infty$. In 2000 Gát proved [4] that the theorem of Möricz, Schipp and Wade above can not be improved. Namely, let $\delta : [0, +\infty) \to [0, +\infty)$ be a measurable function with property $\lim_{t\to\infty} \delta(t) = 0$. Gát proved [4] the existence of a function $f \in L_1(\mathbb{I}^2)$ such that $f \in L\log L\delta(L)$, and $\sigma_{n^1,n^2}(f)$ does not converge to f a.e. as $\min\left\{n^1, n^2\right\} \to \infty$. That is, the maximal convergence space for the $(C,1,1)$ means of two-dimensional partial sums is $L\log L(\mathbb{I}^2)$. On the othar hand, the $(C,1,1)$ means of two-dimensional partial sums of the function $f \in L_1\left(\mathbb{I}^2\right)$, converge in norm $L_1\left(\mathbb{I}^2\right)$ to the function f, as $n^1, n^2 \to \infty$ which imply converge in measure of the $(C,1,1)$ means for all functions $f \in L_1\left(\mathbb{I}^2\right)$.

Almost everywhere convergence of conjugate $(C,1,1)$ means of two-parameter Walsh-Fourier series was investigated by Weisz [18]. In particular, he proved the following theorem.

THEOREM W. *Let t^1, $t^2 \in \mathbb{I}$ and $f \in L\log L\left(\mathbb{I}^2\right)$. Then*

$$\widetilde{\sigma}^{\left(t^1,t^2\right)}_{n^1,n^2}\left(x^1, x^2; f\right) \to \widetilde{f}^{\left(t^1,t^2\right)}\left(x^1, x^2\right)$$

a. e. as $n^1, n^2 \to \infty$.

The main aim of this paper is to prove that when t^1, t^2 are dyadic irational then the Walsh-Fejér Means of two parameter Conjugate Transforms does not improve the convergence in measure. In particular, we prove the following

THEOREM 1. *Let t^1, $t^2 \notin \mathbb{Q}$ and $Q(L)\left(\mathbb{I}^2\right)$ be an Orlicz space such that*

$$Q(L)\left(\mathbb{I}^2\right) \nsubseteq L\log L\left(\mathbb{I}^2\right).$$

Then the set of function from the Orlicz space $Q(L)\left(\mathbb{I}^2\right)$ with Fejér means of conjugate transform $\widetilde{\sigma}^{\left(t^1,t^2\right)}_{n^1,n^2}(f)$ of two-parameter Walsh-Fourier series converges in measure on $\mathbb{I}^2$ is of first Bairy category in $Q(L)\left(\mathbb{I}^2\right)$.

COROLLARY 1. *Let $t^1, t^2 \notin \mathbb{Q}$ and $\varphi : [0,\infty) \to [0,\infty)$ be a nondecreasing function satisfying for $x \to \infty$, the condition*

$$\varphi(x) = o(x\log x).$$

Then there exists a function $f \in L_1\left(\mathbb{I}^2\right)$ such that

a) $\int\limits_{\mathbb{I}^2} \varphi\left(\left|f\left(x^1,x^2\right)\right|\right) dx^1 dx^2 < \infty$;

b) Fejér means of conjugate transform of two-parameter Walsh-Fourier series of f diverge in measure on $\mathbb{I}^2$.

2. Auxiliary results

THEOREM GGT. [3, 2] *Let $\{T_m\}_{m=1}^{\infty}$ be a sequence of linear continuous operators, acting from Orlicz space $Q(L)(\mathbb{I}^2)$ in to the space $L_0(\mathbb{I}^2)$. Suppose that there exists a sequence of functions $\{\xi_k\}_{k=1}^{\infty}$ from unit bull $S_Q(0,1)$ of space $Q(L)(\mathbb{I}^2)$ and an increasing to infinity sequences $\{m_k\}_{k=1}^{\infty}$ and $\{\lambda_k\}_{k=1}^{\infty}$ such that*

$$\varepsilon_0 = \inf_k \mu\{\left(x^1,x^2\right) \in \mathbb{I}^2 : |T_{m_k}\xi_k\left(x^1,x^2\right)| > \lambda_k\} > 0.$$

Then the set of functions f from space $Q(L)(\mathbb{I}^2)$, for which the sequence $\{T_m f\}$ converges in measure to an a. e. finite function is of first Baire category in space $Q(L)(\mathbb{I}^2)$.

THEOREM GGT2. [3, 2] *Let $\Phi(L)(\mathbb{I}^2)$ be an Orlicz space and let $\varphi : [0,\infty) \to [0,\infty)$ be a measurable function with the condition $\varphi(x) = o(\Phi(x))$ as $x \to \infty$. Then there exists an Orlicz space $\omega(L)\left(\mathbb{I}^2\right)$ such that $\omega(x) = o(\Phi(x))$ as $x \to \infty$, and $\omega(x) \geqslant \varphi(x)$ for $x \geqslant c \geqslant 0$.*

3. Proofs

Proof. Since t^1 and t^2 are dyadic irrational there exists two sequences of integers $\left\{a_i^{(k)} : i \in \mathbb{N}\right\}$ and $\left\{b_i^{(k)} : i \in \mathbb{N}\right\}$, $k = 1,2$ such that

$$0 \leqslant a_1^{(k)} \leqslant b_1^{(k)} < a_2^{(k)} \leqslant b_2^{(k)} < \cdots < a_A^{(k)} \leqslant b_A^{(k)} < \cdots$$

and

$$t_j^k = \begin{cases} 1, \text{ if } a_i^{(k)} \leqslant j \leqslant b_i^{(k)} \\ 0, b_i^{(k)} < j < a_{i+1}^{(k)} \end{cases}, \quad i = 1,2,\ldots.$$

Set

$$\Delta_A^{(k)} := I_{b_A^{(k)}+1}\left(x_0^k,\ldots,x_{a_1^{(k)}-1}^k,0,x_{a_1^{(k)}+1}^k,\ldots,x_{b_1^{(k)}-1}^k,0,x_{b_1^{(k)}+1}^k,\ldots,\right.$$
$$\left. x_{a_A^{(k)}-1}^k,0,x_{a_A^{(k)}+1}^k,\ldots,x_{b_A^{(k)}-1}^k,0\right).$$

Define the functions

$$h_A^{(k)}(x) := 2^{2A}\chi_{\Delta_A^{(k)}}(x), \quad k = 1,2$$

and

$$f_A\left(x^1,x^2\right) := h_A^{(1)}\left(x^1\right) h_A^{(2)}\left(x^2\right),$$

where χ_E is characteristic function of the set E.

Since

$$\widetilde{S}_{n^1,n^2}^{\left(t^1,t^2\right)}\left(x^1,x^2;f_A\right) = \widetilde{S}_{n^1}^{\left(t^1\right)}\left(x^1,h_A^{(1)}\right) \widetilde{S}_{n^2}^{\left(t^2\right)}\left(x^1,h_A^{(2)}\right)$$

we obtain

$$\widetilde{\sigma}_{2^{2b_A^{(1)}+1},2^{2b_A^{(2)}+1}}^{\left(t^1,t^2\right)}\left(x^1,x^2;f_A\right) = \widetilde{\sigma}_{2^{2b_A^{(1)}+1}}^{\left(t^1\right)}\left(x^1;h_A^{(1)}\right) \widetilde{\sigma}_{2^{2b_A^{(2)}+1}}^{\left(t^2\right)}\left(x^2;h_A^{(2)}\right). \tag{2}$$

Since for $2^{m-1} \leqslant k < 2^m \quad \left(S_{2^{-1}}(f) = 0\right)$

$$\begin{aligned}
\widetilde{S}_k^{(t)}(f) &= \rho_0(t)\widehat{f}(0)w_0 + \sum_{l=1}^{m-1} r_l(t)\left(S_{2^l}(f) - S_{2^{l-1}}(f)\right) \\
&\quad + r_m(t)\left(S_k(f) - S_{2^{m-1}}(f)\right) \\
&= \sum_{l=0}^{m-1} r_l(t)\left(S_{2^l}(f) - S_{2^{l-1}}(f)\right) + r_m(t)\left(S_k(f) - S_{2^{m-1}}(f)\right)
\end{aligned}$$

we have

$$\begin{aligned}
\widetilde{\sigma}_{2^{2b_A+1}}^{(t)}(f) &= \frac{1}{2^{2b_A+1}} \sum_{m=1}^{2b_A+1} \sum_{k=2^{m-1}}^{2^m-1} \widetilde{S}_k^{(t)}(f) \\
&= \frac{1}{2^{2b_A+1}} \sum_{m=1}^{2b_A+1} 2^{m-1} \widetilde{S}_{2^{m-1}}^{(t)}(f) \\
&\quad + \frac{1}{2^{2b_A+1}} \sum_{m=1}^{2b_A+1} r_m(t)\left(2^m \sigma_{2^m}(f) - 2^{m-1}\sigma_{2^{m-1}}(f)\right) \\
&\quad - \frac{1}{2^{2b_A+1}} \sum_{m=1}^{2b_A+1} r_m(t)\, 2^{m-1} S_{2^{m-1}}(f).
\end{aligned} \tag{3}$$

Set

$$\begin{aligned}
x^1 \in \widetilde{\Delta}_A^{(1)} := I_{b_A^{(k)}+1}\Big(& x_0^1,\ldots,x_{a_1^{(1)}-1}^1,0,x_{a_1^{(1)}+1}^1,\ldots,x_{b_1^{(1)}-1}^1,0,x_{b_1^{(1)}+1}^1,\ldots, \\
& x_{a_A^{(1)}-1}^1,1,x_{a_A^{(1)}+1}^1,\ldots,x_{b_A^{(1)}-1}^1,1\Big).
\end{aligned}$$

Then from (1) we have

$$\begin{aligned}
S_{2^{m-1}}\left(x^1;h_A^{(1)}\right) &= \int_{\mathbb{I}} h_A^{(1)}(s) D_{2^{m-1}}\left(x^1 \dot{+} s\right) ds \\
&= 2^{m-1} \int_{I_{m-1}\left(x^1\right)} h_A^{(1)}(s)\, ds = 0
\end{aligned} \tag{4}$$

if $m>a_A^{(1)}+1$. It is well known that (see [12])

$$\sigma_{2^{m-1}}\left(x^1;h_A^{(1)}\right)=\int_{\mathbb{I}} h_A^{(1)}(s)\,K_{2^{m-1}}\left(x^1\dotplus s\right)ds,$$

where

$$K_{2^n}(x)=\frac{1}{2}\left(2^{-n}D_{2^n}(x)+\sum_{j=0}^{n}2^{j-n}D_{2^n}\left(x\dotplus e_j\right)\right).$$

Let $m>b_i^{(1)}+2$. Then $h_A^{(1)}(s)\neq 0$ imply that there exists at least two coordinates in $x^1\dotplus s=(y_0,y_1,\ldots,)$ which are equal to 1. Consequently,

$$\sigma_{2^{m-1}}\left(x^1;h_A^{(1)}\right)=0 \tag{5}$$

when $m>b_i^{(1)}+2$ and $x^1\in\widetilde{\Delta}_i^{(1)}$.

Let $x^1\in\widetilde{\Delta}_i^{(1)}$. Then Combining (3)–(5) we obtain

$$\begin{aligned}
&\left|\widetilde{\sigma}^{(t^1)}_{2^{2b_A^{(1)}+1}}\left(x^1;h_A^{(1)}\right)\right| \qquad (6)\\
&\geqslant \frac{1}{2^{2b_A^{(1)}+1}}\left|\sum_{m=b_i^{(1)}+3}^{2b_A^{(1)}+1}2^{m-1}\widetilde{S}^{(t^1)}_{2^{m-1}}\left(x^1;h_A^{(1)}\right)\right|\\
&-\frac{1}{2^{2b_A^{(1)}+1}}\left|\sum_{m=1}^{b_i^{(1)}+2}2^{m-1}\widetilde{S}^{(t^1)}_{2^{m-1}}\left(x^1;h_A^{(1)}\right)\right|\\
&-\frac{1}{2^{2b_A^{(1)}+1}}\sum_{m=1}^{b_i^{(1)}+2}\left(2^m\left|\sigma_{2^m}\left(x^1;h_A^{(1)}\right)\right|+2^{m-1}\left|\sigma_{2^{m-1}}\left(x^1;h_A^{(1)}\right)\right|\right)\\
&-\frac{1}{2^{2b_A^{(1)}+1}}\sum_{m=1}^{b_i^{(1)}+2}2^{m-1}\left|S_{2^{m-1}}\left(x^1;h_A^{(1)}\right)\right|.
\end{aligned}$$

Since

$$\left|S_{2^{m-1}}\left(x^1;h_A^{(1)}\right)\right|,\left|\widetilde{S}^{(t^1)}_{2^{m-1}}\left(x^1;h_A^{(1)}\right)\right|,\left|\sigma_{2^m}\left(x^1;h_A^{(1)}\right)\right|\leqslant 2^m$$

from (6) we have

$$\begin{aligned}
&\left|\widetilde{\sigma}^{(t^1)}_{2^{2b_A^{(1)}+1}}\left(x^1;h_A^{(1)}\right)\right| \qquad (7)\\
&\geqslant\left|\frac{1}{2^{2b_A^{(1)}+1}}\left|\sum_{m=b_i^{(1)}+3}^{2b_A^{(1)}+1}2^{m-1}\widetilde{S}^{(t^1)}_{2^{m-1}}\left(x^1;h_A^{(1)}\right)\right|-c\right|.
\end{aligned}$$

Now, we estimate $\widetilde{S}_{2^m}^{(t^1)}\left(x^1;h_A^{(1)}\right)$ when $m\geqslant b_i^{(1)}+3$. It is easy to see that

$$\widetilde{S}_{2^m}^{(t^1)}\left(x^1;h_A^{(1)}\right)=\int_{\mathbb{I}} h_A^{(1)}(s)\,\widetilde{D}_{2^{m-1}}^{(t^1)}\left(x^1\dot{+}s\right)ds, \tag{8}$$

where

$$\widetilde{D}_{2^{m-1}}^{(t^1)}(x):=\sum_{l=1}^{m}(-1)^{t_l}\left(D_{2^l}(x)-D_{2^{l-1}}(x)\right). \tag{9}$$

We can write

$$\begin{aligned}\widetilde{D}_{2^{m-1}}^{(t^1)}(x)&=\sum_{l=1}^{m}(1-2t_l)\left(D_{2^l}(x)-D_{2^{l-1}}(x)\right) \qquad (10)\\ &=(1-2t_m)D_{2^m}(x)-2\sum_{l=0}^{m-2}(t_l-t_{l+1})D_{2^l}(x).\end{aligned}$$

Then from (1) and (8)–(10) we obtain

$$\begin{aligned}&\widetilde{S}_{2^{m-1}}^{(t^1)}\left(x^1;h_A^{(1)}\right)\\ &=-2\sum_{l=0}^{m-2}(t_l-t_{l+1})S_{2^l}\left(x^1;h_A^{(1)}\right)\\ &=\sum_{k=1}^{i-1}\left[2S_{2^{a_k^{(1)}}-1}\left(x^1;h_A^{(1)}\right)-2S_{2^{b_k^{(1)}}}\left(x^1;h_A^{(1)}\right)\right]\\ &\quad+2S_{2^{a_i^{(1)}}-1}\left(x^1;h_A^{(1)}\right)\\ &=\sum_{k=1}^{i-1}\left[2^{a_k^{(1)}+2A}\mu\left(I_{a_k^{(1)}-1}\left(x^1\right)\cap\Delta_A^{(1)}\right)-2^{b_k^{(1)}+1+2A}\mu\left(I_{b_k^{(1)}}\left(x^1\right)\cap\Delta_A^{(1)}\right)\right]\\ &\quad+2^{a_i^{(1)}+2A}\mu\left(I_{a_i^{(1)}-1}\left(x^1\right)\cap\Delta_A^{(1)}\right).\end{aligned}$$

It is easy to calculate that

$$\mu\left(I_{a_k^{(1)}-1}\left(x^1\right)\cap\Delta_A^{(1)}\right)=\frac{2^{b_A^{(1)}-a_k^{(1)}+2-2(A-k+1)}}{2^{b_A^{(1)}+1}}=2^{-a_k^{(1)}-2(A-k)-1}$$

and

$$\mu\left(I_{b_k^{(1)}}\left(x^1\right)\cap\Delta_A^{(1)}\right)=\frac{2^{b_A^{(1)}-\left(b_k^{(1)}-1\right)-[2(A-k)+1]}}{2^{b_A^{(1)}+1}}=2^{-b_k^{(1)}-2(A-k)-1}.$$

Hence

$$\begin{aligned}
&\widetilde{S}_{2^{m-1}}^{(t^1)}\left(x^1;h_A^{(1)}\right)\\
&=\sum_{k=1}^{i-1}\left[2^{a_k^{(1)}+2A}2^{-a_k^{(1)}-2(A-k)-1}-2^{b_k^{(1)}+1+2A}2^{-b_k^{(1)}-2(A-k)-1}\right]\\
&\quad+2^{a_i^{(1)}+2A}2^{-a_i^{(1)}-2(A-i)-1}\\
&=\frac{2^{2i}+2}{3},
\end{aligned}$$

when

$$m\geqslant b_i^{(1)}+3 \text{ and } x^1\in\widetilde{\Delta}_i^{(1)},\ \ i=1,2,\dots,A.$$

Consequently, from (7) we get

$$\left|\widetilde{\sigma}_{2^{2b_A^{(1)}+1}}^{(t^1)}\left(x^1;h_A^{(1)}\right)\right|\geqslant c_1 2^{2i},\ \text{ when }\ x^1\in\widetilde{\Delta}_i^{(1)},\ \ i=1,2,\dots,A. \tag{11}$$

Analogously, we can prove

$$\left|\widetilde{\sigma}_{2^{2b_A^{(2)}+1}}^{(t^2)}\left(x^2;h_A^{(2)}\right)\right|\geqslant c_1 2^{2j},\ \text{ when }\ x^1\in\widetilde{\Delta}_j^{(2)},\ \ j=1,2,\dots,A. \tag{12}$$

Combining (2), (11) and (12) we have

$$\left|\widetilde{\sigma}_{2^{2b_A^{(1)}+1},2^{2b_A^{(2)}+1}}^{(t^1,t^2)}\left(x^1,x^2;f_A\right)\right|\geqslant c_0 2^{2i+2j}$$

when $\left(x^1,x^2\right)\in\widetilde{\Delta}_i^{(1)}\times\widetilde{\Delta}_j^{(2)}$, $i,j=1,2,\dots,A.$

Set

$$\Omega_A:=\bigcup_{i,j=1}^{A}\widetilde{\Delta}_i^{(1)}\times\widetilde{\Delta}_j^{(2)}.$$

Now, we prove that

$$\mu\left(\left\{\left(x^1,x^2\right)\in\mathbb{I}^2:\left|\widetilde{\sigma}_{2^{2b_A^{(1)}+1},2^{2b_A^{(2)}+1}}^{(t^1,t^2)}\left(x^1,x^2;f_A\right)\right|>2^{2A}\right\}\right)\geqslant\frac{cA}{2^{2A}}. \tag{13}$$

Indeed,

$$\begin{aligned}
&\mu\left(\left\{\left(x^1,x^2\right)\in\mathbb{I}^2:\left|\widetilde{\sigma}_{2^{2b_A^{(1)}+1},2^{2b_A^{(2)}+1}}^{(t^1,t^2)}\left(x^1,x^2;f_A\right)\right|>2^{2A}\right\}\right)\\
&\geqslant\mu\left(\left\{\left(x^1,x^2\right)\in\Omega_A:\left|\widetilde{\sigma}_{2^{2b_A^{(1)}+1},2^{2b_A^{(2)}+1}}^{(t^1,t^2)}\left(x^1,x^2;f_A\right)\right|>2^{2A}\right\}\right)\\
&=\sum_{i,j=1}^{A}\mu\left(\left\{\left(x^1,x^2\right)\in\widetilde{\Delta}_i^{(1)}\times\widetilde{\Delta}_j^{(2)}:\left|\widetilde{\sigma}_{2^{2b_A^{(1)}+1},2^{2b_A^{(2)}+1}}^{(t^1,t^2)}\left(x^1,x^2;f_A\right)\right|>2^{2A}\right\}\right)\\
&\geqslant c\sum_{i=1}^{A}\sum_{j=A-i}^{A}\frac{1}{2^{2i+2j}}\geqslant\frac{c_2A}{2^{2A}}.
\end{aligned}$$

Hence (13) is proved.

Next, we prove that there exists $\left(y_1^1, y_1^2\right), \ldots, \left(y_{p(A)}^1, y_{p(A)}^2\right) \in \mathbb{I}^2$, $p(A) := \left[2^{2A}/c_2 A\right] + 1$, such that

$$\mu\left(\bigcup_{j=1}^{p(A)} \left(\Omega_A \dotplus \left(y_j^1, y_j^2\right)\right)\right) \geqslant \frac{1}{2}. \tag{14}$$

Indeed,

$$\begin{aligned}
&\mu\left(\bigcup_{j=1}^{p(A)} \left(\Omega_A \dotplus \left(y_j^1, y_j^2\right)\right)\right) \\
&= 1 - \mu\left(\bigcap_{j=1}^{p(A)} \left(\overline{\Omega_A \dotplus \left(y_j^1, y_j^2\right)}\right)\right) \\
&= 1 - \int_{\mathbb{I}^2} \mathbb{I}_{\overline{\Omega}_A}\left(s^1 \dotplus y_1^1, s^2 \dotplus y_1^2\right) \cdots \mathbb{I}_{\overline{\Omega}_A}\left(s^1 \dotplus y_{p(A)}^1, s^2 \dotplus y_{p(A)}^2\right) ds^1 ds^2.
\end{aligned} \tag{15}$$

Interpreting $\mathbb{I}_{\overline{\Omega_A}}\left(s^1 \dotplus y_1^1, s^2 \dotplus y_1^2\right) \cdots \mathbb{I}_{\overline{\Omega_A}}\left(s^1 \dotplus y_{p(A)}^1, s^2 \dotplus y_{p(A)}^2\right)$ as a function of the $2p(A) + 2$ variables $s^1, s^2, \left(y_1^1, y_1^2\right), \ldots, \left(y_{p(A)}^1, y_{p(A)}^2\right)$ and integrating over all variables, each over $\mathbb{I}^2$, we note that

$$\begin{aligned}
&\int_{\mathbb{I}^2} \cdots \int_{\mathbb{I}^2} \int_{\mathbb{I}^2} \mathbb{I}_{\overline{\Omega}_A}\left(s^1 \dotplus y_1^1, s^2 \dotplus y_1^2\right) \cdots \mathbb{I}_{\overline{\Omega}_A}\left(s^1 \dotplus y_{p(A)}^1, s^2 \dotplus y_{p(A)}^2\right) \\
&\quad ds^1 ds^2 dy_1^1 dy_1^2 \cdots dy_{p(A)}^1 dy_{p(A)}^2 \\
&= \int_{\mathbb{I}^2} \left(\int_{\mathbb{I}^2} \mathbb{I}_{\overline{\Omega}_A}\left(s^1 \dotplus y_1^1, s^2 \dotplus y_1^2\right) dy_1^1 dy_1^2\right) \\
&\quad \cdots \left(\int_{\mathbb{I}^2} \mathbb{I}_{\overline{\Omega}_A}\left(s^1 \dotplus y_{p(A)}^1, s^2 \dotplus y_{p(A)}^2\right) dy_{p(A)}^1 dy_{p(A)}^2\right) ds^1 ds^2 \\
&= \left(\mu\left(\overline{\Omega}_A\right)\right)^{p(A)} = \left(1 - \mu\left(\Omega_A\right)\right)^{p(A)} \\
&\leqslant \left(1 - \frac{1}{p(A)}\right)^{p(A)} \leqslant \frac{1}{2}.
\end{aligned}$$

Consequently, there exists $\left(y_1^1, y_1^2\right), \ldots, \left(y_{p(A)}^1, y_{p(A)}^2\right) \in \mathbb{I}^2$ such that

$$\int_{\mathbb{I}^2} \mathbb{I}_{\overline{\Omega_A}}\left(s^1 \dotplus y_1^1, s^2 \dotplus y_1^2\right) \cdots \mathbb{I}_{\overline{\Omega_A}}\left(s^1 \dotplus y_{p(A)}^1, s^2 \dotplus y_{p(A)}^2\right) ds^1 ds^2 \leqslant \frac{1}{2}. \tag{16}$$

Combining (15) and (16) we conclude that

$$\mu\left(\bigcup_{j=1}^{p(A)}\left(\Omega_A \dot{+} \left(y_j^1, y_j^2\right)\right)\right) \geqslant 1-\frac{1}{2}=\frac{1}{2}.$$

Hence (14) is proved.

Set $\left(s := s^1 \dot{+} s^2 \in \mathbb{I}\right)$

$$\begin{aligned} F_A\left(x^1,x^2,s\right) &:= \frac{1}{p(A)}\sum_{j=1}^{p(A)} r_j\left(s^1 \dot{+} s^2\right) f_A\left(x^1 \dot{+} y_j^1, x^2 \dot{+} y_j^2\right) \\ &= \frac{1}{p(A)}\sum_{j=1}^{p(A)} r_j(s) f_A\left(x^1 \dot{+} y_j^1, x^2 \dot{+} y_j^2\right). \end{aligned}$$

Then it is proved in ([1], pp. 7–12) that there exists $s_0 \in \mathbb{I}$, such that

$$\int_{\mathbb{I}} \left|F_A\left(x^1,x^2,s_0\right)\right| dx^1 dx^2 \leqslant 1 \tag{17}$$

and

$$\mu\left\{\left(x^1,x^2\right)\in\mathbb{I}^2 : \left|\widetilde{\sigma}^{\left(t^1,t^2\right)}_{2^{2b_A^{(1)}+1},2^{2b_A^{(2)}+1}}\left(x^1,x^2;F_A\right)\right| > cA\right\} \geqslant \frac{1}{8}. \tag{18}$$

From the condition of the Theorem 1 we write

$$\liminf_{u\to\infty}\frac{Q(u)}{u\log u}=0.$$

Consequently, there exists a sequence of integers $\{A_k : k \geqslant 1\}$ icreasing to infinity, such that

$$\lim_{k\to\infty}\frac{Q\left(2^{4A_k}\right)}{2^{4A_k}A_k}=0 \tag{19}$$

and

$$\frac{Q\left(2^{4A_k}\right)}{2^{4A_k}} \geqslant 1. \tag{20}$$

Set

$$\xi_k\left(x^1,x^2\right) := \frac{2^{4A_k-1}}{Q\left(2^{4A_k}\right)} F_{A_k}\left(x^1,x^2;s_0\right).$$

Now, we prove that

$$\|\xi_k\|_{Q(L)} \leqslant 1. \tag{21}$$

Indeed, since

$$\left\|f_{A_k}\right\|_\infty \leqslant 2^{4A_k},$$

$$Q(u) \leqslant \frac{Q(u')}{u'}u \qquad \left(0<u<u'\right),$$

$$\frac{2^{4A_k}}{Q\left(2^{4A_k}\right)}\|F_k\|_\infty \leqslant 2^{4A_k}$$

and

$$\|\xi_k\|_{Q(L)} \leqslant \frac{1}{2}\left[\int_{\mathbb{I}^2} Q\left(2\left|\xi_k\left(x^1,x^2\right)\right|\right) dx^1 dx^2 + 1\right]$$

we can write

$$\begin{aligned}\|\xi_k\|_{Q(L)} &\leqslant \frac{1}{2}\left[\int_{\mathbb{I}^2} Q\left(\frac{2^{4A_k}}{Q\left(2^{4A_k}\right)}\left|F_{A_k}\left(x^1,x^2;s_0\right)\right|\right) dx^1 dx^2 + 1\right] \\ &\leqslant \frac{1}{2}\left[\int_{\mathbb{I}^2} \frac{Q\left(2^{4A_k}\right)}{2^{4A_k}}\frac{2^{4A_k}}{Q\left(2^{4A_k}\right)}\left|F_{A_k}\left(x^1,x^2;s_0\right)\right| dx^1 dx^2 + 1\right] \\ &\leqslant 1.\end{aligned}$$

Hence (21) is proved.

On the other hand, from (18) we have

$$\mu\left\{\left(x^1,x^2\right)\in\mathbb{I}^2 : \left|\widetilde{\sigma}^{\left(t^1,t^2\right)}_{2^{2b^{(1)}_{A_k}+1},2^{2b^{(2)}_{A_k}+1}}\left(x^1,x^2;\xi_k\right)\right| > \frac{cA_k 2^{4A_k}}{Q\left(2^{4A_k}\right)}\right\} \geqslant \frac{1}{8}. \tag{22}$$

Combine (19), (21) and (22), from Theorem GGT we complete the proof of Theorem 1.

The validity of Corollary 1 follows immediately from Theorem 1 and Lemma GGT2.

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