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ON CONSTANTS IN COCONVEX APPROXIMATION OF PERIODIC FUNCTIONS

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Abstract. Let 2π-periodic function $f \in \mathbb{C}$ change its convexity finitely even many times, in the period. We are interested in estimating the degree of approximation of f by trigonometric polynomials which are coconvex with it, namely, polynomials that change their convexity exactly at the points where f does. We list established Jackson-type estimates of such approximation where the constants involved depend on the location of the points of change of convexity and show that this dependence is essential by constructing a counterexample.

1. Introduction

Denote by $\mathbb{C}$ and $\mathbb{C}^r$, respectively the space of continuous 2π-periodic functions $f:\mathbb{R}\to\mathbb{R}$, and that of r-times continuously differentiable functions, equipped with the uniform norm

$$\|f\| := \max_{x\in\mathbb{R}} |f(x)|.$$

Denote by $\mathbb{T}_n$ the space of trigonometric polynomials $P_n(x) = a_0 + \sum_{j=1}^n (a_j \times \cos jx + b_j \sin jx)$ of degree not exceeding $n \in \mathbb{N}$ (of order $\leqslant 2n+1$) with $a_j \in \mathbb{R}$ and $b_j \in \mathbb{R}$, and by

$$E_n(f) := \inf_{P_n\in\mathbb{T}_n} \|f - P_n\|$$

the value (error) of the best uniform approximation of the function f by polynomials $P_n \in \mathbb{T}_n$.

Recall that for any bounded on $[a,b]$ function f, and $k \in N$, the k-th symmetric difference of f at the point x with the step $h \geqslant 0$ is defined as

$$\Delta_h^k f(x) := \begin{cases} \sum_{i=0}^k (-1)^{k-i}\binom{k}{i} f(x - \frac{k}{2}h + ih), & x \pm \frac{k}{2}h \in [a,b], \\ 0, & \text{otherwise}, \end{cases}$$

and the (ordinary) k-th modulus of continuity (or smoothness) of the function $f \in \mathbb{C}[a,b]$ is defined as

$$\omega_k(f,t,[a,b]) := \sup_{h\in[0,t]} \|\Delta_h^k f\| = \sup_{h\in[0,t]} \max_{x\in[a,b]} |\Delta_h^k f(x)|, \quad t \in [0,(b-a)/k],$$

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$$\omega_k(f,t,[a,b]) \equiv \omega_k(f,(b-a)/k,[a,b]), \quad t \geqslant (b-a)/k,$$

and in the case of 2π-periodic f, it is defined as

$$\omega_k(f,t) := \omega_k(f,t,\mathbb{R}) := \sup_{a\in\mathbb{R}} \omega_k(f,t,[a,a+2\pi]).$$

Recall the classical estimate of Jackson (the case $k=1$) [12, 13]-Zigmund ($k=2$, $\omega_2(f,t) \leqslant t$) [28]-Akhiezer ($k=2$) [1]-Stechkin ($k \geqslant 3$) [22]:
If a function $f \in \mathbb{C}$, then

$$E_n(f) \leqslant c(k)\,\omega_k(f,\pi/n), \quad n \in \mathbb{N}, \tag{1}$$

where $c(k)$ is a constant that depends only on k, for details see, for example, [10, Section 4]. And hence, in particular, *if $f \in \mathbb{C}^r$, $r \in \mathbb{N}$, then*

$$E_n(f) \leqslant \frac{c(r+k)}{n^r}\,\omega_k(f^{(r)},\pi/n), \quad n \in \mathbb{N}. \tag{2}$$

In 1968 Lorentz and Zeller [17, 18] proved a bell-shaped analogue of the estimate (1) with $k=1$, that is, for approximation of bell-shaped (even and nonincreasing on $[0,\pi]$) functions from $\mathbb{C}$ by bell-shaped polynomials from $\mathbb{T}_n$, and thus gave rise to the search for other its analogues, with other restrictions on the shape of the function and polynomials such as piecewise positivity, monotonicity, convexity (now this is called Shape Preserving Approximation).

In the papers of Popov [19] and Zalizko [25] a coconvex analogue of the inequality (1) is proved with $k=2$ and $k=3$, respectively. To state it we need some notations.

Let on $[-\pi,\pi)$ there are $2s$, $s \in \mathbb{N}$, fixed points

$$y_i: \ -\pi \leqslant y_{2s} < y_{2s-1} < \ldots < y_1 < \pi,$$

and for the remaining $i \in \mathbb{Z}$, the points y_i are defined by the equality $y_i = y_{i+2s} + 2\pi$ (that is, $y_0 = y_{2s} + 2\pi, \ldots, y_{2s+1} = y_1 - 2\pi, \ldots$), and let $Y := Y_{2s} = \{y_i\}_{i\in\mathbb{Z}}$.

Denote by $\Delta^{(2)}(Y_{2s})$ the collection of all functions $f \in \mathbb{C}$ that are convex on $[y_1,y_0]$, concave on $[y_2,y_1]$, convex on $[y_3,y_2]$ and so on. Thus, if $f \in \mathbb{C}^2$ then

$$f \in \Delta^{(2)}(Y_{2s}) \Leftrightarrow f''(x)\Pi(x) \geqslant 0,\ x \in \mathbb{R}, \quad \text{where} \quad \Pi(x) := \Pi(x,Y_{2s}) := \prod_{i=1}^{2s} \sin\frac{1}{2}(x-y_i)$$

($\Pi(x) > 0$, $x \in (y_1,y_0)$, $\Pi \in \mathbb{T}_s$). The functions from $\Delta^{(2)}(Y_{2s})$ are called *piecewise convex* or *coconvex* (each other or between themselves), and the approximation of them by polynomials also from $\Delta^{(2)}(Y_{2s})$ is called coconvex approximation.

Denote by

$$E_n^{(2)}(f,Y_{2s}) := \inf_{P_n \in \mathbb{T}_n \cap \Delta^{(2)}(Y_{2s})} \|f - P_n\|$$

the value (error) of the best uniform approximation of the function f by polynomials $P_n \in \mathbb{T}_n \cap \Delta^{(2)}(Y_{2s})$.

Thus, in [19] and [25] the following estimate (3) is proved.

If a function $f \in \Delta^{(2)}(Y_{2s})$, *then for each* $n \in \mathbb{N}$ *that is greater than some constant* $N(Y_{2s})$, *which depends only on* $\min\limits_{i=1,\dots,2s}\{y_i - y_{i+1}\}$, *the following inequalities hold*

$$E_n^{(2)}(f, Y_{2s}) \leqslant c(s)\,\omega_k(f, \pi/n), \quad k = 2, 3, \quad n > N(Y_{2s}), \tag{3}$$

where $c(s)$ *is a constant that depends only on* s.

Note that the following estimate (4) is a simple consequence of (3) and the Whitney inequality [23] $\|f - f(0)\| \leqslant 3\,\omega_k(f, 2\pi)$, $k \in \mathbb{N}$, ($f(0) \in \Delta^{(2)}(Y_{2s}) \cap \mathbb{T}_0$ and interpolates f, the constant 3 is obtained in the work of Gilewicz, Kraykin, Shevchuk [11]): *if* $f \in \Delta^{(2)}(Y_{2s})$, *then*

$$E_n^{(2)}(f, Y_{2s}) \leqslant C(Y_{2s})\,\omega_3\left(f, \pi/n\right), \quad n \in \mathbb{N}, \tag{4}$$

where $C(Y_{2s})$ *is a constant that depends only on* $\min\limits_{i=1,\dots,2s}\{y_i - y_{i+1}\}$.

Moreover, Zalizko [26] using considerations of the articles Shvedov [21] and DeVore, Leviatan, Shevchuk [3] *for each* $n \in \mathbb{N}$, *constructed a function* $g_n(x) = g_n(x, Y_2, k) \in \Delta^{(2)}(Y_2)$ *such that*

$$E_n^{(2)}(g_n, Y_2) \geqslant C(Y_2, k)\, n^{2\left(\frac{k}{3}-1\right)}\,\omega_k\left(g_n, \pi/n\right), \quad k \in \mathbb{N},\ k \geqslant 4,$$

where $C(Y_2, k)$ *is a constant that depands only on* Y_2 *(i.e. on* $y_2 - y_1$ *) and* k. In other words, for each $n \in \mathbb{N}$, he found a function from $\Delta^{(2)}(Y_2)$, for which the inequality (4) is false with ω_k, $k > 3$, i.e. it cannot be improved in the order of the modulus of smoothness (unlike the inequalities (1) and (2) of approximation without restrictions, which hold for all $k \in \mathbb{N}$).

In the sequel we will have positive constants $c = c(\cdot)$ and $N = N(\cdot)$ that depend only on the arguments in the parentheses.

In this paper we will prove that $N(Y_{2s})$ in (3) and $C(Y_{2s})$ in (4) cannot be replaced by constants that do not depend on Y_{2s} (but depend only on s). That is if $s \geqslant 1$, then even

$$E_n^{(2)}(f, Y_{2s}) \leqslant c(s)\omega_1(f, \pi/n), \quad n \geqslant N,$$

is not valid with $N = N(s)$ replacing $N = N(Y_{2s})$. In fact we prove more, namely,

THEOREM 1. *For every* $k \geqslant 1$, $r = 0, 1, 2, 3$ *and* $s \in \mathbb{N}$, *there do not exist constants* $c = c(k, r, s)$ *and* $N = N(k, r, s)$, *depending only on* k, r *and* s, *such that the inequality*

$$E_n^{(2)}(f, Y_{2s}) \leqslant \frac{c}{n^r}\,\omega_k(f^{(r)}, \pi/n), \tag{5}$$

holds for all $n \geqslant N$ *and for all* $f \in \mathbb{C}^r \cap \Delta^{(2)}(Y_{2s})$.

For $r = 0, 1$, Theorem 1 was proved by Popov [20]. Like in [20] the considerations in the proof of Theorem 1 are inspired by the paper of Leviatan and Shevchuk [16].

REMARK 1. We think that for smoothness of high orders (for $r > 3$) the statement of Theorem 1 is also true but we do not consider this question in the paper. It seems in this case another counterexample needs to be constructed.

We note that corresponding results on coconvex approximation in an interval by algebraic polynomials (including poinwise estimates of Nikolskii-type and interpolatory at the ends of the interval estimates) can be found in the papers of Kopotun, Leviatan, Shevchuk [14, 16], the author with Gilewicz, Leviatan, Shevchuk [4, 5, 6, 7, 8] with Zalizko [9] and in the papers of Wu, Zhou [24, 27].

We refer an interested reader to the very good (comprehensive and qualified) survey on most cases (without periodic, rational, spline and oneside) of Shape Preserving Approximation written by Kopotun, Leviatan, Prymak, Shevchuk [15] where all known positive and negative results in the field are collected with complete truth tables for the validity of Nikolskii-type and Jackson-type estimates, involving the ordinary k-th moduli of smoothness of the r-th derivative of a given function, as well as estimates involving the Ditzian-Totik moduli of smoothness. Many of the methods applied for the proofs of all positive results in these truth tables are modifications of similar ones in the papers by DeVore, Gilewicz, Kopotun, Leviatan, Mania, Shevchuk, Yu and the author (see References there).

2. Counterexample

We will use the well known Bernstein inequality [2] (1912)

$$\|P_n'\| \leqslant n\|P_n\|, \quad P_n \in \mathbb{T}_n.$$

Given $0 < b < 1$, define 2π-periodic even function g_b by setting it on $[-\pi,\pi]$ as

$$g_b(x) := \int_0^x (x-u)g_b''(u)du,$$

where

$$g_b''(x) := \begin{cases} -\dfrac{1}{b^4}(x^2-b^2)^2 + \dfrac{16}{15(1+b)}b, & |x| \leqslant b, \\ \dfrac{32b}{5(1+b)(1-b)^3}\left(\dfrac{|x|^3}{3} - \dfrac{1+b}{2}x^2 + b|x|\right) + \dfrac{16}{15(1+b)}b, & b < |x| \leqslant 1, \\ 0, & 1 < |x| \leqslant \pi. \end{cases}$$

Then it is readily seen that

$$\|g_b\| = -g_b(1) = -\left(\frac{1}{6} - \frac{b}{2(1+b)} + \frac{b^2}{6(1+b)}\right)\frac{16}{15}b < \frac{8}{45}b,$$

$$g_b'(x) = -\frac{1}{b^4}\left(\frac{x^5}{5} + \frac{2b^2}{3}x^3 + b^4x\right) + \frac{16b}{15(1+b)}x, \quad |x| \leqslant b,$$

$$g_b''(\pm x_0) = 0, \quad x_0 = b\sqrt{1-\sqrt{\frac{16b}{15(1+b)}}} < b, \tag{6}$$

$$\|g_b'\| = -g_b'(x_0) < \frac{2b}{3},$$
$$\|g_b''\| = 1 - \frac{16}{15(1+b)}b,$$
$$\|g_b^{(3)}\| = \frac{8}{3\sqrt{3}b} \leqslant \frac{2}{b}.$$

and clearly $g_b \in \mathbb{C}^3 \cap \Delta^{(2)}(\{-x_0, x_0\})$.

LEMMA 1. *Given $n \geqslant 1$, for each polynomial $p_n \in \mathbb{T}_n$, satisfying*

$$\sin\frac{x-x_0}{2}\sin\frac{x+x_0}{2}p_n''(x) \geqslant 0, \quad x \in \Big[-\frac{1}{2}, \frac{1}{2}\Big],$$

with $b = b_n = \frac{1}{\sqrt[3]{n^4}}$, we have

$$\|g_b - p_n\| > \frac{7b}{120}.$$

Proof. First we observe that $p_n''(\pm x_0) = 0$, and that $p_n''(x) \leqslant 0$, for $-x_0 < x < x_0$. Assume that for some $-x_0 < x_* < x_0$, $p_n''(x_*) < -\frac{1}{6}$. Then

$$|[p_n''; -x_0, x_*, x_0]| = \frac{|p_n''(x_*)|}{(x_0 - x_*)(x_0 + x_*)} > \frac{1}{6x_0^2} > \frac{1}{6b^2},$$

where in the square brackets is the divided difference of second order

$$[f; t_0, t_1, t_2] := \frac{f(t_0)}{(t_0-t_1)(t_0-t_2)} + \frac{f(t_1)}{(t_1-t_0)(t_1-t_2)} + \frac{f(t_2)}{(t_2-t_0)(t_2-t_1)}$$

of f at points (knots of the divided difference) $t_0, t_1, t_2 \ : \ t_i \neq t_j$ if $i \neq j$. Since

$$[p_n''; -x_0, x_*, x_0] = \frac{1}{2}p_n^{(4)}(\theta),$$

for some $-x_0 < \theta < x_0 (< \frac{1}{20})$, it follows by the Bernstein inequality that

$$\frac{1}{2}n^4\|p_n\| \geqslant \frac{1}{2}|p_n^{(4)}(\theta)| > \frac{1}{6b^2}.$$

Now by (6) and the prescribed value of b,

$$\|g_b - p_n\| \geqslant \|p_n\| - \|g_b\| > \frac{b}{3n^4b^3} - \frac{8b}{45} = \frac{7b}{45}. \tag{7}$$

If on the other hand, $p_n''(x) \geqslant -\frac{1}{6}$, for all $-x_0 < x < x_0$, then we represent p_n in the form

$$p_n(x) = p_n(0) + xp_n'(0) + \int_0^x (x-u)p_n''(u)\,du.$$

Since $p_n''(x) \geqslant 0$ for $x_0 \leqslant |x| \leqslant \frac{1}{2}$, it follows that

$$\begin{aligned} p_n\Big(-\frac{1}{2}\Big) - 2p_n(0) + p_n\Big(\frac{1}{2}\Big) &= \int_0^{\frac{1}{2}} \Big(\frac{1}{2}-u\Big)p_n''(u)\,du + \int_0^{-\frac{1}{2}} \Big(-\frac{1}{2}-u\Big)p_n''(u)\,du \\ &> \int_0^{x_0} \Big(\frac{1}{2}-u\Big)p_n''(u)\,du + \int_0^{x_0} \Big(\frac{1}{2}-u\Big)p_n''(-u)\,du \\ &> -\frac{x_0}{6} > -\frac{b}{6}. \end{aligned}$$

Directly estimate

$$\begin{aligned} g_b\Big(-\frac{1}{2}\Big) - 2g_b(0) + g_b\Big(\frac{1}{2}\Big) &= 2\left(\Big(\int_0^b + \int_b^{\frac{1}{2}}\Big)\Big(\frac{1}{2}-u\Big)g_b''(u)\,du\right) \\ &\leqslant -\frac{2b}{5}. \end{aligned}$$

Therefore

$$\begin{aligned} 4\|g_b - p_n\| &\geqslant \Big(p_n\Big(-\frac{1}{2}\Big) - g_b\Big(-\frac{1}{2}\Big)\Big) - 2(p_n(0) - g_b(0)) + \Big(p_n\Big(\frac{1}{2}\Big) - g_b(\frac{1}{2})\Big) \\ &\geqslant -\frac{b}{6} + \frac{2b}{5} = \frac{7b}{30}. \end{aligned}$$

Thus together with (7), this concludes the proof of Lemma 1.

As an immediate consequence we get

EXAMPLE 1. For every constant $A > 1$ there exists an $N(A)$ sufficiently large such that if $n > N(A)$, then for any $s \geqslant 1$, there is a function $g = g(x,n) \in \mathbb{C}^3$, which changes convexity $2s$ times in $[-\pi,\pi)$, and such that any polynomial $p_n \in \mathbb{T}_n$ which is coconvex with it, satisfies

$$\|g - p_n\| > \frac{A\|g^{(3)}\|}{n^3},$$

$$\|g - p_n\| > \frac{A\|g''\|}{n^2},$$

and

$$\|g - p_n\| > \frac{A\|g'\|}{n}.$$

Proof. Let $N(A) = (48A)^3$ and let $s \geqslant 1$. We take $b = b_n = \frac{1}{\sqrt[3]{n^4}}$, $n > N(A)$, as in Lemma 1, and let $g = g_b$. The function g changes convexity at $y_2 = -x_0$ and

$y_1 = x_0$, it is convex in $[y_1, \pi)$, and if $s > 1$, then we take $2(s-1)$ arbitrary points satisfying $-\pi \leqslant y_{2s} < \cdots < y_3 < -1$, and regard g as changing convexity at these points too, hence $g \in \Delta^2(Y_{2s})$. If the polynomial p_n is coconvex with g, then it satisfies the requirements of Lemma 1. Therefore, by Lemma 1 and (6) we have

$$\|g - p_n\| > \frac{7b}{120} = \frac{7\|g^{(3)}\| 3\sqrt{3} n^{\frac{1}{3}}}{n^{\frac{8}{3}} n^{\frac{1}{3}} 960} > \frac{A\|g^{(3)}\|}{n^3},$$

$$\|g - p_n\| > \frac{7b}{120} > \frac{7b\|g''\|}{120} > \frac{A\|g''\|}{n^2},$$

and

$$\|g - p_n\| > \frac{7b}{120} > \frac{21n\|g'\|}{240n} > \frac{A\|g'\|}{n}.$$

Example 1 is proved.

REMARK 2. It should be noted that the function g above is independent of A.

We are ready to prove Theorem 1.

Proof of Theorem 1. The proof readily follows from the observation that for all $k \geqslant 1$,

$$\omega_k(f,t) \leqslant 2^{k-1}\omega_1(f,t) \leqslant 2^{k-1} t \|f'\|,$$

which by Example 1 does not allow the case $r = 0$ in (5) and

$$\omega_k(f,t) \leqslant 2^k \|f\|,$$

which takes care of the other cases and completes the proof.

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