Mathematical
Inequalities
& Applications
Volume 24, Number 1 (2021), 209–217 doi:10.7153/mia-2021-24-15

MEAN–TYPE MAPPINGS AND INVARIANCE PRINCIPLE

JANUSZ MATKOWSKI AND PAWEŁ PASTECZKA

(*Communicated by C. P. Niculescu*)

Abstract. In the finite dimensional case, mean-type mappings, their invariant means, relations between the uniqueness of invariant means and convergence of orbits of the mapping, are considered. In particular it is shown, that the uniqueness of an invariance mean implies the convergence of all orbits. A strongly irregular mean-type mapping is constructed and its unique invariant mean is determined. An application in solving a functional equation is presented.

1. Introduction

We deal with mean-type mappings, invariant means with respect to the mean-type mappings, relations between the uniqueness of invariant means and convergence of the orbits of the mean-type mappings, in general finite dimensional case.

The main result of section 2, Theorem 1, says that, without any regularity conditions, the orbits of the mean-type mapping converge if, and only if, the mean-type map has a unique invariant mean. In particular, the uniqueness of invariance mean implies the convergence of the orbits, and each coordinate of the limit mean-type map is just the invariant mean. This result generalizes the suitable result in [7] where two-dimensional case is considered.

In section 3 we show that the continuity of the mean-type mapping together with its weak contractivity are sufficient conditions for the uniqueness of the invariant mean.

In iteration theory of mean-type mappings, the continuity of the invariant mean was assumed to guarantee its uniqueness (see for example [1, 4, 5] and [9, p. 134, Theorem 83]). In a recent paper [6], basing on the fact that every mean is continuous on the main diagonal of its domain, it was shown that this continuity assumption is redundant.

In section 4, making use of the discontinuous additive functions, we construct a mean-type mapping, which is discontinuous at every point outside of the diagonal and, applying Theorem 1, we show that the arithmetic mean is a unique invariant mean for it.

In section 5, again applying Theorem 1, we find all continuous functions which are invariant with respect to a given mean-type mapping and we show that in this case the assumption of continuity is indispensable.

Mathematics subject classification (2010): Primary 26E60, 39B22; Secondary 39B12.
Keywords and phrases: Means, mean-type mapping, invariant mean, functional equation, iteration, orbit.

2. Invariance principle

Let an interval $I \subset \mathbb{R}$ and $p \in \mathbb{N}$ be fixed.

A function $M : I^p \to I$ is called a mean in I if it is internal, that is if

$$\min(v_1,\ldots,v_p) \leqslant M(v_1,\ldots,v_p) \leqslant \max(v_1,\ldots,v_p), \qquad v_1,\ldots,v_p \in I,$$

or, briefly, if

$$\min v \leqslant M(v) \leqslant \max v, \qquad v \in I^p.$$

In the sequel, to avoid the trivial results, we assume that $p > 1$.

A mapping $\mathbf{M}\colon I^p \to I^p$ is referred to as *mean-type* if there exists some means $M_i\colon I^p \to I$, $i = 1,\ldots,p$, such that $\mathbf{M} = (M_1,\ldots,M_p)$.

We say that a function $K\colon I^p \to \mathbb{R}$ is invariant with respect to $\mathbf{M}$ (briefly $\mathbf{M}$-invariant), if $K \circ \mathbf{M} = K$.

Now, following the idea from [8], for a mean-type mapping $\mathbf{M}\colon I^p \to I^p$ we define an orbit $\mathscr{O}_{\mathbf{M}} : I^p \to (I^p)^\infty$ by

$$\mathscr{O}_{\mathbf{M}}(v) := \big(v, \mathbf{M}(v), \mathbf{M}^2(v), \ldots\big)$$

where $\mathbf{M}^n$ is the n-th iterate of $\mathbf{M}$, $n = 0,1,\ldots$. In view of well known isomorphism $(X^Y)^Z \sim X^{Y\times Z}$, the function $\mathscr{O}^*_{\mathbf{M}}\colon I^p \to I^\infty$ is given by

$$\mathscr{O}^*_{\mathbf{M}}(v) := \big(v_1,\ldots,v_p,[\mathbf{M}(v)]_1,\ldots,[\mathbf{M}(v)]_p,[\mathbf{M}^2(v)]_1,\ldots,[\mathbf{M}^2(v)]_p,\ldots\big),$$

where $[\mathbf{M}^n(v)]_i$ stands for the i-th coordinate of the vector $\mathbf{M}^n(v)$, $i \in \{1,\ldots,p\}$.

By $\ell^\infty(I)$ denote the set of all bounded sequences $a = (a_1,a_2,\ldots)$ with values in an interval I.

For $p \in \mathbb{N}$ a function $\phi\colon \ell^\infty(I) \to I$ is called *p-limit-like* if, for every $a = (a_1,a_2,\ldots) \in \ell^\infty(I)$, the following two conditions hold

(i) $\phi(a_1,a_2,a_3,\ldots) = \phi(a_{p+1},a_{p+2},a_{p+3},\ldots)$, and

(ii) $\liminf_{n\to\infty} a_n \leqslant \phi(a_1,a_2,\ldots) \leqslant \limsup_{n\to\infty} a_n$.

Note that whenever the sequence a is convergent, then $\phi(a) = \lim_{n\to\infty} a_n$.

PROPOSITION 1. *Let $\mathbf{M}\colon I^p \to I^p$ be a mean-type mapping, and $\phi\colon \ell^\infty(I) \to I$ be a p-limit-like function. Then the function $\mathscr{M}_\phi\colon I^p \to \mathbb{R}$ given by $\mathscr{M}_\phi := \phi \circ \mathscr{O}^*_{\mathbf{M}}$ is a mean on I, which is $\mathbf{M}$-invariant.*

Conversely, every $\mathbf{M}$-invariant mean equals $\mathscr{M}_\phi$ for some p-limit-like function ϕ.

Proof. By the definition of mean the sequence $(\max \mathbf{M}^n(v))_{n\in\mathbb{N}}$ is nondecreasing and

$$\limsup \mathscr{O}^*_{\mathbf{M}}(v) = \limsup_{n\to\infty} \max \mathbf{M}^n(v) \leqslant \max \mathbf{M}^0(v) = \max(v).$$

Similarly we obtain $\liminf \mathscr{O}^*_{\mathbf{M}}(v) \geqslant \min(v)$. Now, as ϕ is between liminf and limsup, we obtain that $\mathscr{M}_\phi$ is a mean. Moreover

$$\begin{aligned}\mathscr{M}_\phi \circ \mathbf{M}(v) &= \phi \circ \mathscr{O}^*_{\mathbf{M}}(\mathbf{M}(v)) = \phi\left([\mathbf{M}(v)]_1,\ldots,[\mathbf{M}(v)]_p,[\mathbf{M}^2(v)]_1,\ldots,[\mathbf{M}^2(v)]_p,\ldots\right)\\ &= \phi\left(v_1,\ldots,v_p,[\mathbf{M}(v)]_1,\ldots,[\mathbf{M}(v)]_p,[\mathbf{M}^2(v)]_1,\ldots,[\mathbf{M}^2(v)]_p,\ldots\right)\\ &= \phi \circ \mathscr{O}^*_{\mathbf{M}}(v) = \mathscr{M}_\phi(v)\end{aligned}$$

which concludes the proof.

To prove the converse, for an arbitrary $\mathbf{M}$-invariant mean K, we define the function ϕ on the every orbit $\mathscr{O}^*_{\mathbf{M}}(v)$ by

$$\phi(\mathscr{O}^*_{\mathbf{M}}(v)) := K(v) \qquad v \in I^p, \tag{1}$$

For every $v \in I^p$ we have $\phi(\mathscr{O}^*_{\mathbf{M}}(v)) = K(v) = K \circ \mathbf{M}(v) = \phi(\mathscr{O}^*_{\mathbf{M}}(\mathbf{M}(v)))$ what implies that ϕ satisfies (i) on the image $\mathscr{O}^*_{\mathbf{M}}(I^p)$.

To preserve the p-limit-like properly we need to extend this definition to

$$\phi(\mathscr{O}^*_{\mathbf{M}}(v)) := K(v), \qquad v \in I^p \tag{2}$$

$$\phi(I^{cp} \times \mathscr{O}^*_{\mathbf{M}}(v)) := K(v), \qquad v \in I^p,\ c \in \{1,2,\ldots\}. \tag{3}$$

We underline that if for some $w \in I^p$ and $c_0 > 0$ we get $\mathscr{O}^*_{\mathbf{M}}(w) \in I^{c_0 p} \times \mathscr{O}^*_{\mathbf{M}}(v)$ then by the definition $\mathbf{M}^{c_0}(w) = v$ and, consequently, $K(v) = K(w)$. Therefore definitions (1) and (3) are coherent. Moreover, it is easy to check that the set

$$\Gamma := \bigcup_{v \in I^p} \Big(\big\{ \mathscr{O}^*_{\mathbf{M}}(v) \big\} \cup \bigcup_{c=1}^{\infty} I^{cp} \times \mathscr{O}^*_{\mathbf{M}}(v) \Big) \subset \ell^\infty(I)$$

is closed under shifting by p elements (both left and right). Thus so is the set $\Gamma' := \ell^\infty(I) \setminus \Gamma$. Furthermore properties (i) and (ii) are valid on $\phi|_\Gamma$.

As the value of ϕ on Γ' does not affect the value of $\mathscr{M}_\phi$, we can define it in any way, just to keep validity of (i) and (ii) e.g. $\phi(a) := \liminf a$ for $a \in \Gamma'$.

COROLLARY 1. *Let* $\mathbf{M}\colon I^p \to I^p$ *be a mean-type mapping. Then* $\mathscr{L} := \mathscr{M}_{\liminf}$ *and* $\mathscr{U} := \mathscr{M}_{\limsup}$ *are the smallest and the biggest* $\mathbf{M}$*-invariant means, respectively.*

REMARK 1. Let us underline that analogous corollary can be also established if $\mathbb{M}$ is a selfmapping of compactly supported Borel measures (see [2] for details).

THEOREM 1. (Invariance Principle) *Let* $\mathbf{M}\colon I^p \to I^p$ *be a mean-type mapping and* $K : I^p \to I$ *be an arbitrary mean.* K *is a unique* $\mathbf{M}$*-invariant mean if and only if the sequence of iterates* $(\mathbf{M}^n)_{n\in\mathbb{N}}$ *of the mean-type mapping* $\mathbf{M}$ *converges to* $\mathbf{K} := (K,\ldots,K)$ *pointwise on* I^p.

Proof. We have the following equivalent conditions:

$$\begin{aligned}
& K \text{ is a unique } \mathbf{M}\text{-invariant mean} \\
\iff\quad & K = \mathscr{L} = \mathscr{U} \\
\iff\quad & K(v) = \liminf \mathscr{O}^*_{\mathbf{M}}(v) = \limsup \mathscr{O}^*_{\mathbf{M}}(v) \text{ for all } v \in I^p \\
\iff\quad & \mathscr{O}^*_{\mathbf{M}}(v) \text{ is convergent to } K(v) \text{ for all } v \in I^p \\
\iff\quad & \mathscr{O}_{\mathbf{M}}(v) \text{ is convergent to } \mathbf{K}(v) \text{ for all } v \in I^p \\
\iff\quad & \mathscr{O}_{\mathbf{M}} \text{ is convergent to } \mathbf{K} \text{ pointwise on } I^p \\
\iff\quad & \mathbf{M}^n \text{ is convergent to } \mathbf{K} \text{ pointwise on } I^p,
\end{aligned}$$

thus the proof is complete.

3. Weakly contractive mean-type mappings

We say that a mean-type mapping $\mathbf{M}\colon I^p \to I^p$ is *weakly contractive* if for every nonconstant vector $v \in I^p$ there is a positive integer $n_0(v)$ such that

$$\max(\mathbf{M}^n(v)) - \min(\mathbf{M}^n(v)) < \max(v) - \min(v) \qquad \text{for all } n \geqslant n_0(v).$$

Let us emphasize that it is sufficient to verify if the inequality above is valid for $n = n_0(v)$. Moreover in a special case $p = 2$ it was proved [7] that $\mathbf{M}$ is weakly contractive if and only if $\mathbf{M}^2$ is contractive. This is not the case here.

Even for $p = 3$ we can construct weakly contractive mean-type mapping on I^3 such that the function $I^p \ni v \mapsto n_0(v)$ is unbounded.

EXAMPLE 1. Take a continuous and weakly-contractive mean-type mapping $\mathbf{M}_0\colon I^3 \to I^3$ such that

$$\mathbf{M}_0(v_1,v_2,v_3) = \begin{cases} \left(v_1, v_2, \frac{v_1+v_3}{2}\right) & \text{if } |v_3 - v_1| = (v_2 - v_1)^2; \\ (v_1,v_1,v_1) & \text{if } 2\,|v_3 - v_1| = (v_2 - v_1)^2, \end{cases}$$

and the set $\Lambda := \{(v_1,v_2,v_3) \in I^3\colon\ |v_3 - v_1| \geqslant (v_2 - v_1)^2\}$. Define a mapping $\mathbf{M}\colon I^3 \to I^3$ by

$$\mathbf{M}(v_1,v_2,v_3) := \begin{cases} \left(v_1, v_2, \frac{v_1+v_3}{2}\right) & \text{if } (v_1,v_2,v_3) \in \Lambda; \\ \mathbf{M}_0\,(v_1,v_2,v_3) & \text{otherwise.} \end{cases}$$

Obviosly $\mathbf{M}$ is continuous. Moreover for every $x,y,z \in I$ there exists $n_1(x,y,z)$ such that $\mathbf{M}^{n_1(x,y,z)} \in \Lambda$. Thus $\mathbf{M}^{n_1(x,y,z)+k}(x,y,z) = \mathbf{M}_0^k(\mathbf{M}^{n_1(x,y,z)})$. Consequently as $\mathbf{M}_0$ is weakly contractive, so is $\mathbf{M}$.

Now take $x \in I$ and $i \in \mathbb{N}$ such that $x + 2^{-i} \in I$. By simple induction we can describe the $\mathbf{M}$-orbit of the vector $w := (x, x+2^{-i}, x+2^{-i}) \in I^3$. Namely

$$\mathbf{M}^n(w) = \mathbf{M}^n(x, x+2^{-i}, x+2^{-i}) = \begin{cases} (x, x+2^{-i}, x+2^{-i-n}) & \text{for } n \leqslant i+1 \\ (x,x,x) & \text{for } n > i+1. \end{cases}$$

This equality proves that for all $n < i$ we have

$$\max(\mathbf{M}^n(w)) - \min(\mathbf{M}^n(w)) = \max(w) - \min(w)$$

Thus $n_0(w) \geqslant i$. As i can be take arbitrary large (obviously w depends on i) we have that n_0 cannot be bounded.

THEOREM 2. *If $\mathbf{M} : I^p \to I^p$ is a continuous, weakly contractive mean-type mapping then there exists a unique $\mathbf{M}$-invariant mean $K\colon I^p \to I$. Moreover, the sequence of iterates $(\mathbf{M}^n)_{n\in\mathbb{N}}$ converges (pointwise on I^p) to $\mathbf{K} := (K,\dots,K)$.*

Proof. Assume that I is closed.

First observe that in view of Theorem 1 the moreover part is equivalent to our assertion. Second, it is sufficient to prove that $\mathscr{L} = \mathscr{U}$. Assume to the contrary that $\mathscr{L}(v_0) \neq \mathscr{U}(v_0)$ for some $v_0 \in I^p$.

Define a spread $\delta := \mathscr{U}(v_0) - \mathscr{L}(v_0) > 0$ and sets

$$\begin{aligned} X_0 &:= \{v \in I^p :\ \max(v) - \min(v) \geqslant \delta\}, \\ X_k &:= \{v \in I^p :\ \max(v) - \min(v) \in [\delta, \delta + \tfrac{1}{k}]\} \qquad \text{for } k \in \mathbb{N}_+, \\ X_\omega &:= \bigcap_{k=0}^{\infty} X_k = \{v \in I^p :\ \max(v) - \min(v) = \delta\}. \end{aligned}$$

Then for every $k \in \mathbb{N}$ there exists n_k such that $\mathbf{M}^n(v_0) \in X_k$ for all $n > n_k$.

As X_0 is compact, there exists a subsequence (m_k) such that $\mathbf{M}^{m_k}(v_0) \in X_k$ and the sequence $(\mathbf{M}^{m_k}(v_0))$ is convergent to some element $w_0 \in I^p$. By the definition, as the difference $\max \mathbf{M}^{m_k}(v_0) - \min \mathbf{M}^{m_k}(v_0)$ is nonincreasing, we have $w_0 \in X_\omega$.

As $\mathbf{M}$ is weakly contractive, there exists $s_0 \in \mathbb{N}$ such that

$$\max \mathbf{M}^{s_0}(w_0) - \min \mathbf{M}^{s_0}(w_0) < \max(w_0) - \min(w_0) = \delta.$$

By the continuity of $\mathbf{M}$ there exists an open neighbourhood $W \ni w_0$ such that

$$\max \mathbf{M}^{s_0}(w) - \min \mathbf{M}^{s_0}(w) < \delta \qquad \text{for all } w \in W.$$

But, by the definition, there exists $k_0 \in \mathbb{N}$ such that $\mathbf{M}^{m_{k_0}}(v_0) \in W$. Then

$$\max \mathbf{M}^{s_0 + m_{k_0}}(v_0) - \min \mathbf{M}^{s_0 + m_{k_0}}(v_0) < \delta. \tag{4}$$

On the other hand, as both $\mathscr{L}$ and $\mathscr{U}$ are $\mathbf{M}$-invariant means, we have

$$\begin{aligned} \min \mathbf{M}^{s_0 + m_{k_0}}(v_0) &\leqslant \mathscr{L} \circ \mathbf{M}^{s_0 + m_{k_0}}(v_0) = \mathscr{L}(v_0), \\ \max \mathbf{M}^{s_0 + m_{k_0}}(v_0) &\geqslant \mathscr{U} \circ \mathbf{M}^{s_0 + m_{k_0}}(v_0) = \mathscr{U}(v_0) \end{aligned}$$

which implies

$$\max \mathbf{M}^{s_0 + m_{k_0}}(v_0) - \min \mathbf{M}^{s_0 + m_{k_0}}(v_0) \geqslant \mathscr{U}(v_0) - \mathscr{L}(v_0) = \delta,$$

contradicting (4).

4. Examples of highly discontinuous means

EXAMPLE 2. For a discontinuous additive function $\alpha : \mathbb{R} \to \mathbb{R}$ define a function $\lambda_\alpha : \mathbb{R} \to \mathbb{R}$ by

$$\lambda_\alpha(u) := \frac{3|\alpha(u)| + 3}{4|\alpha(u)| + 12}, \quad u \in \mathbb{R}.$$

Since

$$\frac{1}{4} \leqslant \lambda_\alpha(u) < \frac{3}{4}, \qquad u \in \mathbb{R}, \tag{5}$$

the functions $M,N:\mathbb{R}^2\to\mathbb{R}$ defined by

$$M(u,v)=\lambda_\alpha(u)u+(1-\lambda_\alpha(u))v, \qquad N(u,v)=(1-\lambda_\alpha(u))u+\lambda_\alpha(u)v.$$

are means in $\mathbb{R}$. Both M and N, being the means, are continuous at every point of the diagonal $\Delta:=\{(x,x):x\in\mathbb{R}\}$ (see [6]) but, as the graph of α is dense in $\mathbb{R}^2$ (see, for instance [3]), these functions are strongly irregular in $\mathbb{R}^2\backslash\Delta$ (in particular they are discontinuous at every point outside of Δ).

Note that

$$A\circ(M,N)=A,$$

i.e., the arithmetic mean $A:\mathbb{R}^2\to\mathbb{R}$, $A(u,v)=\frac{u+v}{2}$, is invariant with respect to the mean-type mapping $(M,N):\mathbb{R}^2\to\mathbb{R}^2$.

To show that A is a unique (M,N)-invariant mean, first observe that, for all $u,v\in\mathbb{R}$,

$$|M(u,v)-N(u,v)|=|2\lambda_\alpha(u)-1|\,|u-v|,$$

whence, in view of (5),

$$|M(u,v)-N(u,v)|\leqslant\frac{1}{2}|u-v|, \qquad u,v\in\mathbb{R}.$$

Putting $(M_n,N_n):=(M,N)^n$, $n\in\mathbb{N}_0$, we hence get

$$|M_{n+1}(u,v)-N_{n+1}(u,v)|\leqslant\frac{1}{2}|M_n(u,v)-N_n(u,v)|, \qquad u,v\in\mathbb{R},\ n\in\mathbb{N}_0,$$

whence, by induction,

$$|M_n(u,v)-N_n(u,v)|\leqslant\frac{1}{2^n}|u-v|, \qquad u,v\in\mathbb{R},\ n\in\mathbb{N}.$$

This proves that, for every point $(u,v)\in\mathbb{R}^2$, the orbit

$$\mathscr{O}_{\mathbf{M}}((u,v))=((M_n(u,v),N_n(u,v)):n\in\mathbb{N}_0)$$

approaches the diagonal as $n\to\infty$. To show the convergence of the orbit, take arbitrary $(u,v)\in\mathbb{R}^2\backslash\Delta$ and put $c=A(u,v)=\frac{u+v}{2}$, so we have $v=2c-u$. The invariance of A with respect to (M,N) implies that $N(u,v)=2c-M(u,v)$, and, by induction, $N_n(u,v)=2c-M_n(u,v)$ for all $n\in\mathbb{N}_0$, that is, every point $(M_n(u,v),N_n(u,v))$ lays on the straight-line crossing perpendicularly the diagonal Δ at the point (c,c). It follows that

$$\lim_{n\to\infty}(M_n(u,v),N_n(u,v))=(c.c).$$

Applying Theorem 1 we conclude that A is a unique (M,N)-invariant mean.

The result considered in this example can be easily extended to the following

PROPOSITION 2. *Let* $\alpha : \mathbb{R} \to \mathbb{R}$ *be a discontinuous additive function,* $\kappa \in (0,1)$ *and* b,c,d *be real numbers such that*

$$c > 1,\ 0 < b \leqslant d, \quad \frac{2}{1+\kappa} \leqslant c \leqslant \frac{2}{1-\kappa}, \quad \frac{2b}{1+\kappa} \leqslant d \leqslant \frac{2b}{1-\kappa},$$

and let $\lambda_\alpha : \mathbb{R} \to \mathbb{R}$ *be defined by*

$$\lambda_\alpha(u) := \frac{|\alpha(u)| + b}{c|\alpha(u)| + d}, \quad u \in \mathbb{R}.$$

Then

(i) the functions $M,N : \mathbb{R}^2 \to \mathbb{R}$ *defined by*

$$M(u,v) = \lambda_\alpha(u)u + (1-\lambda_\alpha(u))v, \qquad N(u,v) = (1-\lambda_\alpha(u))u + \lambda_\alpha(u)v,$$

are means in $\mathbb{R}$*;*

(ii) the means M,N *are continuous only at the points of the diagonal* $\Delta := \{(x,x) : x \in \mathbb{R}\}$ *and*

$$|M(u,v) - N(u,v)| \leqslant \kappa|u-v|, \qquad u,v \in \mathbb{R};$$

(iii) the arithmetic mean $A : \mathbb{R}^2 \to \mathbb{R}$, $A(u,v) = \frac{u+v}{2}$, *is a unique* (M,N)*-invariant and*

$$\lim_{n\to\infty} (M,N)^n = (A,A) \qquad \textit{(pointwise).}$$

REMARK 2. This result remains true on replacing $\alpha(u)$ by $\alpha(f(u,v))$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is an arbitrary nonconstant regular function.

REMARK 3. It is not difficult to observe that the above proposition can be modified to a result in which A is replaced by an arbitrary quasi-arithmetic mean.

5. An applications in solving a functional equation

Applying Theorem 1 we prove the following

THEOREM 3. *Assume that* $\mathbf{M} : I^p \to I^p$ *is a mean-type mapping and* $K : I^p \to I$ *is its unique* $\mathbf{M}$*-invariant mean. A function* $F : I^p \to \mathbb{R}$ *which is continuous on the diagonal* $\Delta(I^p) := \{(u_1,\dots,u_p) \in I^p : u_1 = \dots = u_p\}$ *is invariant with respect to the mean-type mapping, i.e.* F *satisfies the functional equation*

$$F \circ \mathbf{M} = F, \tag{6}$$

if, and only if, there is a continuous function $\varphi : I \to \mathbb{R}$ *such that*

$$F = \varphi \circ K.$$

Proof. Assume first that $F : I^p \to \mathbb{R}$ that is continuous on the diagonal $\Delta(I)$ and F satisfies (6). From (6) by induction we get

$$F = F \circ \mathbf{M}^n, \quad n \in \mathbb{N}_0.$$

By Theorem 1 the sequence of mean-type mappings $(\mathbf{M}^n)_{n\in\mathbb{N}_0}$ converges pointwise to the mean-type mapping $\mathbf{K} = (K, \ldots, K) : I^p \to I^p$. Since F is continuous on the diagonal $\Delta(I)$, we hence get, for all $u = (u_1, \ldots, u_p) \in I^p$,

$$\begin{aligned} F(u_1, \ldots, u_p) &= \lim_{n\to\infty} F(\mathbf{M}^n(u_1, \ldots, u_p)) = F\left(\lim_{n\to\infty} (\mathbf{M}^n(u_1, \ldots, u_p))\right) \\ &= F(\mathbf{K}(u_1, \ldots, u_p)) = F((K(u_1, \ldots, u_p), \ldots, K(u_1, \ldots, u_p))), \end{aligned}$$

whence, setting

$$\varphi(t) := F(t, \ldots, t), \quad t \in I,$$

we obtain $F(u_1, \ldots, u_p) = \varphi(K(u_1, \ldots, u_p))$ for all $(u_1, \ldots, u_p) \in I^p$, that is $F = \varphi \circ K$.

To prove the converse implication, take an arbitrary function $\varphi : I \to \mathbb{R}$ and put $F := \varphi \circ K$. Then we have

$$F \circ \mathbf{M} = (\varphi \circ K) \circ \mathbf{M} = \varphi \circ (K \circ \mathbf{M}) = \varphi \circ K = F,$$

which completes the proof.

REMARK 4. The assumption of the continuity of the restriction of the function F on the diagonal $\Delta(I^p)$ is essential.

To show it take arbitrary (not necessarily continuous) function $\varphi : I \to \mathbb{R}$ and define $F : I^p \to \mathbb{R}$ by

$$F(u_1, \ldots, u_p) := \varphi(t) \quad \text{if} \quad \lim_{n\to\infty} \mathbf{M}^n(u_1, \ldots, u_p) = (t, \ldots, t).$$

Since $\lim_{n\to\infty} \mathbf{M}^n(u_1, \ldots, u_p) = \lim_{n\to\infty} \mathbf{M}^n(\mathbf{M}(u_1, \ldots, u_p))$, we have for all $u \in I^p$,

$$F(u) = F(\mathbf{M}(u)).$$

REMARK 5. If F is a pre-mean then $\varphi = \mathrm{id}$ and consequently $F = K$. Therefore if there exists a uniquely determined $\mathbf{M}$-invariant mean then it is also the unique $\mathbf{M}$-invariant premean which is continuous on the diagonal.

REFERENCES

[1] J. M. BORWEIN, P. B. BORWEIN, *Pi and the AGM – a Study in Analytic Number Theory and Computational Complexity*, John Wiley & Sons, New York, 1987.

[2] B. DERĘGOWSKA, P. PASTECZKA, *Quasiarithmetic-type invariant means on probability space*, arXiv:2005.02063 [math.FA] (2020).

[3] M. KUCZMA, *An introduction to the theory of functional equations and inequalities. Cauchy's equation and Jensen's inequality*, Prace Naukowe Uniwersytetu Śląskiego w Katowicach [Scientific Publications of the University of Silesia], 489. Uniwersytet Śląski, Katowice; Państwowe Wydawnictwo Naukowe (PWN), Warsaw, 1985.

[4] J. MATKOWSKI, *Iterations of mean-type mappings and invariant means*, European Conference on Iteration Theory (Muszyna-Złockie, 1998). Ann. Math. Sil. No. **13** (1999), 211–226.

[5] J. MATKOWSKI, *Iterations of the mean-type mappings*, Iteration theory (ECIT '08) Grazer Math. Ber., **354**, Institut für Mathematik, Karl-Franzens-Universität Graz, 2009. 158–179.

[6] J. MATKOWSKI, *Iterations of the mean-type mappings and uniqueness of invariant means*, Ann. Univ. Sci. Budapest. Sect. Comput. **41** (2013), 145–158.

[7] J. MATKOWSKI, P. PASTECZKA, *Invariant means and iterates of mean-type mappings*, Aequationes Math. **94** (2020), 405–414.

[8] P. PASTECZKA, *Invariant property for discontinuous mean-type mappings*, Publ. Math. Debrecen. **94** (2019), 409–419.

[9] G. TOADER, I. COSTIN, *Means in Mathematical Analysis. Bivariate Means*, Mathematical Analysis and Its Applications, Academic Press, London, 2018.

(Received May 20, 2020)

Janusz Matkowski
Institute of Mathematics
University of Zielona Góra
Szafrana 4a, PL-65-516 Zielona Góra, Poland
e-mail: j.matkowski@wmie.uz.zgora.pl

Paweł Pasteczka
Institute of Mathematics
Pedagogical University of Krakow
Podchorążych 2, PL-30-084 Kraków, Poland
e-mail: pawel.pasteczka@up.krakow.pl