Mathematical
Inequalities
& Applications
Volume 24, Number 1 (2021), 185–200 doi:10.7153/mia-2021-24-13

THE ORTHOGONAL PROJECTIONS AND SEVERAL INEQUALITIES

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(*Communicated by S. Varošanec*)

Abstract. In this article we study several inequalities related to the orthogonal projections and we established new results related to a pre-Hilbert space. Among these results we will mention the inequality of Ostrowski. We present an improvement of the inequality between the numerical radius of an operator and the norm of an operator and we also show other inequalities for a bounded linear operator. Finally, we show Grüss type inequalities on double ice-cream cones.

1. Introduction

Let X be a linear space endowed with an inner product over the field $\mathbb{K}$ also called pre-Hilbert space, where $\mathbb{K}$ is the set of the real or the complex numbers. For every subspace $U \subset X$, we have the decomposition $X = U \bigoplus U^{\perp}$. Every $x \in X$ can be uniquely written as $x = x_1 + u$, where $x_1 \in U$ and $u \in U^{\perp}$. We define the orthogonal projection $P_U : X \to X$ by $P_U(x) = x_1$, which implies that $P_U^2 = P_U$.

If X is an inner product space over the field $\mathbb{K}$ and $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of U, then the linear operator P_U is given by $x = P_U(x) + u$, where $P_U(x) = \langle x, e_1\rangle e_1 + \langle x, e_2\rangle e_2 + \ldots + \langle x, e_n\rangle e_n$.

It is easy to see that $\langle u, x_1\rangle = 0$, so we have $\langle P_U(x), u\rangle = 0$, which involves the equality $\langle x, u\rangle = \langle u, u\rangle = \|u\|^2$, where the norm $\|\cdot\|$ is generated by the inner product $\langle \cdot, \cdot\rangle$. Sometimes, instead of $P_U(x)$ we use $P_U x$.

The Cauchy-Schwarz inequality in the real case, $|\langle x, y\rangle| \leqslant \|x\| \cdot \|y\|$ (see e.g. [17]), can be obtained by the following identity, as in [18],

$$\langle x, y\rangle = \|x\|\|y\|\left(1 - \frac{1}{2}\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|^2\right), \tag{1}$$

for all $x, y \in X$, $x, y \neq 0$. Other inequalities in a pre-Hilbert space can be found in [3], [6] and [14].

The quantity $d(x, y) = \left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|$ is called the *angular distance* of x and y [2]. In general, $0 \leqslant d(x, y) \leqslant 2$, for any $x, y \in X$. It is easily seen that

$$d^2(x, y) = 2 - 2\left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|}\right\rangle \leqslant 2, \tag{2}$$

Mathematics subject classification (2010): 46C05, 26D10, 26D15.
Keywords and phrases: Pre-Hilbert space, Cauchy-Schwarz inequality, Ostrowski inequality, Grüss type inequality.

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whenever $\langle x,y\rangle \geqslant 0$ and

$$\left|1-\frac{1}{2}d^2(x,y)\right| \leqslant 1, \tag{3}$$

for every $x,y\in X$, $x,y\neq 0$.

Let $(X,\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. $\mathbb{B}(X)$ is the set of all bounded linear operators on the Hilbert space X. For $T\in\mathbb{B}(X)$, we have the operator norm of T which is defined by $\|T\| = \sup_{\|x\|=1}\|Tx\|$ and the numerical radius of T is defined by $\omega(T) = \sup_{\|x\|=1}|\langle Tx,x\rangle|$ (see [25], [12]).

Gustafson and Rao, in [12], showed that for any $T\in\mathbb{B}(X)$ one has

$$\omega(T)\leqslant\|T\|\leqslant 2\omega(T). \tag{4}$$

The purpose of this paper is to study several inequalities related to the orthogonal projections, which involves the problem of minimization. Among these results we established an inequality which characterizes Bessel's inequality and we will mention Ostrowski's inequality as a consequence of our results. We present an improvement of the inequality between the numerical radius of an operator and the norm of an operator and we also show other inequalities for a bounded linear operator. Finally, we derive and interpret Grüss type inequalities on double ice-cream cones in various inner product spaces.

2. Some inequalities related to the orthogonal projections

THEOREM 1. *Let $U\subset X$ be a subspace of an inner product space X over the field $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$. If $x\in X$ and $x\notin U$, then we have*

$$\frac{\|x-P_U(x)\|}{\|x-y\|} = 1-\frac{1}{2}\left\|\frac{x-y}{\|x-y\|}-\frac{x-P_U(x)}{\|x-P_U(x)\|}\right\|^2 = 1-\frac{1}{2}d^2(x-y,x-P_Ux), \tag{5}$$

for every $y\in U$.

Proof. We know that $x=P_U(x)+u$ and $\langle x,u\rangle=\langle u,u\rangle$, which implies the following relations: $\|x-P_U(x)\|^2=\|u\|^2=\langle x,u\rangle=\langle x-y,u\rangle$, for all $y\in U$. For all $x\in X$, $x\notin U$ and for all $y\in U$, we have $x-y\neq 0, x-P_U(x)\neq 0$ and using the identity (1), we obtain

$$\|u\|^2=\langle x-y,u\rangle=\|x-y\|\|u\|\left(1-\frac{1}{2}\left\|\frac{x-y}{\|x-y\|}-\frac{u}{\|u\|}\right\|^2\right),$$

which means that we obtain

$$\|u\|=\|x-y\|\left(1-\frac{1}{2}\left\|\frac{x-y}{\|x-y\|}-\frac{u}{\|u\|}\right\|^2\right).$$

Therefore, we proved the statement. $\square$

REMARK 1. (The Best Approximation Theorem.) Let $U\subset X$ be a subspace of an inner product space X and $x\in X$. Then we have

$$\|x-P_U(x)\|\leqslant\|x-y\|, \tag{6}$$

for every $y \in U$. This can be easily proved, thus: if $x \in U$, then $P_U(x) = x$ and relation (6) is true. For all $x \in X$, $x \notin U$, we apply the relation from Theorem 1 and we deduce inequality (6).

THEOREM 2. *Let $U \subset X$ be a subspace of an inner product space X and $x, y \in X$. Then we have*

$$|\langle x,y\rangle - \langle P_U(x), P_U(y)\rangle|^2 \leqslant \left(\|x\|^2 - \|P_U(x)\|^2\right)\left(\|y\|^2 - \|P_U(y)\|^2\right)$$
$$\leqslant (\|x\| \cdot \|y\| - \|P_U(x)\| \cdot \|P_U(y)\|)^2. \tag{7}$$

Proof. If $x, y \in X$, then we have the decompositions $x = P_U(x) + u$ and $y = P_U(y) + v$, with $\langle P_U(x), u\rangle = 0$, $\langle P_U(y), v\rangle = 0$, $\langle P_U(x), v\rangle = 0$ and $\langle P_U(y), u\rangle = 0$. Therefore, we deduce $\langle x, P_U(y)\rangle = \langle P_U(x), y\rangle = \langle P_U(x), P_U(y)\rangle$. Consequently, we find the following equality:

$$\langle x - P_U(x), y - P_U(y)\rangle = \langle x,y\rangle - \langle P_U(x), P_U(y)\rangle,$$

which involves, for $x = y$, the identity

$$\|x - P_U(x)\|^2 = \|x\|^2 - \|P_U(x)\|^2.$$

But, using above equality and the inequality Cauchy-Schwarz, we obtain

$$|\langle x,y\rangle - \langle P_U(x), P_U(y)\rangle|^2 = |\langle x - P_U(x), y - P_U(y)\rangle|^2$$
$$\leqslant \|x - P_U(x)\|^2 \cdot \|y - P_U(y)\|^2 = \left(\|x\|^2 - \|P_U(x)\|^2\right)\left(\|y\|^2 - \|P_U(y)\|^2\right).$$

Next, we apply a simple inequality $\left(a^2 - b^2\right)\left(c^2 - d^2\right) \leqslant (ac - bd)^2$, for $a = \|x\|$, $b = \|P_U(x)\|$, $c = \|y\|$ and $d = \|P_U(y)\|$ and we find the inequality on the right side. Therefore, the statement is true. □

COROLLARY 1. *Let $U \subset X$ be a subspace of an inner product space X and $x, y \in X$. Then we have*

$$|\langle x,y\rangle - \langle P_U(x), P_U(y)\rangle| \leqslant \|x\| \cdot \|y\| - \|P_U(x)\| \cdot \|P_U(y)\|, \tag{8}$$

for all vectors x and y in X.

Proof. I. From inequality (7) we obtain the statement.

II. Next, we give another proof of inequality (8).

If we take $x, y \in X$, then we have the decompositions $x = P_U(x) + u$ and $y = P_U(y) + v$, which involves $\langle x,y\rangle = \langle P_U(x), P_U(y)\rangle + \langle u,v\rangle$. Therefore, for $x = y$, we find the following equality:

$$\|x\|^2 = \|P_U(x)\|^2 + \|u\|^2.$$

From this, we obtain $\|x\|^2 \cdot \|y\|^2 = \left(\|P_U(x)\|^2 + \|u\|^2\right) \cdot \left(\|P_U(y)\|^2 + \|v\|^2\right)$. Using the Cauchy-Buniakowski-Schwarz inequality, we have

$$\left(\|P_U(x)\|^2 + \|u\|^2\right) \cdot \left(\|P_U(y)\|^2 + \|v\|^2\right) \geqslant (\|P_U(x)\| \cdot \|P_U(y)\| + \|u\| \cdot \|v\|)^2,$$

so, we deduce $\|x\|\cdot\|y\| \geqslant \|P_U(x)\|\cdot\|P_U(y)\| + \|u\|\cdot\|v\|$. Taking into account that $\|u\|\cdot\|v\| \geqslant |\langle u,v\rangle|$, from the Cauchy-Schwarz inequality, it follows that $\|x\|\cdot\|y\| - \|P_U(x)\|\cdot\|P_U(y)\| \geqslant \|u\|\cdot\|v\| \geqslant |\langle u,v\rangle| = |\langle x,y\rangle - \langle P_U(x), P_U(y)\rangle|$. □

REMARK 2. If X is an inner product space over the field $\mathbb{K}$, where $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$ and $\{e_1,e_2,\ldots,e_n\}$ is an orthonormal basis of U, then we have $P_U(x) = \langle x,e_1\rangle e_1 + \langle x,e_2\rangle e_2 + \ldots + \langle x,e_n\rangle e_n$ and $P_U(y) = \langle y,e_1\rangle e_1 + \langle y,e_2\rangle e_2 + \ldots + \langle y,e_n\rangle e_n$ and from the first part of inequality (7), we obtain

$$\left(\|x\|^2 - \sum_{i=1}^{n}|\langle x,e_i\rangle|^2\right)\cdot\left(\|y\|^2 - \sum_{i=1}^{n}|\langle y,e_i\rangle|^2\right) \geqslant \left(\langle x,y\rangle - \sum_{i=1}^{n}\langle x,e_i\rangle\langle y,e_i\rangle\right)^2, \tag{9}$$

for all vectors x and y in X. This inequality represents a refinement of Bessel's inequality [5].

REMARK 3. From inequality (8) and using the inequality $|\langle x,y\rangle - \langle P_U(x), P_U(y)\rangle| \geqslant |\langle x,y\rangle| - |\langle P_U(x), P_U(y)\rangle|$, we find the following inequality:

$$0 \leqslant \|P_U(x)\|\cdot\|P_U(y)\| - |\langle P_U(x), P_U(y)\rangle| \leqslant \|x\|\cdot\|y\| - |\langle x,y\rangle|, \tag{10}$$

for all vectors x and y in X (see [4]).

In [19], Niezgoda proved an inequality for certain orthoprojectors. The operator $P_z : X \to X$ defined by

$$P_z(x) = \left\langle x, \frac{z}{\|z\|}\right\rangle \frac{z}{\|z\|},\ x\in X, z\neq 0,$$

is *the orthoprojector* from X onto span$\{z\}$.

Let X be a real linear space with the inner product $\langle\cdot,\cdot\rangle$. The *Chebyshev functional* is defined by

$$T_z(x,y) = \|z\|^2\langle x,y\rangle - \langle x,z\rangle\langle y,z\rangle,$$

for all $x,y\in X$, where $z\in X$ is a given nonzero vector.

The standard 2-inner product $(\cdot,\cdot|\cdot)$ is defined on the inner product space $X = (X,\langle\cdot,\cdot\rangle)$ by (see [1]):

$$(x,y|z) := \langle x,y\rangle\langle z,z\rangle - \langle x,z\rangle\langle z,y\rangle,$$

for all $x,y,z\in X$.

It is easy to see that we have $T_z(x,y) = (x,y|z)$ and $T_z(x,x)\geqslant 0$, for all $x,y,z\in X$.

If we replace x and y by $x - \frac{\langle x,z\rangle}{\|z\|^2}z$ and $y - \frac{\langle y,z\rangle}{\|z\|^2}z$, $z\neq 0$, in the Cauchy-Schwarz inequality, then we find the Cauchy-Schwarz inequality in terms of the Chebyshev functional, given by:

$$|T_z(x,y)|^2 \leqslant T_z(x,x)T_z(y,y). \tag{11}$$

We also have that $T_z(x,y)$, can be written in terms of orthoprojectors as:

$$T_z(x,y) = \|z\|^2\left(\langle x,y\rangle - \left\langle x,\frac{z}{\|z\|}\right\rangle\left\langle y,\frac{z}{\|z\|}\right\rangle\right) = \|z\|^2(\langle x,y\rangle - \langle P_z(x),P_z(y)\rangle).$$

Consequently, we find a relation for the standard 2-inner product in terms of orthoprojectors as:

$$(x,y|z) = \|z\|^2\Big(\langle x,y\rangle - \langle P_z(x),P_z(y)\rangle\Big). \tag{12}$$

Let $U \subset X$ be a subspace of an inner product space X and $x,y \in X$. Then we have the decompositions $x = P_U(x) + u$ and $y = P_U(y) + v$, which involves

$$(x,y|z) - (P_U(x), P_U(y)|z) = (u,v|z) - \langle u,z\rangle\langle z,P_U(y)\rangle - \langle z,v\rangle\langle P_U(x),z\rangle, \tag{13}$$

where $(x,y|z)$ is the standard 2-inner product on X. We take $\|x|z\| := \sqrt{(x,x|z)}$. For $x = y$ in relation (13), we have $\|x|z\|^2 = \|P_U(x)|z\|^2 - 2Re(\langle u,z\rangle\langle z,P_U(x)\rangle)$.

If $z \notin U$, then $\langle z,P_U(y)\rangle = 0$ and $\langle P_U(x),z\rangle = 0$, which means that relation (13) becomes

$$(x,y|z) = (P_U(x), P_U(y)|z) + (u,v|z),$$

which implies the equality $\|x|z\|^2 = \|P_U(x)|z\|^2 + \|u|z\|^2$, so, we obtain

$$\|x|z\|^2 = \|P_U(x)|z\|^2 + \|x - P_U(x)|z\|^2.$$

COROLLARY 2. *Let z be a nonzero vector of a complex pre-Hilbert space X and $x,y \in X$. Then, the following inequality holds*

$$\begin{aligned}|\langle x,y\rangle\|z\|^2 - \langle x,z\rangle\langle z,y\rangle|^2 &\leqslant \left(\|x\|^2\|z\|^2 - |\langle x,z\rangle|^2\right)\left(\|y\|^2\|z\|^2 - |\langle y,z\rangle|^2\right)\\ &\leqslant \left(\|x\|\|y\|\|z\|^2 - |\langle x,z\rangle\langle z,y\rangle|\right)^2.\end{aligned} \tag{14}$$

Proof. For z a nonzero vector of a complex pre-Hilbert space X and $x \in X$, we take $U = \text{span}\{z\}$, so, we deduce $P_U x = P_z x$, where P_z is the orthoprojector from X onto span$\{z\}$. Consequently, we have $\|x - P_z x\|^2 = \|x\|^2 - \|P_z x\|^2 = \frac{\|x\|^2\|z\|^2 - |\langle x,z\rangle|^2}{\|z\|^2}$ and $\|P_z x\| = \frac{|\langle x,z\rangle|}{\|z\|}$. Using inequality (7), we prove the inequality of the statement. □

REMARK 4. In first part of inequality (14), if we take $\langle x,y\rangle = 1$ and $\langle z,y\rangle = 0$, then we proved the inequality of Ostrowski for an inner product space [16],

$$0 \leqslant \frac{\|z\|^2}{\|y\|^2} \leqslant \|x\|^2\|z\|^2 - |\langle x,z\rangle|^2, \tag{15}$$

for all $x,y,z \in X$.

Next we present an improvement of first part of inequality (4) given by:

THEOREM 3. *For any $T \in \mathbb{B}(X)$ one has*

$$0 \leqslant \frac{1}{2\gamma} \leqslant \|T\| - \omega(T), \tag{16}$$

where $\gamma = \sup_{\|x\|=1} \|y\|^2\|Tx\|^2$, with $\langle Tx,y\rangle = 1$ and $\langle x,y\rangle = 0$.

Proof. If in Ostrowski's inequality for a pre-Hilbert space, we replace x and z, by Tx and x, it follows that

$$0 \leqslant \frac{\|x\|^2}{\|y\|^2} \leqslant \|Tx\|^2\|x\|^2 - |\langle Tx,x\rangle|^2,$$

for all $x,y \in X$, with $\langle Tx,y\rangle = 1$ and $\langle x,y\rangle = 0$. The above inequality implies the inequality

$$0 \leqslant \frac{1}{2\|y\|^2\|Tx\|} \leqslant \frac{1}{\|y\|^2\left(\|Tx\| + |\langle Tx,x\rangle|\right)} \leqslant \|Tx\| - |\langle Tx,x\rangle|.$$

So, we obtain

$$\frac{1}{2\|y\|^2\|Tx\|} + |\langle Tx,x\rangle| \leqslant \|Tx\|,$$

for all $x,y \in X$, $\|x\| = 1$, with $\langle Tx,y\rangle = 1$ and $\langle x,y\rangle = 0$. Taking the supremum in above inequality over $\|x\| = 1$, we deduce the inequality of the statement. $\square$

THEOREM 4. *Let $U \subset X$ be a subspace of a complex Hilbert space and $x \in X$. If $T \in \mathbb{B}(X)$, such that $\|x - Tx\| \leqslant \|x - y\|$, for every $y \in U$, then we have*

$$\|(T - P_U)(x)\| \leqslant \|T\|\|x - y\|, \tag{17}$$

for every $y \in U$.

Proof. For every subspace $U \subset X$, we have the decomposition $X = U \bigoplus U^{\perp}$, so every $x \in X$ can be uniquely written as $x = P_U x + u$, where $u \in U^{\perp}$. If $x \in U$ and $y = x$, then, from inequality of the hypothesis, we obtain $Tx = x$, so we say that every vector $y \in U$ is a fixed point of the operator T. Since $P_U x \in U$, we have $T(P_U x) = P_U x$. Therefore, if $x \in U$, then $x = P_U x = T(P_U x) = Tx$, so inequality (17) is true.

If $x \in X - U$, then because $x = P_U x + u$, we deduce $Tx = TP_U x + Tu$, it follows $Tx = P_U x + Tu$ and passing to the norm, we find

$$\|(T - P_U)(x)\| = \|Tu\| \leqslant \|T\|\|u\| = \|T\|\|x - P_U x\|.$$

Therefore, using the Best Approximation Theorem, we prove

$$\|(T - P_U)(x)\| \leqslant \|T\|\|x - P_U x\| \leqslant \|T\|\|x - y\|,$$

for every $y \in U$. Consequently, the proof is complete. $\square$

REMARK 5. Let z be a nonzero vector of a complex Hilbert space X and $x \in X$. For $U = \operatorname{span}\{z\}$, we have $P_U x = P_z x$, where P_z is the orthoprojector from X onto $\operatorname{span}\{z\}$. Therefore, using inequality (17), we deduce for an operator $T \in \mathbb{B}(X)$, such that $\|x - Tx\| \leqslant \|x - z\|$, the following inequality:

$$\|(T - P_z)(x)\| \leqslant \|T\|\|x - z\|. \tag{18}$$

THEOREM 5. *If $T \in \mathbb{B}(X)$, then the following inequality holds*

$$\|T - I\| \geqslant \|T\| - \omega(T), \tag{19}$$

where I *is the identity operator.*

Proof. From [15], in an inner product space X over the field of complex numbers $\mathbb{C}$, we have the identity

$$\|2\langle u,v\rangle v - \|v\|^2 u\| = \|u\|\|v\|^2,$$

for every $u,v \in X$. Therefore, applying the triangle inequality, we find

$$|\langle u,v\rangle|\|v\| + \|\langle u,v\rangle v - \|v\|^2 u\| \geqslant \|u\|\|v\|^2, \tag{20}$$

for every $u,v \in X$. If we choose in relation (20) $u = Tx$ and $v = x$, with $\|x\| = 1$, then we obtain

$$|\langle Tx,x\rangle| + \|\langle Tx,x\rangle x - Tx\| \geqslant \|Tx\|. \tag{21}$$

But, using the following identity from [13]: $\|u - \langle u,e\rangle e\|^2 = \inf_{\lambda\in\mathbb{C}} \|u - \lambda e\|^2$, with $\|e\| = 1$, we deduce that

$$\|Tx - \langle Tx,x\rangle x\| \leqslant \|Tx - x\|, \tag{22}$$

with $\|x\| = 1$. Combining relations (21) and (22), we obtain

$$|\langle Tx,x\rangle| + \|Tx - x\| \geqslant \|Tx\|,$$

and taking the supremum in above inequality over $\|x\| = 1$, we deduce the inequality of the statement. $\square$

3. Applications for Grüss type inequalities

Grüss' inequality [11] says that

$$\left|\frac{1}{b-a}\int_a^b f(t)g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt\right| \leqslant \frac{1}{4}(\beta_0 - \alpha_0)(\delta_0 - \gamma_0), \tag{23}$$

for two bounded integrable functions $f,g : [a,b] \to \mathbb{R}$ and four real scalars $\alpha_0, \beta_0, \gamma_0, \delta_0$ such that

$$\alpha_0 \leqslant f(t) \leqslant \beta_0 \quad \text{and} \quad \gamma_0 \leqslant g(t) \leqslant \delta_0, \quad \text{for all } t \in [a,b].$$

It is our aim to establish a class of inequalities similar to (23) in the framework of an arbitrary inner product space. Throughout this section X is a real linear space endowed with an inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\| = \langle\cdot,\cdot\rangle^{1/2}$. It is also assumed that $e \in X$ is a given vector of norm one. We consider the subspace $U = \operatorname{span} e$ and its orthogonal complement $U^\perp = (\operatorname{span} e)^\perp$ to X. We denote by $P = P_U$ the orthogonal projection from X onto U and by $Q = \mathrm{id}_X - P$ the orthogonal projection from X onto the subspace $U^\perp$, where id_X is the identity map on X.

A *Grüss type inequality* gives an estimate from above for the expression

$$|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle|, \tag{24}$$

where $x,y \in X$ (see [7, 8, 9, 10, 20, 21, 22, 23, 24, 26]).

Usually Grüss type inequalities are intended for vectors $x, y \in X$ in some subsets of balls described by requirement of type (25) (see [7, 8, 20, 21, 22, 24]). In Theorem 6 we continue this approach and restrict ourselves to the subsets formed by double ice-cream cones defined by mutual conditions (25) and (26) (see Remarks 6 and 7 for details). This allows to achieve an improvement of the standard Grüss type inequality by using an additional factor $0 < km \leqslant 1$ in the upper bound of Grüss functional (24) (see inequality (28)).

THEOREM 6. *Let $x, y \in X$ with $x \notin \operatorname{span} e$ and $y \notin \operatorname{span} e$ (i.e., $Qx \neq 0$ and $Qy \neq 0$). If there exist $x_0 \in \operatorname{span} e$ and $y_0 \in \operatorname{span} e$ such that $\widetilde{x} = x - x_0$ and $\widetilde{y} = y - y_0$, and there exist $R > 0$ and $S > 0$ such that*

$$\|x - x_0\| \leqslant R \quad \text{and} \quad \|y - y_0\| \leqslant S, \tag{25}$$

and there exist $k, m \in (0, 1]$ such that

$$\|Qx\| \leqslant k\|x - x_0\| \quad \text{and} \quad \|Qy\| \leqslant m\|y - y_0\|, \tag{26}$$

then

$$\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle = \|\widetilde{x}\|\,\|\widetilde{y}\| \left(1 - \frac{1}{2}d^2(\widetilde{x}, Q\widetilde{x})\right)\left(1 - \frac{1}{2}d^2(\widetilde{y}, Q\widetilde{y})\right)\left(1 - \frac{1}{2}d^2(Qx, Qy)\right), \tag{27}$$

and

$$|\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle| \leqslant RSkm\left|1 - \frac{1}{2}d^2(Qx, Qy)\right| \leqslant RSkm. \tag{28}$$

Proof. It is easy to see that $\widetilde{x} \neq 0$ and $\widetilde{y} \neq 0$. It follows that $Px = \langle x, e\rangle e$ and $Qx = x - \langle x, e\rangle e$. It is clear that $Q\widetilde{x} = Q(x - x_0) = Qx$ and $Q\widetilde{y} = Q(y - y_0) = Qy$.

Having in mind identity (1), we establish

$$\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle = \langle x - \langle x, e\rangle e, y - \langle y, e\rangle e\rangle = \langle Qx, Qy\rangle = \|Qx\|\,\|Qy\| \left(1 - \frac{1}{2}d^2(Qx, Qy)\right), \tag{29}$$

where

$$d(Qx, Qy) = \left\| \frac{Qx}{\|Qx\|} - \frac{Qy}{\|Qy\|} \right\|.$$

By making use of equality (5) we get

$$\|Qx\| = \|Q\widetilde{x}\| = \|\widetilde{x}\| \left(1 - \frac{1}{2}d^2(\widetilde{x}, Q\widetilde{x})\right). \tag{30}$$

Likewise, we have

$$\|Qy\| = \|Q\widetilde{y}\| = \|\widetilde{y}\| \left(1 - \frac{1}{2}d^2(\widetilde{y}, Q\widetilde{y})\right). \tag{31}$$

By combining (29), (30) and (31) we deduce that

$$\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle = \|\widetilde{x}\|\,\|\widetilde{y}\|\left(1-\frac{1}{2}d^2(\widetilde{x},Q\widetilde{x})\right)\left(1-\frac{1}{2}d^2(\widetilde{y},Q\widetilde{y})\right)\left(1-\frac{1}{2}d^2(Qx,Qy)\right), \tag{32}$$

which proves (27).

By using the facts $\langle \widetilde{x},Q\widetilde{x}\rangle = \langle Q\widetilde{x},Q\widetilde{x}\rangle = \|Q\widetilde{x}\|^2 = \|Qx\|^2 > 0$, we shall show that

$$1-\frac{1}{2}d^2(\widetilde{x},Q\widetilde{x}) > 0, \tag{33}$$

whence

$$0 \leqslant d(\widetilde{x},Q\widetilde{x}) < \sqrt{2}.$$

To see (33), we proceed as follows.

$$\begin{aligned} d^2(\widetilde{x},Q\widetilde{x}) &= \left\|\frac{\widetilde{x}}{\|\widetilde{x}\|} - \frac{Q\widetilde{x}}{\|Q\widetilde{x}\|}\right\|^2 = \left\langle \frac{\widetilde{x}}{\|\widetilde{x}\|} - \frac{Q\widetilde{x}}{\|Q\widetilde{x}\|}, \frac{\widetilde{x}}{\|\widetilde{x}\|} - \frac{Q\widetilde{x}}{\|Q\widetilde{x}\|}\right\rangle \\ &= \left\langle \frac{\widetilde{x}}{\|\widetilde{x}\|}, \frac{\widetilde{x}}{\|\widetilde{x}\|}\right\rangle - \left\langle \frac{\widetilde{x}}{\|\widetilde{x}\|}, \frac{Q\widetilde{x}}{\|Q\widetilde{x}\|}\right\rangle - \left\langle \frac{Q\widetilde{x}}{\|Q\widetilde{x}\|}, \frac{\widetilde{x}}{\|\widetilde{x}\|}\right\rangle + \left\langle \frac{Q\widetilde{x}}{\|Q\widetilde{x}\|}, \frac{Q\widetilde{x}}{\|Q\widetilde{x}\|}\right\rangle \\ &= \left\|\frac{\widetilde{x}}{\|\widetilde{x}\|}\right\|^2 - 2\left\langle \frac{\widetilde{x}}{\|\widetilde{x}\|}, \frac{Q\widetilde{x}}{\|Q\widetilde{x}\|}\right\rangle + \left\|\frac{Q\widetilde{x}}{\|Q\widetilde{x}\|}\right\|^2 = 2-2\left\langle \frac{\widetilde{x}}{\|\widetilde{x}\|}, \frac{Q\widetilde{x}}{\|Q\widetilde{x}\|}\right\rangle \\ &= 2-2\frac{\langle Q\widetilde{x},Q\widetilde{x}\rangle}{\|\widetilde{x}\|\,\|Q\widetilde{x}\|} = 2-2\frac{\|Q\widetilde{x}\|^2}{\|\widetilde{x}\|\,\|Q\widetilde{x}\|} < 2, \end{aligned}$$

as claimed.

In conclusion, we get

$$d^2(\widetilde{x},Q\widetilde{x}) = 2-2\frac{\|Q\widetilde{x}\|^2}{\|\widetilde{x}\|\,\|Q\widetilde{x}\|}.$$

Hence,

$$1-\frac{1}{2}d^2(\widetilde{x},Q\widetilde{x}) = \frac{\|Q\widetilde{x}\|^2}{\|\widetilde{x}\|\,\|Q\widetilde{x}\|}. \tag{34}$$

By (26) we have

$$\|Qx\| \leqslant k\|x-x_0\|, \tag{35}$$

which can be rewritten as

$$\|Q\widetilde{x}\| \leqslant k\|\widetilde{x}\|, \tag{36}$$

or, equivalently,

$$\frac{\|Q\widetilde{x}\|^2}{\|\widetilde{x}\|\,\|Q\widetilde{x}\|} \leqslant k,$$

since $\|\widetilde{x}\| > 0$ and $\|Q\widetilde{x}\| > 0$.

Now, by utilizing (34) we obtain

$$1-\frac{1}{2}d^2(\widetilde{x},Q\widetilde{x}) \leqslant k, \tag{37}$$

as wanted.

In a similar manner, we find that

$$1-\frac{1}{2}d^2(\widetilde{y},Q\widetilde{y})\leqslant m. \tag{38}$$

Therefore the former inequality of (28) can be easily deduced from (3), (25), (27), (37) and (38). This completes the proof. □

REMARK 6. a) From relations (3), (27) and (33), we deduce

$$\left|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle\right|\leqslant\|\widetilde{x}\|\|\widetilde{y}\|\left(1-\frac{1}{2}d^2(\widetilde{x},Q\widetilde{x})\right)\left(1-\frac{1}{2}d^2(\widetilde{y},Q\widetilde{y})\right).$$

But, using the algebraic inequality $\left(a^2-b^2\right)\left(c^2-d^2\right)\leqslant(ac-bd)^2$, we find

$$\left(1-\frac{1}{2}d^2(\widetilde{x},Q\widetilde{x})\right)\left(1-\frac{1}{2}d^2(\widetilde{y},Q\widetilde{y})\right)\leqslant\left(1-\frac{1}{2}d(\widetilde{x},Q\widetilde{x})d(\widetilde{y},Q\widetilde{y})\right)^2,$$

so, we prove the following inequality:

$$\left|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle\right|\leqslant\|\widetilde{x}\|\|\widetilde{y}\|\left(1-\frac{1}{2}d(\widetilde{x},Q\widetilde{x})d(\widetilde{y},Q\widetilde{y})\right)^2. \tag{39}$$

b) We now interpret conditions (25) and (26). So, for given $R>0$ and $k\in(0,1]$ we are interested in the set

$$\begin{aligned}C(R,k)&=\{x\in X:\|x-x_0\|\leqslant R\ \text{ and }\ \|Qx\|\leqslant k\|x-x_0\|\}\\&=\bigcup_{R_1\in(0,R]}\bigcup_{k_1\in(0,k]}\{x\in X:\|x-x_0\|=R_1\ \text{ and }\ \|Qx\|=k_1\|x-x_0\|\}\\&=\bigcup_{R_1\in(0,R]}\bigcup_{k_1\in(0,k]}\{x\in X:\|x-x_0\|=R_1\ \text{ and }\ \|Qx\|=k_1R_1\}.\end{aligned}$$

Observe that the set

$$\{x\in X:\|x-x_0\|=R_1\ \text{ and }\ \|Qx\|=k_1R_1\}$$

is the circle on the sphere $\{x\in X:\|x-x_0\|=R_1\}$ whose projection onto the subspace $(\operatorname{span}e)^\perp$ is the circle of radius k_1R_1 centered at the origin. Thus the set $C(R,k)$ is the union of two *"ice-cream cones"* with the same symmetry axis $\operatorname{span}e$ characterized by parameters R and k as above. This shows the geometry of $C(R,k)$ defined by conditions (25) and (26). In consequence, Grüss type inequality (28) in Theorem 6 is designed to work for vectors x and y in some double ice-cream cones with their symmetry axis $\operatorname{span}e$ and with their apices at some points $x_0\in\operatorname{span}e$ and $y_0\in\operatorname{span}e$, respectively. The case $k=m=1$ is mentioned in Remark 7.

REMARK 7. Inequality (28) relies on numbers k and m satisfying requirement (26). Certainly, (26) holds for $k=m=1$, since $Qx=Q(x-x_0)$ and Q is an orthogonal projection. In context of Remark 6, this case corresponds to the trivial double ice-cream cones reduced to the unions of two half-balls giving whole balls $B(x_0,R)$ and $B(y_0,S)$, respectively. Therefore Corollary 3 concerns vectors x and y from balls defined by condition (40), only.

COROLLARY 3. *Let $x,y \in X$ with $x \notin \operatorname{span} e$ and $y \notin \operatorname{span} e$. If there exist $x_0 \in \operatorname{span} e$ and $y_0 \in \operatorname{span} e$ and there exist $R > 0$ and $S > 0$ such that*

$$\|x - x_0\| \leqslant R \quad \text{and} \quad \|y - y_0\| \leqslant S, \tag{40}$$

then

$$|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle| \leqslant RS\left|1 - \frac{1}{2}d^2(Qx,Qy)\right| \leqslant RS. \tag{41}$$

Proof. We consider the vectors $\widetilde{x} = x - x_0$ and $\widetilde{y} = y - y_0$. It is easy to see that $\widetilde{x} \neq 0$ and $\widetilde{y} \neq 0$. It is sufficient to employ inequality (28) in Theorem 6 with the specification $k = m = 1$. In fact, condition (26) is satisfied in the form

$$\|Qx\| = \|Q\widetilde{x}\| \leqslant \|\widetilde{x}\| = \|x - x_0\| \quad \text{and} \quad \|Qy\| = \|Q\widetilde{y}\| \leqslant \|\widetilde{y}\| = \|y - y_0\|. \qquad \square$$

A nonempty set $K \subset X$ is said to be a *convex cone*, if **(i)** $x,y \in K$ implies $x+y \in K$, and **(ii)** $0 \leqslant t \in \mathbb{R}$ and $x \in K$ imply $tx \in K$.

If $K \subset X$ is a convex cone, then the *dual cone* of K is defined by

$$\operatorname{dual} K = \{z \in X : \langle z,v\rangle \geqslant 0 \quad \text{for all } v \in K\}.$$

If $\operatorname{dual} K = K$ then K is called *self-dual*.

In what follows we use the following cone preorders $\leqslant_K$ and $\leqslant_{\operatorname{dual} K}$ on X defined by

$$y \leqslant_K x \text{ iff } x - y \in K,$$

$$y \leqslant_{\operatorname{dual} K} x \quad \text{iff} \quad x - y \in \operatorname{dual} K.$$

LEMMA 1. ([9, Lemma 2.1], [20, Lemma 4.1]) *For any vectors $\alpha, \beta, x \in X$, the following statements are mutually equivalent:*

(i) *There exists a convex cone $K \subset X$ such that $\alpha \leqslant_K x \leqslant_{\operatorname{dual} K} \beta$.*

(ii) $\langle \beta - x, x - \alpha\rangle \geqslant 0$.

(iii) $\|x - \frac{1}{2}(\alpha+\beta)\| \leqslant \|\frac{1}{2}(\beta - \alpha)\|$.

The next result is an extension of [20, Theorem 4.2]. The pair of conditions **(ii)**-**(iii)** below says that vectors x and y belong to the corresponding double ice-cream cones $C(R,k)$ and $C(S,m)$ for suitable R, k, S, m. (The same concerns the forthcoming corollaries after Theorem 7.)

THEOREM 7. *Let $x,y,\alpha,\beta,\gamma,\delta \in X$ be vectors such that $x \notin \operatorname{span} e$ and $y \notin \operatorname{span} e$. Assume that*

(i) $x \neq \frac{\alpha+\beta}{2} \in \operatorname{span} e$ *and* $y \neq \frac{\gamma+\delta}{2} \in \operatorname{span} e$,

(ii) *for some convex cones* $K_1, K_2 \subset X$,

$$\alpha \leqslant_{K_1} x \leqslant_{\operatorname{dual} K_1} \beta \quad \text{and} \quad \gamma \leqslant_{K_2} y \leqslant_{\operatorname{dual} K_2} \delta \tag{42}$$

(iii) *for some* $k, m \in (0,1]$,

$$\|Qx\| \leqslant k \left\| x - \frac{\alpha+\beta}{2} \right\| \quad \text{and} \quad \|Qy\| \leqslant m \left\| x - \frac{\gamma+\delta}{2} \right\|. \tag{43}$$

Then

$$|\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle| \leqslant \frac{1}{4} km \|\beta - \alpha\| \|\delta - \gamma\|. \tag{44}$$

Proof. By **(ii)** we have

$$\langle \beta - x, x - \alpha\rangle \geqslant 0 \quad \text{and} \quad \langle \delta - y, y - \gamma\rangle \geqslant 0.$$

By Lemma 1 we infer that

$$\left\| x - \frac{\alpha+\beta}{2} \right\| \leqslant \left\| \frac{\beta-\alpha}{2} \right\| \quad \text{and} \quad \left\| y - \frac{\gamma+\delta}{2} \right\| \leqslant \left\| \frac{\delta-\gamma}{2} \right\|. \tag{45}$$

We introduce

$$x_0 = \frac{\alpha+\beta}{2} \quad \text{and} \quad y_0 = \frac{\gamma+\delta}{2}$$

and

$$R = \left\| \frac{\beta-\alpha}{2} \right\| \quad \text{and} \quad S = \left\| \frac{\delta-\gamma}{2} \right\|.$$

Thus condition (25) is satisfied by (45).

By virtue of **(i)** one sees that $x \neq x_0 \in \operatorname{span} e$ and $y \neq y_0 \in \operatorname{span} e$.

It is now sufficient to apply inequality (28) in Theorem 6. □

The case $k = m = 1$ of the next corollary leads to some results due to Dragomir [7, Theorem 1], [9, Theorem 2.5].

COROLLARY 4. *Let* $x, y \in X$ *be vectors such that* $x \notin \operatorname{span} e$ *and* $y \notin \operatorname{span} e$, *and* $\alpha_0, \beta_0, \gamma_0, \delta_0 \in \mathbb{R}$. *Assume that*

(i) $x \neq \frac{\alpha_0+\beta_0}{2} e$ *and* $y \neq \frac{\gamma_0+\delta_0}{2} e$,

(ii) *for some convex cones* $K_1, K_2 \subset X$,

$$\alpha_0 e \leqslant_{K_1} x \leqslant_{\operatorname{dual} K_1} \beta_0 e \quad \text{and} \quad \gamma_0 e \leqslant_{K_2} y \leqslant_{\operatorname{dual} K_2} \delta_0 e, \tag{46}$$

or, equivalently,

$$\left\| x - \frac{\alpha_0+\beta_0}{2} e \right\| \leqslant \left| \frac{\beta_0-\alpha_0}{2} \right| \quad \text{and} \quad \left\| y - \frac{\gamma_0+\delta_0}{2} e \right\| \leqslant \left| \frac{\delta_0-\gamma_0}{2} \right|,$$

(iii) *for some* $k, m \in (0,1]$,

$$\|Qx\| \leqslant k \left\| x - \frac{\alpha_0+\beta_0}{2} e \right\| \quad \text{and} \quad \|Qy\| \leqslant m \left\| y - \frac{\gamma_0+\delta_0}{2} e \right\|,$$

then

$$|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle| \leqslant \frac{1}{4}km|\beta_0-\alpha_0||\delta_0-\gamma_0|.$$

Proof. Use Theorem 7 for the vectors $\alpha=\alpha_0 e$, $\beta=\beta_0 e$, $\gamma=\gamma_0 e$ and $\delta=\delta_0 e$. □

We now specialize Theorem 7. In Corollary 5 we deal with the space $X=\mathbb{R}^n$ endowed with the standard inner product of $\mathbb{R}^n$, and with the vector $e=\frac{1}{\sqrt{n}}(1,\dots,1)\in\mathbb{R}^n$ of norm one (cf. [20, Corollary 4.4]).

COROLLARY 5. *Let $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)$, $\alpha=(\alpha_1,\dots,\alpha_n)$, $\beta=(\beta_1,\dots,\beta_n)$, $\gamma=(\gamma_1,\dots,\gamma_n)$ and $\delta=(\delta_1,\dots,\delta_n)$ be vectors in $\mathbb{R}^n$ such that $x\notin \operatorname{span} e$ and $y\notin\operatorname{span} e$ (i.e., x and y are not of constant entries). Assume that*

(i) $x\neq\frac{\alpha+\beta}{2}\in\operatorname{span} e$ *and* $y\neq\frac{\gamma+\delta}{2}\in\operatorname{span} e$,

(ii) $\alpha_i\leqslant x_i\leqslant\beta_i$ *and* $\gamma_i\leqslant y_i\leqslant\delta_i$ *for all* $i=1,\dots,n$, *or more generally*

$$\sum_{i=1}^{n}(\beta_i-x_i)(x_i-\alpha_i)\geqslant 0 \quad and \quad \sum_{i=1}^{n}(\delta_i-y_i)(y_i-\gamma_i)\geqslant 0, \tag{47}$$

(iii) *for some* $k,m\in(0,1]$,

$$\left(\sum_{i=1}^{n}(x_i-\bar{x})^2\right)^{1/2}\leqslant k\left(\sum_{i=1}^{n}\left(x_i-\frac{\alpha_i+\beta_i}{2}\right)^2\right)^{1/2} \tag{48}$$

and

$$\left(\sum_{i=1}^{n}(y_i-\bar{y})^2\right)^{1/2}\leqslant m\left(\sum_{i=1}^{n}\left(y_i-\frac{\gamma_i+\delta_i}{2}\right)^2\right)^{1/2}, \tag{49}$$

where $\bar{x}=\frac{1}{n}\sum_{i=1}^{n}x_i$ *and* $\bar{y}=\frac{1}{n}\sum_{i=1}^{n}y_i$.

Then

$$\left|\sum_{i=1}^{n}x_iy_i-\frac{1}{n}\sum_{i=1}^{n}x_i\sum_{i=1}^{n}y_i\right|\leqslant\frac{1}{4}km\left(\sum_{i=1}^{n}(\beta_i-\alpha_i)^2\right)^{1/2}\left(\sum_{i=1}^{n}(\delta_i-\gamma_i)^2\right)^{1/2}. \tag{50}$$

Proof. It is not hard to verify that (42) and (47) are equivalent (see Lemma 1). Moreover, the orthogonal projection P from $\mathbb{R}^n$ onto $\operatorname{span} e$ is given by $Pz=(\bar{z},\dots,\bar{z})\in\mathbb{R}^n$, where $\bar{z}=\sum_{i=1}^{n}z_i$ for $z=(z_1,\dots,z_n)\in\mathbb{R}^n$. Therefore the orthogonal projection Q from $\mathbb{R}^n$ onto $(\operatorname{span} e)^{\perp}$ is given by

$$Qz=z-Pz=(z_1,\dots,z_n)-(\bar{z},\dots,\bar{z})=(z_1-\bar{z},\dots,z_n-\bar{z})\in\mathbb{R}^n.$$

For this reason, in light of (48)–(49) we see that (43) holds valid. Thus all assumptions **(i)**, **(ii)** and **(iii)** of Theorem 7 are met. On account of Theorem 7 we obtain (44), which can be restated as (50), completing the proof. □

We now focus on the space $X = L^2_{[a,b]}$ of real functions integrable with second power on an interval $[a,b] \subset \mathbb{R}$. As usual, we equip X with the inner product $\langle f,g\rangle = \int_a^b f(t)g(t)\,dt$. We apply the constant function $e:[a,b]\to\mathbb{R}$ given by $e(t) = \frac{1}{\sqrt{b-a}}$ for $t\in[a,b]$. It is obvious that $\|e\| = 1$.

Using Theorem 7 we obtain the following result (cf. [20, [Corollary 4.5]).

COROLLARY 6. *Let $x, y, \alpha, \beta, \gamma, \delta \in L^2_{[a,b]}$ be functions such that $x \notin \operatorname{span} e$ and $y \notin \operatorname{span} e$ (i.e., x and y are not constant functions). Assume that*

(i) $x \neq \frac{\alpha+\beta}{2} \in \operatorname{span} e$ *and* $y \neq \frac{\gamma+\delta}{2} \in \operatorname{span} e$,

(ii) $\alpha(t) \leqslant x(t) \leqslant \beta(t)$ *and* $\gamma(t) \leqslant y(t) \leqslant \delta(t)$ *for all* $t\in[a,b]$, *or more generally*

$$\int_a^b (\beta(t)-x(t))(x(t)-\alpha(t))\,dt \geqslant 0 \quad \text{and} \quad \int_a^b (\delta(t)-y(t))(y(t)-\gamma(t))\,dt \geqslant 0, \tag{51}$$

(iii) *for some* $k, m \in (0,1]$,

$$\left(\int_a^b (x(t)-\bar{x})^2\,dt\right)^{1/2} \leqslant k\left(\int_a^b \left(x(t)-\frac{\alpha(t)+\beta(t)}{2}\right)^2 dt\right)^{1/2} \tag{52}$$

and

$$\left(\int_a^b (y(t)-\bar{y})^2\,dt\right)^{1/2} \leqslant m\left(\int_a^b \left(y(t)-\frac{\gamma(t)+\delta(t)}{2}\right)^2 dt\right)^{1/2}, \tag{53}$$

where $\bar{x} = \frac{1}{b-a}\int_a^b x(t)\,dt$ *and* $\bar{y} = \frac{1}{b-a}\int_a^b y(t)\,dt$.

Then

$$\left|\int_a^b x(t)y(t)\,dt - \frac{1}{b-a}\int_a^b x(t)\,dt\int_a^b y(t)\,dt\right| \leqslant \frac{1}{4}km\left(\int_a^b (\beta(t)-\alpha(t))^2\,dt\right)^{1/2}\left(\int_a^b (\delta(t)-\gamma(t))^2\,dt\right)^{1/2}. \tag{54}$$

Proof. First of all, we note that (42) and (51) are equivalent (see Lemma 1). Furthemore, the orthogonal projection P from $L^2_{[a,b]}$ onto the subspace $\operatorname{span} e$ of constant functions is given by $(Pz)(t) = \frac{1}{b-a}\int_a^b z(s)\,ds = \bar{z}$ for $t \in [a,b]$. Therefore the orthogonal projection Q from $L^2_{[a,b]}$ onto $(\operatorname{span} e)^{\perp}$ is given by $Qz = z - Pz$ for $z \in L^2_{[a,b]}$, where $(Qz)(t) = z(t) - \bar{z}$ for $t \in [a,b]$. In consequence, by making use of (52) and (53) we see that (43) is true. In conclusion, assumptions **(i)**, **(ii)** and **(iii)** of Theorem 7 are fulfilled. According to Theorem 7 we deduce that (44) holds. From this we get (54), as desired. □

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(Received May 9, 2020)

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