Mathematical
Inequalities
& Applications
Volume 24, Number 1 (2021), 219–234 doi:10.7153/mia-2021-24-16

TRANSFERENCE METHOD FOR CONE–LIKE RESTRICTED SUMMABILITY OF THE TWO–DIMENSIONAL WALSH–LIKE SYSTEMS

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(*Communicated by T. Erdélyi*)

Abstract. In the present paper we investigate the boundedness of the maximal operator of some d-dimensional means, provided that the set of the indeces is inside a cone-like set L. Applying some assumptions on the summation kernels $P_{n_1,\ldots,n_d}$ we state that the cone-like restricted maximal operator T^{γ}_{CLR} is bounded from the Hardy space H^{γ}_p to the Lebesgue space L^p for $p > p_0$. In the end point p_0 assuming some natural conditions on one-dimensional kernels we show that the maximal operator T^{γ}_{CLR} is not bounded from the Hardy space $H^{\gamma}_{p_0}$ to the Lebesgue space L^{p_0}.

1. Definitions and notation

We follow the standard notions of dyadic analysis introduced by Schipp, Simon, Wade and Pál in [20] (see also [1]). Let $\mathbb{N}$ denote the set of natural numbers and let $\mathbb{P} := \mathbb{N}\backslash\{0\}$. The cyclic group of order 2 will be denoted by $\mathbb{Z}_2$. The topology is given by that every subset is open. The Haar measure on $\mathbb{Z}_2$ is given such that

$$\mu(\{0\}) = \mu(\{1\}) = 1/2.$$

Let G be the complete direct product of countable infinite copies of the compact group $\mathbb{Z}_2$. The elements of G are sequences of the form $x = (x_0, x_1, \ldots, x_k, \ldots)$ with components $x_k \in \{0,1\}\,(k \in \mathbb{N})$. On G the group operation is the component-wise addition, the measure μ is the product measure and the topology is the product topology. Such compact Abelian group G is called the Walsh group.

A base for the neighbourhoods of G can be given by

$$I_0(x) := G,$$
$$I_n(x) := I_n(x_0, \ldots, x_{n-1}) := \{y \in G : y = (x_0, \ldots, x_{n-1}, y_n, y_{n+1}, \ldots)\},$$

$(x \in G, n \in \mathbb{N})$, $I_n(x)$ are called dyadic intervals. Let $0 = (0 : i \in \mathbb{N}) \in G$ denote the null element of G, and for the simplicity we write $I_n := I_n(0)$ $(n \in \mathbb{N})$. Set $e_n := (0, \ldots, 0, 1, 0, \ldots) \in G$, the nth component of which is 1 and the rest are zeros.

Mathematics subject classification (2010): 42C10, 43A75, 42B08, 42B30.
Keywords and phrases: Walsh-Paley system, Walsh-Kaczmarz system, maximal operator, multi-dimensional system, restricted summability, almost everywhere convergence, Cesàro means, group of 2-adic integers, Hardy space.
Research supported by project UAEU UPAR 2017 Grant G00002599
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The Fine's map $\|.\|: G \to [0,1]$ is defined by

$$\|x\| := \sum_{n=0}^{\infty} x_n 2^{-(n+1)}. \tag{1}$$

Backwards, each $x \in [0,1[$ can be expressed in number system based 2 in the form

$$x = \sum_{j=0}^{\infty} x_j 2^{-j-1}, \qquad \text{where } x_j \in \{0,1\} \text{ for all } j.$$

This expansion is unique except for dyadic rational numbers $x \in \{\frac{p}{2^n} : p,n \in \mathbb{P}\}$. In this case we choose the expansion which terminates in 0 's. For $n \in \mathbb{N}$ let n_k be the kth coordinate of n with respect to number system based 2. That is, we write

$$n = \sum_{k=0}^{\infty} n_k 2^k,$$

where $n_k \in \{0,1\}$ $k \in \mathbb{N}$. We use the notation $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$, that is $2^{|n|} \leqslant n < 2^{|n|+1}$, where $|n|$ is called the order of natural number n.

Let r_k denote the k-th Rademacher function, it is defined by

$$r_k(x) := (-1)^{x_k} \quad (k \in \mathbb{N},\ x \in G).$$

The Walsh-Paley system (simply we say Walsh system) is defined as the product system of Rademacher functions. Namely,

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum\limits_{k=0}^{|n|-1} n_k x_k} \quad (x \in G, n \in \mathbb{P}).$$

The Walsh-Kaczmarz functions are defined by $\kappa_0 := 1$ and for $n \geqslant 1$

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}}.$$

The Walsh-Kaczmarz system was introduced by Šneider [26] in 1948. Some basic result with respect to Walsh-Kaczmarz system can be found in [3, 21, 23, 24, 25, 29], for current results see also [12, 16, 27, 28, 35]. We give more details later.

It is known that the set of Walsh-Kaczmarz functions and the set of Walsh-Paley functions are equal in dyadic blocks. Namely,

$$\{\kappa_n : 2^k \leqslant n < 2^{k+1}\} = \{w_n : 2^k \leqslant n < 2^{k+1}\}$$

for all $k \in \mathbb{P}$. Moreover, $\kappa_0 = w_0$.

The relation between the Walsh-Paley and Walsh-Kaczmarz system is not a simple relation, it is given by a coordinate transformation (for more details see [25] written by V.A. Skvortsov).

For both system we define the one-dimensional Dirichlet kernels and Cesàro kernels (see [10, 24, 35, 36]) by

$$D_n^{\psi} := \sum_{k=0}^{n-1} \psi_k, \quad K_n^{\psi,\alpha}(x) := \frac{1}{A_n^{\alpha}} \sum_{k=1}^{n} A_{n-k}^{\alpha-1} D_k^{\psi}(x),$$

($\psi_n = w_n$ $(n\in\mathbb{N})$ or $\psi_n = \kappa_n$ $(n\in\mathbb{N})$, and $0<\alpha$), where

$$A_j^\alpha := \binom{j+\alpha}{j} = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+j)}{j!} \quad (j\in\mathbb{N};\ \alpha\neq -1,-2,\dots).$$

Choosing $\alpha = 1$ we get back the Fejér kernels

$$K_n^\psi = K_n^{\psi,1} = \frac{1}{n}\sum_{k=1}^{n} D_k^\psi = \sum_{k=0}^{n-1}\left(1-\frac{k}{n}\right)\psi_k.$$

These functions has some good properties, useful in the following investigations. First, we mention a simple result with respect to the Dirichlet kernels, which play a central role in the Walsh-Fourier analysis (see [20]):

$$D_{2^n}^w(x) = D_{2^n}^\kappa(x) = D_{2^n}(x) = \begin{cases} 0, & \text{if } x\notin I_n, \\ 2^n, & \text{if } x\in I_n. \end{cases} \tag{2}$$

For functions $f,g\in L^1(G)$ the dyadic convolution $f*g$ is defined by

$$(f*g)(x) := \int_G f(t)g(x+t)d\mu(t).$$

For $f\in L^1$ let us denote the nth partial sum of f by $S_n^\psi(f)$, the nth Fejér means of f by $\sigma_n^\psi(f)$ and the nth (C,α) mean of f by $\sigma_n^{\psi,\alpha}(f)$. It is clear that $S_n^\psi(f) = f*D_n^\psi$, $\sigma_n^\psi(f) = f*K_n^\psi$ and $\sigma_n^{\psi,\alpha}(f) = f*K_n^{\psi,\alpha}$ $(n\in\mathbb{N})$. We remark that the Fejér kernels and (C,α) kernels ($0<\alpha$) are uniformly bounded for both systems, that is,

$$\sup_n \|K_n^{w,\alpha}\|_1 < \infty \quad \text{and} \quad \sup_n \|K_n^{\kappa,\alpha}\|_1 < \infty$$

hold (see [3, 20, 22, 23, 24, 25]).

Further, we assume that the summation kernels $P_n := \sum_{k=0}^{n-1}\lambda_{n,k}\psi_k$ with real coefficients $\lambda_{n,k}$ $(n,k\in\mathbb{N})$ (here $\{\psi_k : k\in\mathbb{N}\}$ denotes the Walsh-Paley or the Walsh-Kaczmarz system) satisfy the inequality

$$\sup_n \|P_n\|_1 < \infty. \tag{3}$$

If we consider the maximal operator

$$T(f) := \sup_n |f*P_n| \quad (f\in L^1(G)), \tag{4}$$

then $T: L^\infty(G)\to L^\infty(G)$ is evidently bounded.

The σ-algebra generated by the dyadic intervals $I_n(x)$ $(x\in G)$ will be denoted by $\mathbf{F}_n$ $(n\in\mathbb{N})$. Let us denote by $f = \left(f^{(n)}, n\in\mathbb{N}\right)$ a martingale with respect to $(\mathbf{F}_n, n\in\mathbb{N})$ (for details see, e. g. [31]). The maximal function of a martingale f is defined by

$$f^* = \sup_{n\in\mathbb{N}}\left|f^{(n)}\right|.$$

In case $f \in L^1$, the maximal function can also be given by

$$f^*(x) = \sup_{n\in\mathbb{N}} \frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u)\, d\mu(u) \right|, \quad x \in G.$$

For $0 < p \leqslant \infty$ the Hardy martingale space H_p consists of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f \in L^1$, then it is easily seen that the sequence $(S_{2^n}(f) : n \in \mathbb{N})$ is a martingale. If f is a martingale, that is, $f = (f^{(0)}, f^{(1)}, \ldots)$ then the Fourier coefficients with respect to both systems Walsh-Paley and Walsh-Kaczmarz, must be defined in a little bit different way:

$$\widehat{f}^{\psi}(i) = \lim_{k\to\infty} \int_G f^{(k)}(x)\, \psi_i(x)\, d\mu(x).$$

The Fourier coefficients of $f \in L^1$ are the same as the ones of the martingale $(S_{2^n}(f) : n \in \mathbb{N})$ obtained from f. The atomic decomposition is a useful characterization of the Hardy space H_p. Let $0 < p \leqslant 1$. A bounded measurable function a is a p-atom, if either a is identically equal to 1, or there exists a dyadic interval I for which

a) $\operatorname{supp} a \subseteq I$,

b) $\|a\|_\infty \leqslant \mu(I)^{-1/p}$,

c) $\int_I a\, d\mu = 0$.

We say that the atom a is supported on the dyadic interval I. Then a martingale $f = (f_n : n \in \mathbb{N})$ belongs to the Hardy space H_p $(0 < p \leqslant 1)$ if and only if there exists a sequence $(\lambda_k : k \in \mathbb{N})$ of real numbers such that $\sum_{k=0}^{\infty} |\lambda_k|^p < \infty$ and

$$f = \sum_{k=0}^{\infty} \lambda_k a_k. \tag{5}$$

Moreover, the following equivalence of norms (quasi-norms) holds:

$$c_p \|f\|_{H_p} \leqslant \inf \left(\sum_{k=0}^{\infty} |\lambda_k|^p \right)^{1/p} \leqslant C_p \|f\|_{H_p} \quad (f \in H_p),$$

where the infimum is taken over all decompositions of f of the form (5). We note that here and later c_p and C_p, denote positive constants depending only on p, although not always the same in different occurrences.

In this paper we use the next Lemma of Weisz [31]:

LEMMA 1. (Weisz [31]) *Suppose that the operator T is σ-sublinear and p-quasi-local for some $0 < p < 1$. If T is bounded from L^∞ to L^∞, then*

$$\|Tf\|_p \leqslant c_p \|f\|_{H_p} \quad \textit{for all } f \in H_p.$$

The Kronecker product $\left(\psi_n : n \in \mathbb{N}^d\right)$ of d Walsh-(Kaczmarz) system is said to be the d-dimensional Walsh-(Kaczmarz) system. That is,

$$\psi_n(x) = \psi_{n_1}\left(x^1\right)\ldots\psi_{n_d}\left(x^d\right),$$

where $n = (n_1, \ldots, n_d)$ and $x = (x^1, \ldots, x^d)$.

If $f \in L^1\left(G^d\right)$, then the number $\widehat{f}^{\psi}(n) := \int_{G^d} f\psi_n \quad \left(n \in \mathbb{N}^d\right)$ is said to be the nth Walsh-(Kaczmarz)-Fourier coefficient of f. We can extend this definition to martingales in the usual way (see Weisz [30, 31]).

For $x = (x^1, \ldots, x^d) \in G^d$ and $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$ the d-dimensional rectangles are defined by $I_n(x) := I_{n_1}(x^1) \times \ldots \times I_{n_d}(x^d)$. For $n \in \mathbb{N}^d$ the σ-algebra generated by the rectangles $\{I_n(x), x \in G^d\}$ is denoted by $\mathscr{F}_n$. The conditional expectation operators relative to $\mathscr{F}_n$ are denoted by E_n (with $n = (n_1, \ldots, n_d)$).

Suppose that the functions $\gamma_j\colon [1,+\infty) \to [1,+\infty)$ are strictly monotone increasing continuous functions with properties $\lim_{x\to+\infty} \gamma_j(x) = +\infty$ and $\gamma_j(1) = 1$ for all $j = 2, \ldots, d$. Moreover, suppose that for $j = 2, \ldots, d$ there exist $\zeta, c_{j,1}, c_{j,2} > 1$ such that the inequality

$$c_{j,1}\gamma_j(x) \leqslant \gamma_j(\zeta x) \leqslant c_{j,2}\gamma_j(x) \tag{6}$$

holds for each $x \geqslant 1$. In this case the functions γ_j are called CRF (cone-like restriction functions) [4, 8]. Let us introduce the notion $\gamma := (\gamma_2, \ldots, \gamma_d)$ and set $\beta_j \geqslant 1$ be fixed for $j = 2, \ldots, d$. We define the d-dimensional cone-like set L (with respect to the first dimension) by

$$L := \{n \in \mathbb{N}^d : \gamma_j(n_1)/\beta_j \leqslant n_j \leqslant \beta_j\gamma_j(n_1), j = 2, \ldots, d\}.$$

The d-dimensional Dirichlet kernels, Fejér kernels and (C,α) kernels can be given as the Kronecker product of one-dimensional kernels. That is,

$$D_n^{\psi}(x) = D_{n_1}^{\psi}(x^1)\ldots D_{n_d}^{\psi}(x^d), \quad K_n^{\psi}(x) = K_{n_1}^{\psi}(x^1)\ldots K_{n_d}^{\psi}(x^d),$$

$$K_n^{\psi,\alpha}(x) = K_{n_1}^{\psi,\alpha_1}(x^1)\ldots K_{n_d}^{\psi,\alpha_d}(x^d),$$

where $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$, $x = (x^1, \ldots, x^d) \in G^d$, and $\alpha = (\alpha_1, \ldots, \alpha_d)$.

In the present paper we investigate the boundedness of the maximal operator of some d-dimensional means, provided that the set of the indeces is inside a cone-like set L and the convergence over a cone-like set L. Namely, we consider the d-parameter analogue of T (see (4)) as follows. Let $P_n = P_{n_1,\ldots,n_d}$ ($n = (n_1, \ldots, n_d) \in \mathbb{N}^d$) be the Kronecker product of summation kernels $P_{n_1}^1, \ldots, P_{n_d}^d$, that is $P_{n_1,\ldots,n_d}(x^1, \ldots, x^d) := P_{n_1}^1(x^1)\ldots P_{n_d}^d(x^d)$. For a fixed CRF function γ we define the cone-like restricted maximal operator T_{CLR}^{γ} by

$$T_{CLR}^{\gamma}(f) := \sup_{n\in L} |f * P_n| \quad (f \in L^1(G^d)). \tag{7}$$

If γ is the identical function (in each coordinate) then we get a d-dimensional cone. The cone-like sets were introduced by Gát in [4]. The condition (6) on the function γ is natural, because Gát [4] proved that to each cone-like set with respect to the first dimension there exists a larger cone-like set with respect to the second dimension and reversely, if and only if the inequality (6) holds.

Connecting to the work of Gát [4] Weisz defined a new type martingale Hardy space depending on the function γ (see [32]). For a given $n_1 \in \mathbb{N}$ let us set $n_j := \lfloor \gamma_j(2^{n_1}) \rfloor$, that is, n_j is the order of $\gamma_j(2^{n_1})$ (this means that $2^{n_j} \leqslant \gamma_j(2^{n_1}) < 2^{n_j+1}$) for $j = 2,\dots,d$. Let us set $\overline{n}_1 := (n_1,\dots,n_d)$. Since, the function γ is increasing (for each coordinate function γ_j), the sequence $(\overline{n}_1,\ n_1 \in \mathbb{N})$ is increasing (in each coordinate), as well. A class of one-parameter martingales $f = (f_{\overline{n}_1},\ n_1 \in \mathbb{N})$ is given with respect to the σ-algebras $(\mathscr{F}_{\overline{n_1}},\ n_1 \in \mathbb{N})$. The maximal function of a martingale f is defined by $f^* = \sup\limits_{n_1 \in \mathbb{N}} |f_{\overline{n}_1}|$. For $0 < p \leqslant \infty$ the martingale Hardy space $H_p^\gamma(G^d)$ consists of all martingales for which the L_p-norm of the maximal function f^* is finite, that is, $\|f\|_{H_p^\gamma} := \|f^*\|_p < \infty$. It is known that $H_p^\gamma \sim L^p$ for $1 < p \leqslant \infty$, where $\sim$ denotes the equivalence of the norms and spaces (see [31]).

If $f \in L^1(G^d)$ then it is easily shown that the sequence $(S_{2^{n_1},\dots,2^{n_d}}(f) : \overline{n_1} = (n_1,\dots,n_d), n_1 \in \mathbb{N})$ is a one-parameter martingale with respect to the σ-algebras $(\mathscr{F}_{\overline{n_1}},\ n_1 \in \mathbb{N})$. In this case the maximal function can also be given by

$$f^*(x) = \sup_{n_1 \in \mathbb{N}} \frac{1}{\operatorname{mes}(I_{\overline{n_1}}(x))} \left| \int_{I_{\overline{n_1}}(x)} f(u)\,d\mu(u) \right| = \sup_{n_1 \in \mathbb{N}} |S_{2^{n_1},\dots,2^{n_d}}(f,x)|$$

for $x \in G^d$. The Hardy space H_p^γ has atomic structure also. The atoms a are supported on the dyadic rectangles I from the σ-algebras $\mathscr{F}_{\overline{n_1}}$.

If γ is the identical function (in each coordinate) then L is a d-dimensional cone. In the two-dimensional case, using a two-dimensional cone restriction set, the properties of the maximal operator of Walsh-Paley-Fejér means was discussed by Gát [5] and Weisz [33], separately. Later, the cone restricted maximal operator of two-dimensional Walsh-Kaczmarz-Fejér means was investigated by Simon [22]. Namely, they showed that the maximal operator σ^* of Fejér means is bounded from the Hardy space H_p to the Lebesgue space L^p for $p > 1/2$. It was shown also that the end point $p = 1/2$ is essential. Further properties in the end point $p = \frac{1}{2}$ was discussed later. Namely, in 2007 Goginava and the first author proved that the cone-restricted maximal operator is not bounded from the Hardy space $H_{1/2}$ to the space weak-$L^{1/2}$ [13, 14].

Connecting to the original paper [4] on trigonometric system, Gát asked the following. ”What could we state for other systems for example Walsh-Paley, Walsh-Kaczmarz and Vilenkin systems and for other means for example logarithmic means, Riesz means, (C,α) means?” Some parts of Gát's question was answered by Weisz [32], Blahota and the authors [2, 17, 18, 19] and naturally in paper [8]. Moreover, there are some results in papers [5, 8] about divergence of two-dimensional Fejér means. Namely, if we suppose that β is a function and not just a constant, then we have two cases. Either β is bounded, then we have a.e. convergence for each integrable function, or β is not bounded, then the maximal convergence space is $L\log^+ L$.

In 2011, the properties of the maximal operator of the (C,α) and Riesz means of multi-dimensional Vilenkin-Fourier series with cone-like restriction set, was discussed by Weisz [32]. Namely, it was shown that the maximal operator is bounded from dyadic

Hardy space H_p to the space L^p for $p_0 < p \leqslant \infty$ ($p_0 := \max\{1/(1+\alpha_k) : k = 1,\dots,d\}$) and is of weak type $(1,1)$. Recently, it was shown that the index p_0 is sharp. Namely, it was proven that the maximal operator is not bounded from the dyadic Hardy space H_{p_0} to the space L^{p_0} [2] (see also [19]). A detailed list of the reached results for one- and several dimensional Walsh-like systems can be found in [34]. For Walsh-Kaczmarz system the properties of cone-like restricted two-dimensional maximal operator of Fejér and (C,α) means was discussed in [17, 18]. Namely, it is proven that the maximal operator is bounded from dyadic Hardy space H_p to the Lebesgue space L_p for $p_0 < p \leqslant \infty$ (with the same p_0 as Weisz showed) and is of weak type $(1,1)$. Moreover, at the end point $p = p_0$, it is showed that the maximal operator $\sigma_L^{\kappa,\alpha,*}$ is not bounded from the Hardy space $H_{p_0}^{\gamma}$ to the space L^{p_0}.

Our work is motivated by the work of the authors [17, 18] mentioned above and the paper of Simon [22]. In the last paper Simon improved a so called transference method. Namely, he considered the maximal operator T of a sequence of summations (see (4)) and showed that the p-quasi-locality of T implies the same statement for its two-dimensional version T_{CR}^{Id} where Id notes that the CRF function γ is the identical function (in each coordinate, see (7), as well). The main aim of this paper is to extend the transference method of Simon for all CRF functions γ and cone-like sets L defined by γ. We mention that our proof is mainly based on the papers [18, 22], we generalize the method improved in [18] and in the same time we extend the method of paper [22] for cone-like sets. Applying Lemma 1 of Weisz and some assumptions on the summation kernels $P_{n_1,\dots,n_d}$ we state that the maximal operator T_{CLR}^{γ} is bounded from the Hardy space H_p^{γ} to the Lebesgue space L^p for $p > p_0$. In the end point p_0 assuming some natural conditions on one-dimensional kernels we show that the maximal operator T_{CLR}^{γ} is not bounded from the Hardy space $H_{p_0}^{\gamma}$ to the Lebesgue space L^{p_0}. After proving our main theorems we give an application, we get some new results unknown until the present days.

2. Improved transference method

In the one-dimensional case to apply the previous Lemma of Weisz (see Lemma 1) we show usually that the operator $T : L^\infty(G) \to L^\infty(G)$ in question is bounded. It immediately follows from inequality (3). The operator T is called p-quasi-local if for arbitrary p-atom a supported on the dyadic interval I the inequality

$$\int_{G\setminus I} |T(a)|^p d\mu \leqslant C_p \tag{8}$$

holds [30]. Hence, p-quasi-locality together with (L^∞, L^∞)-boundedness of the σ-sublinear operator T implies that the operator T is bounded from the Hardy space H_p to the Lebesgue space L^p.

If $I = I_n(x)$ is a one-dimensional dyadic interval for some $x \in G$ and $n \in \mathbb{N}$, then for all $r = 0,1,\dots,n$ we define I^r by $I^r := I_{n-r}(x)$. Furthermore, the definition of p-quasi-locality of operator T can be modified as follows: there exists $r = 0,1,\dots$ such that

$$\int_{G\setminus I^r} |T(a)|^p d\mu \leqslant C_p \tag{9}$$

holds for every p-atom a with support I. Analogical idea can be applied in multi-dimensional case. In most of the proofs of the p-quasi-locality we could realize the inequality

$$\int_{G\setminus I_n} \left(\sup_{n\geqslant 2^n} \int_{I_n} |P_n(x+t)| d\mu(t) \right)^p d\mu(x) \leqslant C_p 2^{-N} \quad (n\in\mathbb{N}) \tag{10}$$

which immediately implies inequalities (8) or (9) (see [22]).

In the next Theorem, we prove that the inequality (10) for one-dimensional kernels $P^i_{n_i} := \sum_{k=0}^{n_i-1} \lambda^i_{n_i,k}\psi_k$ $(i=1,\dots,d)$ (that is the p-quasi-locality of the one-dimensional operators $T^1,\dots,T^d$ defined by the kernels $P^1_{n_1},\dots P^d_{n_d}$, respectively) implies the same property for cone-like restricted operator T^γ_{CLR}.

THEOREM 1. *Let the function γ be CRF. Assume that the kernels $P^i_{n_i}$ satisfy the inequalities* (3) *and* (10) *for all $0<p_i<p\leqslant 1$ $(i=1,\dots,d)$. Then the maximal operator T^γ_{CLR} is bounded from the Hardy space H^γ_p to the Lebesgue space L^p for $p_0<p\leqslant 1$ (where $p_0 := \max\{p_1,\dots,p_d\}$). Moreover, the maximal operator T^γ_{CLR} is of weak type (1,1).*

Proof. The operator T^γ_{CLR} is bounded from the space L^∞ to the space L^∞. It immediately follows from the inequality (3). Moreover, it can be seen easily that the operator T^γ_{CLR} is σ-sublinear.

Let a be a p-atom. Let it be supported on the dyadic rectangle I. Without loss of generality we can assume that $I = I_{N_1}\times\dots\times I_{N_d}$ (with $N_j := |\gamma_j(2^{N_1})|$, $j=2,\dots,d$). The atom a satisfies $\|a\|_\infty \leqslant 2^{(N_1+\dots+N_d)/p}$ and $\int_I a d\mu = 0$. Furthermore, it follows that $a * P_n = 0$ $(n=(n_1,\dots,n_d))$, when $n_j < 2^{N_j}$ for $j=1,\dots,d$.

In the next steps, we use the following inequality and the monotonicity of CRF functions γ_j $(j=2,\dots,d)$.

$$c^l_{j,1}\gamma_j\left(\frac{2^{N_1}}{\zeta^l}\right) \leqslant \gamma_j(2^{N_1}) = \gamma_j\left(\frac{2^{N_1}}{\zeta^l}\zeta^l\right) \leqslant c^l_{j,2}\gamma_j\left(\frac{2^{N_1}}{\zeta^l}\right)$$

holds for all $l\in\mathbb{P}$ $(j=2,\dots,d)$. That is,

$$\frac{\gamma_j(2^{N_1})}{c^l_{j,2}} \leqslant \gamma_j\left(\frac{2^{N_1}}{\zeta^l}\right) \leqslant \frac{\gamma_j(2^{N_1})}{c^l_{j,1}} \quad (j=2,\dots,d). \tag{11}$$

First, we apply the right side of inequality (11) for any positive real number x,

$$\gamma_j\left(\frac{2^{N_1}}{\zeta^x}\right) \leqslant \gamma_j\left(\frac{2^{N_1}}{\zeta^{[x]}}\right) \leqslant \frac{\gamma_j(2^{N_1})}{c^{[x]}_{j,1}} \leqslant \frac{c_{j,1}\gamma_j(2^{N_1})}{c^x_{j,1}} \quad (j=2,\dots,d),$$

where $[x]$ denotes the integer part of x. Now, let us set $\delta := \max\{\zeta^{\log_{c_{j,1}} 2\beta_j+1} : j = 2,\dots,d\}$, as we did in paper [18].

If $n_1 \leqslant 2^{N_1}/\delta$, then

$$n_j \leqslant \beta_j\gamma_j(n_1) \leqslant \beta_j\gamma_j(2^{N_1}\zeta^{-\log_{c_{j,1}}2\beta_j-1}) \leqslant \beta_j \frac{c_{j,1}}{c_{j,1}^{\log_{c_{j,1}}2\beta_j+1}}\gamma_j(2^{N_1}) \leqslant \frac{\gamma_j(2^{N_1})}{2} < 2^{N_j}.$$

$\zeta, c_{j,1}, c_{j,2} > 1$, $\beta_j \geqslant 1$ imply $n_1 < 2^{N_1}$ and $n_j \leqslant \gamma_j(2^{N_1})/2 < 2^{N_j}$ $(j=2,\ldots,d)$. This yields $a * P_n = 0$ for $n = (n_1,\ldots,n_d)$.

That is, we could suppose that $n_1 > 2^{N_1}/\delta$. Now, we apply the left side of inequality (11) for any positive real number x,

$$\gamma_j\left(\frac{2^{N_1}}{\zeta^x}\right) \geqslant \gamma_j\left(\frac{2^{N_1}}{\zeta^{\lceil x\rceil}}\right) \geqslant \frac{\gamma_j(2^{N_1})}{c_{j,2}^{\lceil x\rceil}} \geqslant \frac{\gamma_j(2^{N_1})}{c_{j,2}^{x+1}} \quad (j=2,\ldots,d),$$

where $\lceil x\rceil$ denotes the upper integer part of x. This yields that

$$n_j \geqslant \frac{\gamma_j(n_1)}{\beta_j} \geqslant \frac{\gamma_j(2^{N_1}/\delta)}{\beta_j} \geqslant \frac{1}{\beta_j c_{j,2}^{\max\{\log_{c_{j,1}}2\beta_j+1:j=2,\ldots,d\}+1}}\gamma_j(2^{N_1}) \geqslant \frac{\gamma_j(2^{N_1})}{\delta'_j} \geqslant \frac{2^{N_j}}{\delta'}$$

with $\delta' := \max_{j=2,\ldots,d}\delta'_j$ for all $j=2,\ldots,d$. $\delta' > 1$ can be assumed.

The proof will be complete, if we show that the maximal operator T^{γ}_{CLR} satisfies the modified version of p-quasi-locality (9) for $p_0 < p \leqslant 1$, where $p_0 := \max\{p_i : i=1,\ldots,d\}$. For a p-atom a in H^{γ}_p with support $I = I_{N_1}\times\ldots\times I_{N_d}$ (with $\overline{N_1} = (N_1,\ldots,N_d)$) the multidimensional version of inequality (9) reads as follows: There exist a constant $c_p > 0$ and $r = 0,1,\ldots,$ such that

$$\int_{\overline{I^r}} |T^{\gamma}_{CLR}(a)|^p\,d\mu \leqslant c_p < \infty \tag{12}$$

holds for each atom a, where $I^r := I^r_{N_1}\times\ldots\times I^r_{N_d} := I_{N_1-r}\times\ldots\times I_{N_d-r}$ ($N_j - r \geqslant 0$ for all $j=1,\ldots,d$). We will give the value of r later.

Let us set $x = (x^1,\ldots,x^d) \in \overline{I^r}$.

$$\begin{aligned}|(a*P_n)(x)| &= \left|\int_I a(t^1,\ldots,t^d)P^1_{n_1}(x^1+t^1)\ldots P^d_{n_d}(x^d+t^d)\,d\mu(t)\right| \\ &\leqslant 2^{(N_1+\ldots+N_d)/p}\int_{I_{N_1}}|P^1_{n_1}(x^1+t^1)|\,d\mu(t^1)\ldots\int_{I_{N_d}}|P^d_{n_d}(x^d+t^d)|\,d\mu(t^d) \quad (n\in\mathbb{N}^d).\end{aligned}$$

Now, we decompose the set $\overline{I^r} = \overline{I^r_{N_1}\times\ldots\times I^r_{N_d}}$ as the following disjoint union (see [18] or for $d=2$ [22])

$$\begin{aligned}\overline{I^r} =& (\overline{I^r_{N_1}}\times\ldots\times\overline{I^r_{N_d}})\cup \\ &\cup(I^r_{N_1}\times\overline{I^r_{N_2}}\times\ldots\times\overline{I^r_{N_d}})\cup\ldots\cup(\overline{I^r_{N_1}}\times\ldots\times\overline{I^r_{N_{d-1}}}\times I^r_{N_d})\cup \\ &\vdots \\ &\cup(\overline{I^r_{N_1}}\times I^r_{N_2}\times\ldots\times I^r_{N_d})\cup\ldots\cup(I^r_{N_1}\times\ldots\times I^r_{N_{d-1}}\times\overline{I^r_{N_d}}).\end{aligned} \tag{13}$$

Let us set $\delta'' := \max\{\delta,\delta'\}$ and set $r\in\mathbb{P}$ such that $2^{-r} \leqslant 1/\delta'' < 2^{-r+1}$. Moreover, we set $L^{r,l} := I^r_{N_1} \times \ldots \times I^r_{N_l} \times \overline{I^r_{N_{l+1}}} \times \ldots \times \overline{I^r_{N_d}}$ for $l = 0, \ldots, d-1$. Let us define

$$J_i := \int_{I^r_{N_i}} \left(\sup_{n_i \geqslant 2^{N_i}/\delta''} \int_{I_{N_i}} |P^i_{n_i}(x^i + t^i)| d\mu(t^i) \right)^p \mu(x^i), \quad i = 1, \ldots, l,$$

$$\overline{J_j} := \int_{\overline{I^r_{N_j}}} \left(\sup_{n_j \geqslant 2^{N_j}/\delta''} \int_{I_{N_j}} |P^j_{n_j}(x^j + t^j)| d\mu(t^j) \right)^p \mu(x^j), \quad j = l+1, \ldots, d.$$

We immediately get

$$\int_{L^{r,l}} |T^\gamma_{CLR}(a)|^p d\mu \leqslant 2^{N_1 + \ldots + N_d} J_1 \cdot \ldots \cdot J_l \cdot \overline{J_{l+1}} \cdot \ldots \cdot \overline{J_d}. \tag{14}$$

First, we discuss the integrals J_i $(i = 1, \ldots, l)$. Inequality (3) and the definitions of δ'', r immediately yield that

$$J_i \leqslant 2^{-(N_i - r)} \left(\sup_{n_1 \in \mathbb{N}} \|P^i_{n_i}\|_1 \right)^p \leqslant c_p 2^{-N_i} \quad (i = 1, \ldots, l). \tag{15}$$

Second, we discuss the integrals $\overline{J_j}$ $(j = l+1, \ldots, d)$. Inequality (10) implies

$$\overline{J_j} \leqslant \int_{\overline{I^r_{N_j}}} \left(\sup_{n_j \geqslant 2^{N_j - r}} \int_{I^r_{N_j}} |P^j_{n_j}(x^j + t^j)| d\mu(t^j) \right)^p \mu(x^j) \leqslant c_p 2^{-N_j}, \text{ if } p > p_j \tag{16}$$

$(j = l+1, \ldots, d)$. Inequalities (14)–(16) yield

$$\int_{L^{r,l}} |T^\gamma_{CLR}(a)|^p d\mu \leqslant c_p \quad \text{if } p > p_0$$

for all $l = 0, \ldots, d-1$. The decomposition (13) of $\overline{I^r}$ written above gives

$$\int_{\overline{I^r}} |T^\gamma_{CLR}(a)|^p d\mu \leqslant c_p 2^d \quad \text{if } p > p_0.$$

Lemma 1 completes the proof of Theorem 1. The maximal operator T^γ_{CLR} is of weak type $(1,1)$ follows by Marcinkiewicz interpolation theorem.

It follows that the a.e. convergence holds for polynomials, they form a dense subset of L^1. Thus, by standard argument the next Corollary holds.

COROLLARY 1. *Let γ be CRF and L be a cone-like set. Let $f \in L^1(G^d)$. Assume that the kernels $P^i_{n_i}$ satisfy the inequalities (3) and (10) for all $0 < p_i < p \leqslant 1$ $(i = 1, \ldots, d)$. Then*

$$\lim_{\substack{\wedge n \to \infty \\ n \in L}} T_n(f) = f$$

holds almost everywhere.

Fejér and (C, α) means were investigated with respect to Walsh-Paley sysetem in papers [8, 32] and Walsh-Kaczmarz system in [18].

In many cases the following question arises. Is the border point p_0 sharp or not? In the next Theorem, we prove that if we could construct a one-dimensional counterexample martingale sequence in $H_{p_0}(G)$ which shows that the one-dimensional maximal operator T is not bounded from the Hardy space H_{p_0} to the space L^{p_0}. Then this enable us to construct a new d-dimensional counterexample martingale sequence in $H^{\gamma}_{p_0}$, as well, which has the same property from the Hardy space $H^{\gamma}_{p_0}$ to the space L^{p_0}.

In many papers the one-dimensional martingale

$$f_A(x) := D_{2^{A+1}}(x) - D_{2^A}(x) \tag{17}$$

is applied as a counterexample martingale [9, 22]. Since,

$$\hat{f}^{\psi}_A(k) = \begin{cases} 1, & \text{if } k = 2^A, \dots, 2^{A+1}-1, \\ 0, & \text{otherwise} \end{cases}$$

and

$$S^{\psi}_j(f_A)(x) = \begin{cases} D^{\psi}_j(x) - D_{2^A}(x), & \text{if } j = 2^A+1, \dots, 2^{A+1}-1, \\ f_A(x), & \text{if } j \geqslant 2^{A+1}, \\ 0, & \text{otherwise.} \end{cases}$$

It could be concluded that

$$\|f_A\|_{H_p} = \|f^*_A\|_p = \|D_{2^A}\|_p = 2^{A(1-1/p)}$$

and $f \in H_p$. Moreover, f_A is a p-atom and satisfies

$$S_{2^{A+1}}(f_A) = f_A. \tag{18}$$

The next inequality usually is proved

$$\frac{\|T(f_n)\|_{p_0}}{\|f_n\|_{H_{p_0}}} \geqslant a_n, \tag{19}$$

where T is defined by one-dimensional summation kernels P_n (see (4)) and (a_n) is a positive real valued sequence, which tends to $+\infty$ monotone increasingly. Inequality (19) and $\lim_{n\to+\infty} a_n = +\infty$ yield that the maximal operator T is not bounded from the Hardy space H_{p_0} to the space L^{p_0}. In the end point p_0 assuming some natural conditions for one-dimensional kernels in the next Theorem, we show that the maximal operator T^{γ}_{CLR} is not bounded from the Hardy space $H^{\gamma}_{p_0}$ to the Lebesgue space L^{p_0}. Although, we have to require conditions not only for the kernels $P^1_{n_1}$, but for kernels $P^l_{n_l}$ $(l = 2, \dots, d)$, as well. Namely, there exist positive constants c^*_l such that

$$|P^l_{n_l} * \psi_{2^k-1}| = |\lambda^l_{n_l, 2^k-1}| \geqslant c^*_l \tag{20}$$

hold for all $n_l \geqslant 2^{k+1}$ $(l = 2, \dots, d)$. Inequality (20) hold automatically, if the kernel functions $P^l_{n_l}$ are the Fejér kernels $K^{\psi}_{n_l}$ or the (C, α) kernels $K^{\psi,\alpha}_{n_l}$. In the first case $c^*_l = \frac{1}{2}$ and in the second case it is easily seen that there exist positive constants c^*_l such that

$$\frac{1}{A^{\alpha}_{n_l}} \sum_{i=2^k}^{n_l-1} A^{\alpha-1}_{n_l-i} \geqslant c^*_l > 0 \text{ hold for all } n_l \geqslant 2^{k+1}.$$

So, inequality (20) is a natural condition for many type of kernels $P^l_{n_l}$ $(l = 2, \dots, d)$.

THEOREM 2. *Let γ be CRF and $p_0 = p_1 \geqslant p_i$ ($i = 1, \dots, d$). Let f_A be a p_0-atom in $H_{p_0}(G)$ with support I_A which satisfies inequalities (18) and (19) with a positive real valued sequence (a_n) which tends to $+\infty$ monotone increasingly. Moreover, the kernels $P^l_{n_l}$ satisfy inequality (20) for $l = 2, \dots, d$.*

Then the maximal operator T^γ_{CLR} is not bounded from the Hardy space $H^\gamma_{p_0}(G^d)$ to the space $L^{p_0}(G^d)$.

We note that the fact that f_A is a p_0-atom in $H_{p_0}(G)$ with support I_A and satisfies equality (18) determine f_A having the form of (17) multiplied by a constant $c \neq 0$, where $|c| \leqslant 1$. This yields that

$$\|f_A\|_{H_{p_0}} = \|f_A^*\|_{p_0} = \|f_A\|_{p_0}.$$

Proof. We note that some idea of this proof is coming from papers [11, 2]. Let $f_A \in H_{p_0}(G)$ such that it satisfies the conditions of our theorem. We define a d-dimensional martingale $F_{\overline{n_1}}$ in $H^\gamma_{p_0}(G^d)$ by

$$F_{\overline{n_1}}(x) := f_{n_1}(x^1) \prod_{j=2}^{d} \psi_{2^{n_j-1}-1}(x^j),$$

where $n_2, \dots, n_d$ is defined to n_1, earlier, that is, $\overline{n_1} = (n_1, \dots, n_d)$ and $x = (x^1, \dots, x^d) \in G^d$.

We have

$$\hat{F}^\psi_{\overline{n_1}}(k) = \begin{cases} \hat{f}^\psi_{n_1}(k_1), & \text{if } k_j = 2^{n_j-1} - 1 \text{ for all } j = 2, \dots, d; \\ 0, & \text{otherwise.} \end{cases}$$

for $k = (k_1, \dots, k_d)$. Now, we calculate $S^\psi_j(F_{\overline{n_1}})$. Using equality (18) and the Fourier coefficients of $F_{\overline{n_1}}$ we may write

$$S^\psi_j(F_{\overline{n_1}}, x) = \begin{cases} S^\psi_{j_1}(f_{n_1}, x^1) \prod_{l=2}^d \psi_{2^{n_l-1}-1}(x^l), & \text{if } j_l \geqslant 2^{n_l-1} \text{ for all } l = 2, \dots, d; \;\; j_1 < 2^{n_1+1} \\ F_{\overline{n_1}}(x), & \text{if } j_1 \geqslant 2^{n_1+1} \text{ and } j_l \geqslant 2^{n_l-1} \\ & \text{for all } l = 2, \dots, d; \\ 0, & \text{otherwise.} \end{cases} \tag{21}$$

We immediately have that

$$F^*_{\overline{n_1}}(x) = \sup_{m_1 \in \mathbb{N}} |S_{2^{m_1}, \dots, 2^{m_d}}(F_{\overline{n_1}}, x)| = |F_{\overline{n_1}}(x)| = |f_{n_1}(x^1)|,$$

where $\overline{m_1} = (m_1, \dots, m_d)$. Moreover,

$$\|F_{\overline{n_1}}\|_{H^\gamma_{p_0}(G^d)} = \|F^*_{\overline{n_1}}\|_{p_0} = \|f^*_{n_1}\|_{p_0} = \|f_{n_1}\|_{H_{p_0}(G)} < \infty. \tag{22}$$

That is, $F_{\overline{n_1}} \in H^\gamma_{p_0}(G^d)$.

First, we set $L^N_1 := 2^{n_1} + N$, where $0 < N$ and $L^N_j := [\gamma_j(2^{n_1} + N)]$ for $j = 2, \dots, d$ (where $[x]$ denotes the integer part of x). In this case $L^N := (L^N_1, \dots, L^N_d) \in L$. Let us calculate $F_{\overline{n_1}} * P_{L^N}$.

$$(F_{\overline{n_1}} * P_{L^N})(x) = (f_{n_1} * P^1_{L^N_1})(x^1)\prod_{l=2}^{d}\lambda^l_{L^N_l,2^{n_l-1}-1}\psi_{2^{n_l-1}-1}(x^l)$$

From this and inequality (20) we write

$$\begin{aligned}|(F_{\overline{n_1}} * P_{L^N})(x)| &= |(f_{n_1} * P^1_{L^N_1})(x^1)\prod_{l=2}^{d}\lambda^l_{L^N_l,2^{n_l-1}-1}| \\ &\geqslant |(f_{n_1} * P^1_{L^N_1})(x^1)|\prod_{l=2}^{d}c^*_l \geqslant c^*|(f_{n_1} * P^1_{L^N_1})(x^1)| \end{aligned} \tag{23}$$

with a positive constant c^*. For the maximal operator T^{γ}_{CLR} we get

$$T^{\gamma}_{CLR}F_{\overline{n_1}}(x) = \sup_{n\in L}|(F_{\overline{n_1}} * P_n)(x)| \geqslant \sup_{L^N_1}|(F_{\overline{n_1}} * P_{L^N})(x)| \geqslant c^*\sup_{0<N}|(f_{n_1} * P^1_{L^N_1})(x^1)| \tag{24}$$

Since, $f_{n_1} \in H_{p_0}(G)$ is a p_0-atom with support I_{n_1}, we have that

$$f_{n_1} * P_m = 0 \quad \text{ for } m \leqslant 2^{n_1}.$$

This together with inequality (24) yield that

$$T^{\gamma}_{CLR}(F_{\overline{n_1}})(x) \geqslant c^*\sup_{m}|(f_{n_1} * P^1_m)(x^1)| = c^*T^1(f_{n_1})(x^1) \tag{25}$$

and

$$\frac{\|T^{\gamma}_{CLR}(F_{\overline{n_1}})\|_{p_0}}{\|F_{\overline{n_1}}\|_{H^{\gamma}_{p_0}(G^d)}} \geqslant \frac{c^*\|T^1(f_{n_1})\|_{p_0}}{\|f_{n_1}\|_{H_{p_0}(G)}} \geqslant c^*a_{n_1}. \tag{26}$$

$n_1 \to \infty$ and inequality (19) complete the proof of Theorem 2.

At last, we note that condition (20) can be weakened a little bit. Applying inequality (20) more precisely in the form

$$|P^l_{n_l} * \psi_{2^k-1}| = |\lambda^l_{n_l,2^k-1}| \geqslant c^k_l \tag{27}$$

in inequality (23). That is, instead of $\prod_{l=2}^{d}c^*_l$ we could write $\prod_{l=2}^{d}c_l^{n_l-1}$. Thus, on the right side of inequality (26) the expression $a^*_{n_1} := \prod_{l=2}^{d}c_l^{n_l-1}a_{n_1}$ appears (where $n_2,\ldots,n_d$ defined to n_1 earlier). If $a^*_{n_1} \to +\infty$, while $n_1 \to \infty$ our Theorem remain valid. It means that the absolute value of some coefficients in inequality (20) is not necessarily bounded from below by a positive constant.

3. Application

It is well-known (see Fine's map in the first section, equation (1)) that there is a direct connection between the Walsh group G and the interval $I := [0,1[$. The coordinate wise addition in G defines a so-called dyadic addition denoted by $\dot{+}$ in the interval $[0,1[$. The characters of the Walsh group G are the Walsh-Paley functions. If we change the operation $\dot{+}$ on the interval $I := [0,1[$ by the usual arithmetical sum denoted by $+$, then we get the so-called group of 2-adic integers. It is denoted by $(I,+)$. The character system belongs to $(I,+)$ will change, as well. Namely, the 2-adic (or arithmetic) sum $a+b := \sum_{n=0}^{\infty} r_n 2^{-(n+1)}$ $(a,b \in I)$, where bits $q_n, r_n \in \{0,1\} (n \in \mathbb{N})$ are defined recursively as follows : $q_{-1} := 0$, $a_n + b_n + q_{n-1} = 2q_n + r_n$ for $n \in \mathbb{N}$. (Since q_n, r_n take on only the values $0,1$, these equations uniquely determine the coefficients q_n and r_n.) Set

$$\varepsilon(t) := \exp(2\pi \imath t) \quad (t \in \mathbb{R}), \quad (\imath = (-1)^{\frac{1}{2}})$$

and

$$v_{2^n}(x) := \varepsilon\left(\frac{x_n}{2} + \ldots + \frac{x_0}{2^{n+1}}\right) \quad (x \in I, n \in \mathbb{N}).$$

We define the product system by

$$v_n := \prod_{n=0}^{\infty} v_{2^j}^{n_j},$$

where n_j is the jth coordinate of natural number n, for more details see the notion of product system in the first section. It is known [15] that the system $(v_n, n \in \mathbb{N})$ is the character system of $(I,+)$. In the present section, the system $\{\psi_k : k \in \mathbb{N}\}$ is defined by $\psi_k := v_k$ for all $k \in \mathbb{N}$. The Fourier coefficients, the Dirichlet and the Fejér kernels and (C,α) kernels are defined in the same way as we did in the first part of this paper. The partial sums, Fejér means and (C,α) means of an integrable function f are defined by the convolution of f with the kernel functions. For system $(v_n, n \in \mathbb{N})$ equality (2) remain valid. In the most cases the proving methods derived from the dyadic case.

In one-dimensional case the behaviour of the Cesàro means of the Fourier series on the group of 2-adic integers was discussed, the a.e. convergence and $(H.L)$ issue were treated by Gát [6]. In paper [9] Gát and the first author investigated the maximal operator $\sigma^* := \sup_n |\sigma_n^1|$ of the Fejér means with respect to the character system of 2-adic integers. Among others, they proved that this operator is bounded from the Hardy space H_p to the Lebesgue space L^p if and only if $1/2 < p < \infty$. They showed inequality (10) holds for the Fejér kernels for all $1/2 < p < 1$, as well (see inequality (4) in [9, page 75]). Inequality (3) for Fejér kernels follows from paper [7]. Applying our transference method (Theorem 1), we immediately get the following Theorem.

THEOREM 3. *Let γ be CRF. Set $\beta > 1$. The maximal operator σ_{CLR}^* is bounded from the Hardy space $H_p^{\gamma}(I^d)$ to the space $L^p(I^d)$ for all $\frac{1}{2} < p \leqslant 1$. Moreover, the maximal operator is of weak type $(1,1)$.*

According to Corollary 1 we could state that the d-dimensional Fejér means $\sigma_n(f)$ of an integrable function $f \in L^1(I^d)$ converge almost everywhere to the function f itself provided that the indeces are inside a cone-like set L.

In the end point case $p_0 = \frac{1}{2}$ the one-dimensional martingale $f_A(x) = D_{2^{A+1}}(x) - D_{2^A}(x)$ (see equality (17)) is applied to show that the maximal operator σ^* of Fejér means is not bounded from Hardy space $H_{1/2}(I)$ to the Lebesque space $L_{1/2}(I)$ [9, Theorem 2.2]. During the proof of this result it was proven, that

$$\frac{\|\sigma^*(f_n)\|_{1/2}}{\|f_n\|_{H_{1/2}}} \geqslant cn^2,$$

(see inequality (19), as well). For Fejér kernel functions

$$K_n^{\psi} * \varphi_{2^k - 1} \geqslant \frac{1}{2} \quad \text{hold for all } n \geqslant 2^{k+1}$$

(see inequality (20), as well). Applying Theorem 2 we immediately have the following Theorem.

THEOREM 4. *The cone-like restricted maximal operator* σ_{CLR}^* *is not bounded from the Hardy space* $H_{1/2}^{\gamma}(I^d)$ *to the space* $L^{1/2}(I^d)$.

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(Received May 22, 2020)

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