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SHARP INEQUALITIES FOR THE NUMERICAL RADIUS OF HILBERT SPACE OPERATORS AND OPERATOR MATRICES

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Abstract. We present new upper and lower bounds for the numerical radius of a bounded linear operator defined on a complex Hilbert space, which improve on the existing bounds. Among many other inequalities proved in this article, we show that for a non-zero bounded linear operator T on a complex Hilbert space H,

$$w(T) \geqslant \frac{\|T\|}{2} + \frac{m(T^2)}{2\|T\|},$$

where $w(T)$ is the numerical radius of T and $m(T^2)$ is the Crawford number of T^2. This substantially improves on the existing inequality $w(T) \geqslant \frac{\|T\|}{2}$. We also obtain some upper and lower bounds for the numerical radius of operator matrices and illustrate with numerical examples that these bounds are better than the existing bounds.

1. Introduction

Computation of the numerical radius of a bounded linear operator defined on a complex Hilbert space is an interesting embroiled problem. Till date one can compute the exact numerical radius for certain special class of operators and for this reason estimation of bounds for the numerical radius is a very important problem. Our aim in this article to present better estimation of the numerical radius of bounded linear operators and operator matrices. The following notations and terminologies are necessary to begin with.

Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be two complex Hilbert spaces with inner product $\langle .,. \rangle$. Let $B(\mathbb{H}_1,\mathbb{H}_2)$ denote the set of all bounded linear operators from $\mathbb{H}_1$ to $\mathbb{H}_2$, if $\mathbb{H}_1 = \mathbb{H}_2 = \mathbb{H}$ (say), then we write $B(\mathbb{H}_1,\mathbb{H}_2) = B(\mathbb{H})$. For $T \in B(\mathbb{H})$, let $\|T\|$ and $c(T)$ denote the usual operator norm and the minimum norm of T, respectively, defined as

$$\|T\| = \sup\{\|Tx\| : x \in \mathbb{H}, \|x\| = 1\}$$

and

$$c(T) = \inf\{\|Tx\| : x \in \mathbb{H}, \|x\| = 1\},$$

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where $\|.\|$ is the norm on $\mathbb{H}$ induced from the inner product $\langle .,. \rangle$. Let $\sigma(T)$ denote the spectrum of T, and $r(T)$, the spectral radius of T, defined as

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

The numerical range of T, denoted as $W(T)$, is defined as

$$W(T) = \{\langle Tx, x\rangle : x \in \mathbb{H}, \|x\| = 1\}.$$

Let $w(T)$ and $m(T)$ denote the numerical radius and the Crawford number of T, respectively, defined as

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\}$$

and

$$m(T) = \inf\{|\lambda| : \lambda \in W(T)\}.$$

It is well-known that the numerical range is a convex subset of the scalar field and closure of the numerical range contains the spectrum, i.e., $\sigma(T) \subseteq \overline{W(T)}$, so $r(T) \leqslant w(T)$. The numerical radius $w(.)$ acts as a norm on $B(\mathbb{H})$ and is equivalent to the usual operator norm $\|.\|$, satisfying the following inequality

$$\frac{\|T\|}{2} \leqslant \max\left\{r(T), \frac{\|T\|}{2}\right\} \leqslant w(T) \leqslant \|T\|.$$

The weak unitarily invariance property of the numerical radius states that

$$w(U^*TU) = w(T), \quad \text{for all unitary operators } U \in B(\mathbb{H}),$$

which will be used repeatedly in this article. For further properties of the numerical range and the numerical radius, we refer the interested readers to [3, 8].

Over the years many eminent mathematicians have studied and improved on the above inequality, to cite a few of them are [6, 7, 9, 11, 12, 15, 16]. Recently we [1, 4, 5, 13, 14] have developed some bounds for the numerical radius and applied them to estimate zeros of polynomials. In 1963, Bernau and Smithies [2] gave an elegant proof of the inequality $w(T) \geqslant \frac{1}{2}\|T\|$ using parallelogram law. In this paper we improve on this inequality to prove that

$$w(T) \geqslant \frac{1}{2}\|T\| + \frac{m(T^2)}{2\|T\|}.$$

We generalize the inequality [2, Lemma 3] substantially to obtain new inequalities for the numerical radius. Further we obtain bounds for the numerical radius of $n \times n$ operator matrices defined on the complex Hilbert space $\mathbb{H}_1 \oplus \mathbb{H}_2 \oplus \ldots \oplus \mathbb{H}_n$. Here $\mathbb{H}_1, \mathbb{H}_2, \ldots, \mathbb{H}_n$ are complex Hilbert spaces. We show that the bounds obtained here generalize and improve on the existing bounds given in [9, 11].

2. Inequalities for the numerical radius of product operators

We begin this section with the following inequality proved in [2, Lemma 3].

LEMMA 1. *Let* $T \in B(\mathbb{H})$. *Then*

$$\|Tx\|^2 + |\langle T^2x, x\rangle| \leqslant 2w(T)\|Tx\|\|x\|, \tag{1}$$

for all $x \in \mathbb{H}$.

We generalize the inequality (1) in the following lemma.

LEMMA 2. *Let* $A, T, B \in B(\mathbb{H})$. *Then*

$$|\langle A^*TBx, x\rangle| + |\langle B^*TAx, x\rangle| \leqslant 2w(T)\|Ax\|\|Bx\|, \tag{2}$$

for all $x \in \mathbb{H}$.

Proof. Let $x \in \mathbb{H}$ and θ, ϕ be real numbers such that $e^{i\phi}\langle B^*TAx, x\rangle = |\langle B^*TAx, x\rangle|$, $e^{2i\theta}\langle e^{-i\phi}A^*TBx, x\rangle = |\langle e^{-i\phi}A^*TBx, x\rangle| = |\langle A^*TBx, x\rangle|$. Then for any non-zero real number λ, we have

$$\begin{aligned}
&2e^{2i\theta}\langle TBx, e^{i\phi}Ax\rangle + 2e^{i\phi}\langle TAx, Bx\rangle \\
&\quad = \left\langle e^{i\theta}T\left(\lambda e^{i\theta}Bx + \frac{1}{\lambda}e^{i\phi}Ax\right), \lambda e^{i\theta}Bx + \frac{1}{\lambda}e^{i\phi}Ax\right\rangle \\
&\qquad - \left\langle e^{i\theta}T\left(\lambda e^{i\theta}Bx - \frac{1}{\lambda}e^{i\phi}Ax\right), \lambda e^{i\theta}Bx - \frac{1}{\lambda}e^{i\phi}Ax\right\rangle \\
\Rightarrow\ &2e^{2i\theta}\langle e^{-i\phi}A^*TBx, x\rangle + 2e^{i\phi}\langle B^*TAx, x\rangle \\
&\quad = \left\langle e^{i\theta}T\left(\lambda e^{i\theta}Bx + \frac{1}{\lambda}e^{i\phi}Ax\right), \lambda e^{i\theta}Bx + \frac{1}{\lambda}e^{i\phi}Ax\right\rangle \\
&\qquad - \left\langle e^{i\theta}T\left(\lambda e^{i\theta}Bx - \frac{1}{\lambda}e^{i\phi}Ax\right), \lambda e^{i\theta}Bx - \frac{1}{\lambda}e^{i\phi}Ax\right\rangle \\
\Rightarrow\ &2|\langle A^*TBx, x\rangle| + 2|\langle B^*TAx, x\rangle| \\
&\quad = \left\langle e^{i\theta}T\left(\lambda e^{i\theta}Bx + \frac{1}{\lambda}e^{i\phi}Ax\right), \lambda e^{i\theta}Bx + \frac{1}{\lambda}e^{i\phi}Ax\right\rangle \\
&\qquad - \left\langle e^{i\theta}T\left(\lambda e^{i\theta}Bx - \frac{1}{\lambda}e^{i\phi}Ax\right), \lambda e^{i\theta}Bx - \frac{1}{\lambda}e^{i\phi}Ax\right\rangle \\
\Rightarrow\ &2|\langle A^*TBx, x\rangle| + 2|\langle B^*TAx, x\rangle| \\
&\quad \leqslant \left|\left\langle e^{i\theta}T\left(\lambda e^{i\theta}Bx + \frac{1}{\lambda}e^{i\phi}Ax\right), \lambda e^{i\theta}Bx + \frac{1}{\lambda}e^{i\phi}Ax\right\rangle\right| \\
&\qquad + \left|\left\langle e^{i\theta}T\left(\lambda e^{i\theta}Bx - \frac{1}{\lambda}e^{i\phi}Ax\right), \lambda e^{i\theta}Bx - \frac{1}{\lambda}e^{i\phi}Ax\right\rangle\right|
\end{aligned}$$

$$
\begin{aligned}
&\Rightarrow\ 2|\langle A^*TBx,x\rangle|+2|\langle B^*TAx,x\rangle| \\
&\qquad \leqslant w(T)\left(\left\|\lambda e^{i\theta}Bx+\frac{1}{\lambda}e^{i\phi}Ax\right\|^2+\left\|\lambda e^{i\theta}Bx-\frac{1}{\lambda}e^{i\phi}Ax\right\|^2\right) \\
&\Rightarrow\ |\langle A^*TBx,x\rangle|+|\langle B^*TAx,x\rangle| \leqslant w(T)\left(\lambda^2\|Bx\|^2+\frac{1}{\lambda^2}\|Ax\|^2\right).
\end{aligned}
$$

This holds for all non-zero real λ. If $\|Bx\| \neq 0$, then we choose $\lambda^2 = \frac{\|Ax\|}{\|Bx\|}$. So, we get

$$|\langle A^*TBx,x\rangle|+|\langle B^*TAx,x\rangle| \leqslant 2w(T)\|Ax\|\|Bx\|.$$

Clearly, this inequality holds also when $\|Bx\| = 0$. This completes the proof of the lemma.

REMARK 1. If we take $A = T$ and $B = I$ in Lemma 2, then we get the inequality [2, Lemma 3].

Now using the inequality in Lemma 2, we obtain the following inequalities involving the numerical radius, the Crawford number and the usual operator norm of bounded linear operators.

THEOREM 1. *Let $A,T,B \in B(\mathbb{H})$. Then the following inequalities hold:*

$$
\begin{aligned}
m(A^*TB)+w(B^*TA) &\leqslant 2w(T)\|A\|\|B\|, \\
w(A^*TB)+m(B^*TA) &\leqslant 2w(T)\|A\|\|B\|.
\end{aligned}
$$

Proof. Taking $\|x\| = 1$ in the inequality (2), we get

$$
\begin{aligned}
&|\langle A^*TBx,x\rangle|+|\langle B^*TAx,x\rangle| \leqslant 2w(T)\|A\|\|B\| \\
&\Rightarrow m(A^*TB)+|\langle B^*TAx,x\rangle| \leqslant 2w(T)\|A\|\|B\|.
\end{aligned}
$$

Taking supremum over $\|x\| = 1$, we get

$$m(A^*TB)+w(B^*TA) \leqslant 2w(T)\|A\|\|B\|.$$

Again taking $\|x\| = 1$ in the inequality (2), we get

$$
\begin{aligned}
&|\langle A^*TBx,x\rangle|+|\langle B^*TAx,x\rangle| \leqslant 2w(T)\|A\|\|B\| \\
&\Rightarrow |\langle A^*TBx,x\rangle|+m(B^*TA)+ \leqslant 2w(T)\|A\|\|B\|.
\end{aligned}
$$

Taking supremum over $\|x\| = 1$, we get

$$w(A^*TB)+m(B^*TA) \leqslant 2w(T)\|A\|\|B\|.$$

This completes the proof of the theorem.

Taking $B = I, T = A$ and $A = B$ in the inequalities in Theorem 1, we get the following upper bounds for the numerical radius of product of two operators, which improve on the existing bounds.

COROLLARY 1. *Let $A,B \in B(\mathbb{H})$. Then the following inequalities hold:*

$$w(AB) \leqslant 2w(A)\|B\| - m(B^*A),$$
$$w(AB) \leqslant 2w(B)\|A\| - m(BA^*).$$

REMARK 2. It is clear that both the inequalities obtained in Corollary 1 improve on the existing inequalities, namely, $w(AB) \leqslant 2w(A)\|B\| \leqslant 4w(A)w(B)$ and $w(AB) \leqslant 2w(B)\|A\| \leqslant 4w(A)w(B)$, respectively, (see [8, Th. 2.5-2]).

Next using Lemma 1, we establish some new inequalities for the numerical radius of 2×2 operator matrices with the zero operator as main diagonal entries.

THEOREM 2. *Let $A,B \in B(\mathbb{H})$. Then the following inequalities hold:*

$$(i)\ \ \|A\|^2 + m(BA) \leqslant 2w\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\|A\|,$$

$$(ii)\ \ c^2(A) + w(BA) \leqslant 2w\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\|A\|,$$

$$(iii)\ \ \|B\|^2 + m(AB) \leqslant 2w\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\|B\|,$$

$$(iv)\ \ c^2(B) + w(AB) \leqslant 2w\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\|B\|.$$

Proof. Putting $T = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in B(\mathbb{H} \oplus \mathbb{H})$ and $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{H} \oplus \mathbb{H}$ with $\|x\| = 1$, i.e., $\|x_1\|^2 + \|x_2\|^2 = 1$ in the inequality (1), we get

$$\|Ax_2\|^2 + \|Bx_1\|^2 + |\langle ABx_1, x_1\rangle + \langle BAx_2, x_2\rangle| \leqslant 2w(T)\left(\|Ax_2\|^2 + \|Bx_1\|^2\right)^{\frac{1}{2}}. \quad (3)$$

Taking $x_1 = 0$ in (3), we get

$$\|Ax_2\|^2 + |\langle BAx_2, x_2\rangle| \leqslant 2w\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\|Ax_2\|$$
$$\Rightarrow \|Ax_2\|^2 + |\langle BAx_2, x_2\rangle| \leqslant 2w\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\|A\|$$
$$\Rightarrow \|Ax_2\|^2 + m(BA) \leqslant 2w\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\|A\|$$

Taking supremum over $\|x_2\| = 1$, we get the inequality (i), i.e.,

$$\|A\|^2 + m(BA) \leqslant 2w\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\|A\|.$$

Again from the inequality

$$\|Ax_2\|^2 + |\langle BAx_2, x_2\rangle| \leqslant 2w\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\|A\|,$$

we get

$$c^2(A)+|\langle BAx_2,x_2\rangle| \leqslant 2w\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\|A\|.$$

Taking supremum over $\|x_2\|=1$, we get the inequality (ii), i.e.,

$$c^2(A)+w(BA) \leqslant 2w\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\|A\|.$$

Similarly, considering $x_2=0$ in (3), we can prove the remaining inequalities of the theorem.

Considering $A=B=T$ in Theorem 2 and using the fact that $w\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}=w(A)$, we get the following lower bounds for the numerical radius of non-zero bounded linear operators.

THEOREM 3. *Let $T\in B(\mathbb{H})$ be non-zero. Then the following inequalities hold:*

$$w(T) \geqslant \frac{\|T\|}{2}+\frac{m(T^2)}{2\|T\|}, \tag{4}$$

$$w(T) \geqslant \frac{c^2(T)}{2\|T\|}+\frac{w(T^2)}{2\|T\|}. \tag{5}$$

REMARK 3. The inequality (4) improves on the existing inequality $w(T)\geqslant\frac{\|T\|}{2}$ substantially. Also from the inequality (4), it follows that if $w(T)=\frac{\|T\|}{2}$ then $m(T^2)=0$. There are operators for which $m(T^2)=0$ but $w(T)\neq\frac{\|T\|}{2}$.

Next, we prove a necessary and sufficient condition for $w(T)=\frac{\|T\|}{2}$, where T is an $n\times n$ complex matrix.

THEOREM 4. *Let T be an $n\times n$ complex matrix. Then $w(T)=\frac{\|T\|}{2}$ if and only if T is unitarily similar to a matrix of the form $\begin{pmatrix} 0 & \|T\| \\ 0 & 0 \end{pmatrix}\oplus\|T\|B$, where B is a matrix of order $n-2$ and $w(B)\leqslant\frac{1}{2}$.*

Proof. The necessary part follows from [8, Th. 1.3-5] and the sufficient part is obvious.

REMARK 4. The inequalities (4) and (5) obtained by us in Theorem 3 are incomparable. Consider $T=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then it is easy to see that , (4) gives $w(T)\geqslant\frac{1}{2}$ and (5) gives $w(T)\geqslant 0$, whereas if we consider $T=\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$, then (4) gives $w(T)\geqslant\frac{1}{2}$ and (5) gives $w(T)\geqslant 1$.

Using Theorem 3 and noting the Remark 4, we obtain the following lower bound for the numerical radius of non-zero bounded linear operators.

COROLLARY 2. *Let $T \in B(\mathbb{H})$ be non-zero. Then*

$$w(T) \geqslant \frac{1}{2\|T\|} \max\left\{\|T\|^2 + m(T^2), c^2(T) + w(T^2)\right\}.$$

In the next theorem we prove another inequality for the numerical radius of sum of product operators.

THEOREM 5. *Let $A, T, B \in B(\mathbb{H})$. Then*

$$w(A^*TB \pm B^*TA) \leqslant 2w(T)\|A\|\|B\|.$$

Proof. Using Lemma 2, we get

$$|\langle (A^*TB \pm B^*TA)x, x\rangle| \leqslant |\langle A^*TBx, x\rangle| + |\langle B^*TAx, x\rangle| \leqslant 2w(T)\|Ax\|\|Bx\|,$$

for all $x \in \mathbb{H}$. Therefore,

$$|\langle (A^*TB \pm B^*TA)x, x\rangle| \leqslant 2w(T)\|A\|\|B\|.$$

Taking supremum over $\|x\| = 1$, we get the required inequality.

REMARK 5. The inequality in Theorem 5 was already proved by Hirzallah et al. in [9] using different technique. If we consider $B = I$ in Theorem 5, then we get the well-known inequality, namely, $w(A^*T \pm TA) \leqslant 2w(T)\|A\|$, i.e., $w(AT \pm TA^*) \leqslant 2w(T)\|A\|$.

Our final result in this section is to compute an upper bound for the numerical radius of a bounded linear operator T in terms of $\|\mathrm{Re}(T)\|, \|\mathrm{Im}(T)\|, m(\mathrm{Re}(T))$ and $m(\mathrm{Im}(T))$.

THEOREM 6. *Let $T \in B(\mathbb{H})$. Then*

$$\begin{aligned} w^4(T) \leqslant \max&\left\{\left|\|\mathrm{Re}(T)\|^2 - m^2(\mathrm{Im}(T))\right|^2, \left|\|\mathrm{Im}(T)\|^2 - m^2(\mathrm{Re}(T))\right|^2\right\} \\ &+4\|\mathrm{Re}(T)\|^2\|\mathrm{Im}(T)\|^2. \end{aligned}$$

Proof. Let $x \in \mathbb{H}$ with $\|x\| = 1$. Then from the Cartesian decomposition of T, we have

$$\begin{aligned} \langle Tx, x\rangle &= \langle \mathrm{Re}(T)x, x\rangle + i\langle \mathrm{Im}(T)x, x\rangle \\ \Rightarrow \langle Tx, x\rangle^2 &= \langle \mathrm{Re}(T)x, x\rangle^2 - \langle \mathrm{Im}(T)x, x\rangle^2 + 2i\langle \mathrm{Re}(T)x, x\rangle\langle \mathrm{Im}(T)x, x\rangle \\ \Rightarrow \left|\langle Tx, x\rangle^2\right|^2 &= \left|\langle \mathrm{Re}(T)x, x\rangle^2 - \langle \mathrm{Im}(T)x, x\rangle^2\right|^2 + 4\langle \mathrm{Re}(T)x, x\rangle^2\langle \mathrm{Im}(T)x, x\rangle^2 \\ \Rightarrow \left|\langle Tx, x\rangle\right|^4 &\leqslant \max\left\{\left|\|\mathrm{Re}(T)\|^2 - m^2(\mathrm{Im}(T))\right|^2, \left|\|\mathrm{Im}(T)\|^2 - m^2(\mathrm{Re}(T))\right|^2\right\} \\ &\quad +4\|\mathrm{Re}(T)\|^2\|\mathrm{Im}(T)\|^2. \end{aligned}$$

Taking supremum over $x, \|x\| = 1$, we get the desired inequality.

3. Upper bounds for the numerical radius of operator matrices

In this section we obtain bounds for the numerical radius of $n\times n$ operator matrices. We begin with the estimation of an upper bound for the $n\times n$ operator matrix for which entires of all rows are zero operators except first row. For this we need the following inequality [4, Remark 2.8], which gives an upper bound for the numerical radius of a bounded linear operator T in terms of $\|\mathrm{Re}(T)\|$ and $\|\mathrm{Im}(T)\|$, where $\mathrm{Re}(T)=\frac{1}{2}(T+T^*)$ and $\mathrm{Im}(T)=\frac{1}{2i}(T-T^*)$.

LEMMA 3. *Let $T\in B(\mathbb{H})$. Then*

$$w^2(T)\leqslant \|\mathrm{Re}(T)\|^2+\|\mathrm{Im}(T)\|^2.$$

Also we need the following lemma, proof of which can be found in [10, Th. 1.1].

LEMMA 4. *Let $A_{ij}\in B(\mathbb{H}_j,\mathbb{H}_i)$ and $A=\big(A_{ij}\big)$ be an $n\times n$ operator matrix. Then*

$$\|A\|\leqslant \big(\|A_{ij}\|\big).$$

THEOREM 7. *Let $A_{11}\in B(\mathbb{H}_1,\mathbb{H}_1), A_{12}\in B(\mathbb{H}_2,\mathbb{H}_1),\ldots,A_{1n}\in B(\mathbb{H}_n,\mathbb{H}_1)$. Then*

$$w\begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1n}\\ 0 & 0 & \ldots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \ldots & 0\end{pmatrix}\leqslant \frac{1}{2}\sqrt{\alpha^2+\beta^2},$$

where

$$\alpha = \|\mathrm{Re}(A_{11})\|+\sqrt{\|\mathrm{Re}(A_{11})\|^2+\sum_{j=2}^{n}\|A_{1j}\|^2},$$

$$\beta = \|\mathrm{Im}(A_{11})\|+\sqrt{\|\mathrm{Im}(A_{11})\|^2+\sum_{j=2}^{n}\|A_{1j}\|^2}.$$

Proof. Let

$$T=\begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1n}\\ 0 & 0 & \ldots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \ldots & 0\end{pmatrix}.$$

Then

$$\mathrm{Re}(T)=\begin{pmatrix} \mathrm{Re}(A_{11}) & \frac{A_{12}}{2} & \ldots & \frac{A_{1n}}{2}\\ \frac{A_{12}^*}{2} & 0 & \ldots & 0\\ \vdots & \vdots & & \vdots\\ \frac{A_{1n}^*}{2} & 0 & \ldots & 0\end{pmatrix} \text{ and } \mathrm{Im}(T)=\begin{pmatrix} \mathrm{Im}(A_{11}) & \frac{A_{12}}{2i} & \ldots & \frac{A_{1n}}{2i}\\ -\frac{A_{12}^*}{2i} & 0 & \ldots & 0\\ \vdots & \vdots & & \vdots\\ -\frac{A_{1n}^*}{2i} & 0 & \ldots & 0\end{pmatrix}.$$

Consider

$$S=\begin{pmatrix} \|\mathrm{Re}(A_{11})\| & \frac{\|A_{12}\|}{2} & \dots & \frac{\|A_{1n}\|}{2} \\ \frac{\|A_{12}\|}{2} & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \frac{\|A_{1n}\|}{2} & 0 & \dots & 0 \end{pmatrix}.$$

Then S is an $n\times n$ Hermitian matrix with non-negative real entries. Note that $r(S)=\max\{|\lambda| : \lambda \text{ is an eigenvalue of } S\}$ and so finding the eigenvalues of S we get

$$r(S)=\frac{1}{2}\left(\|\mathrm{Re}(A_{11})\|+\sqrt{\|\mathrm{Re}(A_{11})\|^2+\sum_{j=2}^{n}\|A_{1j}\|^2}\right).$$

Now S being a Hermitian matrix, we have $\|S\|=r(S)$ and using Lemma 4 we get $\|\mathrm{Re}(T)\|\leqslant\|S\|$. Thus we have,

$$\|\mathrm{Re}(T)\|\leqslant\frac{1}{2}\left(\|\mathrm{Re}(A_{11})\|+\sqrt{\|\mathrm{Re}(A_{11})\|^2+\sum_{j=2}^{n}\|A_{1j}\|^2}\right). \tag{6}$$

Proceeding in the same way we can show that,

$$\|\mathrm{Im}(T)\|\leqslant\frac{1}{2}\left(\|\mathrm{Im}(A_{11})\|+\sqrt{\|\mathrm{Im}(A_{11})\|^2+\sum_{j=2}^{n}\|A_{1j}\|^2}\right). \tag{7}$$

Now using the inequalities (6) and (7) in Lemma 3, we get the desired inequality.

In the next theorem we compute an upper bound for the numerical radius of $n\times n$ operator matrices.

THEOREM 8. *Let $T=(A_{ij})$ be an $n\times n$ operator matrix with $A_{ij}\in B(\mathbb{H})$. Then*

$$w(T)\leqslant\frac{1}{2}\sum_{k=1}^{n}\sqrt{\alpha_k^2+\beta_k^2},$$

where

$$\alpha_k=\|\mathrm{Re}(A_{kk})\|+\sqrt{\|\mathrm{Re}(A_{kk})\|^2+\sum_{j=1,j\neq k}^{n}\|A_{kj}\|^2},$$

$$\beta_k=\|\mathrm{Im}(A_{kk})\|+\sqrt{\|\mathrm{Im}(A_{kk})\|^2+\sum_{j=1,j\neq k}^{n}\|A_{kj}\|^2}.$$

Proof. Let $T=T_1+T_2+\ldots+T_n$, where

$$T_1=\begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix},\ T_2=\begin{pmatrix} 0 & 0 & \dots & 0 \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \end{pmatrix},\dots,T_n=\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}.$$

So by the triangle inequality, we get

$$w(T) \leqslant w(T_1) + w(T_2) + \ldots + w(T_n). \tag{8}$$

For each $i = 2, 3, \ldots, n$, let U_i be the $n \times n$ permutation operator matrix obtained by interchanging the 1st and the ith rows of the identity operator matrix. Then U_i is a unitary operator and so using the weak unitarily invariance property of the numerical radius it follows from (8) that

$$\begin{aligned} w(T) &\leqslant w(T_1) + w(U_2^* T_2 U_2) + w(U_3^* T_3 U_3) + \ldots + w(U_n^* T_n U_n) \\ &= w\begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1n} \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix} + w\begin{pmatrix} A_{22} & A_{21} & \ldots & A_{2n} \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix} \\ &\quad + \ldots + w\begin{pmatrix} A_{nn} & A_{n2} & \ldots & A_{n1} \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}. \end{aligned}$$

Now using Theorem 7, we get the desired inequality and this completes the proof of the theorem.

Next we obtain new upper bounds for the numerical radius of 2×2 operator matrices. For this we need the following lemma.

LEMMA 5. *Let $T \in B(H)$, then*

$$w(T) = \sup_{\theta \in \mathbb{R}} \left\| Re(e^{i\theta} T) \right\|.$$

By replacing T by iT in the above equality, also we have

$$w(T) = \sup_{\theta \in \mathbb{R}} \left\| Im(e^{i\theta} T) \right\|.$$

THEOREM 9. *Let $T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$, where $A \in B(\mathbb{H}_1), B \in B(\mathbb{H}_2, \mathbb{H}_1)$. Then*

$$w(T) \leqslant \sqrt{w^2(A) + \frac{1}{2}\|B\| \left(w(A) + \frac{1}{2}\|B\| \right)}.$$

Proof. By elementary calculations we have, for every $\theta \in \mathbb{R}$

$$\begin{aligned} \mathrm{Re}(e^{i\theta} T) &= \begin{pmatrix} \mathrm{Re}(e^{i\theta} A) & \frac{1}{2} e^{i\theta} B \\ \frac{1}{2} e^{-i\theta} B^* & 0 \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{Re}(e^{i\theta} A) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{2} e^{i\theta} B \\ \frac{1}{2} e^{-i\theta} B^* & 0 \end{pmatrix}. \end{aligned}$$

This implies that

$$\left(\mathrm{Re}(e^{i\theta}T)\right)^2 = \begin{pmatrix} (\mathrm{Re}(e^{i\theta}A))^2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{4}BB^* & 0 \\ 0 & \frac{1}{4}B^*B \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{2}e^{i\theta}\mathrm{Re}(e^{i\theta}A)B \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \frac{1}{2}e^{-i\theta}B^*\mathrm{Re}(e^{i\theta}A) & 0 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} \|\mathrm{Re}(e^{i\theta}T)\|^2 &\leqslant \|\mathrm{Re}(e^{i\theta}A)\|^2 + \frac{1}{4}\|B\|^2 + \frac{1}{2}\|\mathrm{Re}(e^{i\theta}A)\|\|B\| \\ &\leqslant w^2(A) + \frac{1}{4}\|B\|^2 + \frac{1}{2}w(A)\|B\|. \end{aligned}$$

Taking supremum over θ, we get

$$w^2(T) \leqslant w^2(A) + \frac{1}{4}\|B\|^2 + \frac{1}{2}w(A)\|B\|.$$

This completes the proof of the theorem.

Now using Theorem 9, we give an upper bound for the numerical radius of 2×2 operator matrices.

COROLLARY 3. *Let* $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, *where* $A,B,C,D \in B(\mathbb{H})$. *Then*

$$w(T) \leqslant \sqrt{w^2(A) + \frac{1}{2}\|B\|\left(w(A) + \frac{1}{2}\|B\|\right)} + \sqrt{w^2(D) + \frac{1}{2}\|C\|\left(w(D) + \frac{1}{2}\|C\|\right)}.$$

Proof. Let us consider an operator matrix $U = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. Then U is an unitary operator, so using weak unitarily invariance property of the numerical radius we get,

$$\begin{aligned} w(T) &\leqslant w\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} + w\begin{pmatrix} 0 & 0 \\ C & D \end{pmatrix} \\ &= w\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} + w\left(U^*\begin{pmatrix} 0 & 0 \\ C & D \end{pmatrix}U\right) \\ &= w\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} + w\begin{pmatrix} D & C \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Therefore, using Theorem 9, we get the required inequality of the theorem.

In the following theorem we provide a new upper bound for 2×2 operator matrices, in which the entries in second row are all the zero operator.

THEOREM 10. *Let* $T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$, *where* $A \in B(\mathbb{H}_1), B \in B(\mathbb{H}_2, \mathbb{H}_1)$. *Then*

$$w(T) \leqslant \sqrt{2w^2(A) + \frac{1}{2}(\|A^*B\| + \|B\|^2)}.$$

Proof. For $\theta \in \mathbb{R}$, it is easy to see that $\mathrm{Re}(e^{i\theta}T) = \begin{pmatrix} \mathrm{Re}(e^{i\theta}A) & \frac{1}{2}e^{i\theta}B \\ \frac{1}{2}e^{-i\theta}B^* & 0 \end{pmatrix}$ and $\mathrm{Im}(e^{i\theta}T) = -i\begin{pmatrix} i\mathrm{Im}(e^{i\theta}A) & \frac{1}{2}e^{i\theta}B \\ -\frac{1}{2}e^{-i\theta}B^* & 0 \end{pmatrix}$.

Therefore, by elementary calculations, we get

$$\mathrm{Re}^2(e^{i\theta}T)+\mathrm{Im}^2(e^{i\theta}T) = \begin{pmatrix} \mathrm{Re}^2(e^{i\theta}T)+\mathrm{Im}^2(e^{i\theta}T) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{A^*B}{2} \\ \frac{B^*A}{2} & 0 \end{pmatrix} + \begin{pmatrix} \frac{BB^*}{2} & 0 \\ 0 & \frac{B^*B}{2} \end{pmatrix}.$$

Since $\mathrm{Im}^2(e^{i\theta}T) \geqslant 0$, so we get,

$$\mathrm{Re}^2(e^{i\theta}T) \leqslant \begin{pmatrix} \mathrm{Re}^2(e^{i\theta}T)+\mathrm{Im}^2(e^{i\theta}T) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{A^*B}{2} \\ \frac{B^*A}{2} & 0 \end{pmatrix} + \begin{pmatrix} \frac{BB^*}{2} & 0 \\ 0 & \frac{B^*B}{2} \end{pmatrix}.$$

Taking norm on both sides, we get

$$\begin{aligned} \|\mathrm{Re}(e^{i\theta}T)\|^2 &\leqslant \|\mathrm{Re}^2(e^{i\theta}A)+\mathrm{Im}^2(e^{i\theta}A)\| + \frac{1}{2}\|A^*B\| + \frac{1}{2}\|B\|^2 \\ &\leqslant 2w^2(A) + \frac{1}{2}\left(\|A^*B\| + \|B\|^2\right). \end{aligned}$$

Taking supremum over $\theta \in \mathbb{R}$, we get

$$w^2(T) \leqslant 2w^2(A) + \frac{1}{2}\left(\|A^*B\| + \|B\|^2\right).$$

This completes the proof.

Now, using Theorem 10 and using the same technique as in the proof of Corollary 3, we can obtain the following bound for numerical radius of 2×2 operator matrices.

COROLLARY 4. *Let* $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, *where* $A,B,C,D \in B(\mathbb{H})$. *Then*

$$w(T) \leqslant \sqrt{2w^2(A) + \frac{1}{2}(\|A^*B\| + \|B\|^2)} + \sqrt{2w^2(D) + \frac{1}{2}(\|D^*C\| + \|C\|^2)}.$$

REMARK 6. Considering the operator $T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$, where $A = \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$, it is easy to see that Theorem 10 gives $w(T) \leqslant \sqrt{8+\sqrt{10}}$ whereas the bound obtained by Shebrawi in [15, Th. 3.2] gives $w(T) \leqslant \frac{1}{4}(12+\sqrt{10})$. This indicates that for this operator the bound obtained by us is better than that obtained by Shebrawi.

4. Lower bounds for the numerical radius of operator matrices

In this section we first obtain a new lower bound for the numerical radius of a special class of $n \times n$ operator matrices.

THEOREM 11. *Let*

$$T = \begin{pmatrix} 0 & 0 & \dots & 0 & A_1 \\ 0 & 0 & \dots & A_2 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ A_n & 0 & \dots & 0 & 0 \end{pmatrix},$$

where $A_i \in B(\mathbb{H})$ *for each* $i = 1, 2, \dots, n$*. Then*

$$w(T) \geqslant \frac{1}{\sqrt{2}} \max_{1 \leqslant i \leqslant n} \left\{ \sqrt{w(A_iA_{n-i+1} + A_{n-i+1}A_i)}, \sqrt{w(A_iA_{n-i+1} - A_{n-i+1}A_i)} \right\}.$$

Proof. Consider the unitary operator $U = \begin{pmatrix} 0 & 0 & \dots & 0 & I \\ 0 & 0 & \dots & I & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ I & 0 & \dots & 0 & 0 \end{pmatrix}$.

Then it is easy to see that,

$$T^2 + (U^*TU)^2 = \begin{pmatrix} A_1A_n + A_nA_1 & 0 & \dots 0 & 0 \\ 0 & A_2A_{n-1} + A_{n-1}A_2 & \dots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots 0 & A_nA_1 + A_1A_n \end{pmatrix} = D_1 \quad \text{(say)}.$$

Therefore,

$$\begin{aligned} w(D_1) &= w(T^2 + (U^*TU)^2) \\ &\leqslant w(T^2) + w\big((U^*TU)^2)\big) \\ &\leqslant w^2(T) + w^2(U^*TU) \\ &= 2w^2(T) \end{aligned}$$

This shows that

$$\max\{w(A_iA_{n-i+1} + A_{n-i+1}A_i) : 1 \leqslant i \leqslant n\} \leqslant 2w^2(T).$$

Now, we calculate $T^2 - (U^*TU)^2$ and then using the same arguments as above we can prove that

$$\max\{w(A_iA_{n-i+1} - A_{n-i+1}A_i) : 1 \leqslant i \leqslant n\} \leqslant 2w^2(T).$$

Therefore we conclude that

$$w(T) \geqslant \frac{1}{\sqrt{2}} \max_{1 \leqslant i \leqslant n} \left\{ \sqrt{w(A_iA_{n-i+1} + A_{n-i+1}A_i)}, \sqrt{w(A_iA_{n-i+1} - A_{n-i+1}A_i)} \right\}.$$

Now using Theorem 11 and the pinching inequalities (see [3, p. 107]),

$$w\begin{pmatrix} A & B \\ C & D \end{pmatrix} \geqslant w\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \text{ and } w\begin{pmatrix} A & B \\ C & D \end{pmatrix} \geqslant w\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix},$$

where $A,B,C,D \in B(\mathbb{H})$, we obtain the following lower bound for the numerical radius of arbitrary 2×2 operator matrices.

COROLLARY 5. *Let* $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, *where* $A,B,C,D \in B(\mathbb{H})$. *Then*

$$w(T) \geqslant \max\left\{ w(A), w(D), \sqrt{\frac{1}{2}w(BC+CB)}, \sqrt{\frac{1}{2}w(BC-CB)} \right\}.$$

REMARK 7. The inequality obtained in Corollary 5 and the first inequality in [9, Th. 3.7] obtained by Hirzallah et al. are incomparable. Consider $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A = D = (0), B = (1), C = (2)$. Then Corollary 5 gives $w(T) \geqslant \sqrt{2}$ and [9, Th. 3.7] gives $w(T) \geqslant \frac{3}{2}$. Again, if we consider $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A = D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix}$, then Corollary 5 gives $w(T) \geqslant \sqrt{3}$ and [9, Th. 3.7] gives $w(T) \geqslant \frac{3}{2}$.

We next prove an inequality which gives a lower bound for the numerical radius of 2×2 operator matrices of the form $\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$, where $A,B \in B(\mathbb{H})$. To do so we need the following lemma, proof of which follows from the weak unitarily invariance property of the numerical radius.

LEMMA 6. *Let* $T = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$, *where* $A,B \in B(\mathbb{H})$. *Then*

$$w(T) = \max\{w(A+B), w(A-B)\}.$$

Now we prove the theorem.

THEOREM 12. *Let* $A,B \in B(\mathbb{H})$. *Then*

$$w\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \geqslant \frac{1}{2}\max\{w(A+B), w(A-B)\}.$$

Proof. Let $T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$. We consider an unitary operator matrix $U = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. Then we get,

$$\begin{aligned} \begin{pmatrix} A & B \\ B & A \end{pmatrix} &= T + U^*TU \\ \Rightarrow w\begin{pmatrix} A & B \\ B & A \end{pmatrix} &\leqslant w(T) + w(U^*TU) \\ &= 2w(T) \\ \Rightarrow \max\{w(A+B), w(A-B)\} &\leqslant 2w(T), \quad \text{using Lemma } 6. \end{aligned}$$

This completes the proof.

We end this section with the following theorem, in which we obtain an inequality for the lower bound of numerical radius of 2×2 operator matrix, which generalizes the inequality $w(T) \geqslant \|Re(T)\|$ and $w(T) \geqslant \|Im(T)\|$, obtained by Kittaneh et al. [11].

THEOREM 13. *Let* $T = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$, *where* $A, B \in B(\mathbb{H})$. *Then*

$$w(T) \geqslant \frac{1}{2} \sup_{\theta\in\mathbb{R}} \left\| \mathrm{Re}(e^{i\theta}A) \pm \mathrm{Re}(e^{i\theta}B) \right\|,$$

$$w(T) \geqslant \frac{1}{2} \sup_{\theta\in\mathbb{R}} \left\| \mathrm{Im}(e^{i\theta}A) \pm \mathrm{Im}(e^{i\theta}B) \right\|.$$

Proof. Let $\theta \in \mathbb{R}$ and let $H_\theta = \mathrm{Re}(e^{i\theta}T)$. Let $U = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ be an unitary operator. Then we get,

$$H_\theta + U^*H_\theta U = \begin{pmatrix} 0 & \mathrm{Re}(e^{i\theta}A) + \mathrm{Re}(e^{i\theta}B) \\ \mathrm{Re}(e^{i\theta}A) + \mathrm{Re}(e^{i\theta}B) & 0 \end{pmatrix}.$$

Taking norm on both sides we get,

$$\begin{aligned} \|\mathrm{Re}(e^{i\theta}A) + \mathrm{Re}(e^{i\theta}B)\| &= \|H_\theta + U^*H_\theta U\| \\ &\leqslant \|H_\theta\| + \|U^*H_\theta U\| \\ &= 2\|H_\theta\| \\ &\leqslant 2w(T). \end{aligned}$$

Since this holds for all $\theta \in \mathbb{R}$, so we have

$$w(T) \geqslant \frac{1}{2} \sup_{\theta\in\mathbb{R}} \left\| \mathrm{Re}(e^{i\theta}A) + \mathrm{Re}(e^{i\theta}B) \right\|.$$

Next we consider $K_\theta = \mathrm{Im}(e^{i\theta}T)$. Then we get,

$$K_\theta + U^*K_\theta U = \begin{pmatrix} 0 & \mathrm{Im}(e^{i\theta}A) + \mathrm{Im}(e^{i\theta}B) \\ \mathrm{Im}(e^{i\theta}A) + \mathrm{Im}(e^{i\theta}B) & 0 \end{pmatrix}.$$

Taking norm on both sides we get,

$$\begin{aligned}\|\mathrm{Im}(e^{i\theta}A)+\mathrm{Im}(e^{i\theta}B)\| &= \|K_\theta+U^*K_\theta U\|\\ &\leqslant \|K_\theta\|+\|U^*K_\theta U\|\\ &= 2\|K_\theta\|\\ &\leqslant 2w(T).\end{aligned}$$

Since this holds for all $\theta\in\mathbb{R}$, so we have

$$w(T)\geqslant \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|\mathrm{Im}(e^{i\theta}A)+\mathrm{Im}(e^{i\theta}B)\right\|.$$

Considering $H_\theta-U^*H_\theta U$ and $K_\theta-U^*K_\theta U$ and using similar arguments as above we can prove the remaining inequalities.

REMARK 8. If we take $A=B$ and $\theta=0$ in Theorem 13, then we get, $w(A)\geqslant \|\mathrm{Re}(A)\|$ and $w(A)\geqslant\|\mathrm{Im}(A)\|$.

REMARK 9. There was a minor error in the calculation of bound in Remark 2.4 of [4], the estimation of bound obtained there should be 1.86317171 instead of 1.784. This was pointed out by the reviewer while reviewing the paper for Mathematical Reviews (MR3933295), we thank him/her for that.

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