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A NOTE ON INTEGRAL REPRESENTATION OF SOME GENERALIZED ZETA FUNCTIONS AND ITS CONSEQUENCES

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Abstract. The main focus of the present note is to establish new integral representation for the Hurwitz-Lerch zeta and the multi-parameter Hurwitz-Lerch zeta functions. In particular, new integral expression of the polylogarithm function and the Fox-Wright function are derived. In addition, closed integral form expression of the moment generating function of a zeta distribution is established. As application, we derive the complete monotonicity properties of two classes of function related to the Hurwitz-Lerch zeta and the polylogarithm function. Moreover, some inequalities involving these two functions are proved.

1. Introduction

The Hurwitz-Lerch zeta function $\Phi(z,s,a)$ is defined by [17, p. 121]

$$\Phi(z,s,a)=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s} \tag{1}$$

$$\left(a\in\mathbb{C}\setminus\mathbb{Z}_0^-; s\in\mathbb{C} \text{ when } |z|<1;\ \Re(s)>1 \text{ when } |z|=1\right),$$

where $\mathbb{C}$ is the set of complex numbers, $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^+$ is the set of positive real numbers, $\mathbb{Z}$ is the set of integers and

$$\mathbb{Z}_0^- := \{0,-1,-2,-3,\ldots\}.$$

The Hurwitz-Lerch zeta function contains some special functions such as the Riemann zeta function $\zeta(s)$, the Hurwitz zeta function $\zeta(s,a)$, the polylogarithmic function (or de Jonquière's function) $\mathrm{Li}_s(z)$, the Lipschitz-Lerch zeta function $L(\xi,a,s)$ and the Lerch zeta function $l_s(\xi)$ defined by (see for example [2, p. 27–31])

$$\zeta(s):=\sum_{k=1}^{\infty}\frac{1}{k^s},\quad (\Re(s)>1), \tag{2}$$

$$\zeta(s,a):=\sum_{k=0}^{\infty}\frac{1}{(k+a)^s},\quad \left(\Re(s)>1,\ a\in\mathbb{C}\setminus\mathbb{Z}_0^-\right), \tag{3}$$

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$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s}, \quad (\Re(s) > 0;\ z \in \mathbb{C} \ \text{ when } |z| < 1) \tag{4}$$

$$L(\xi, a, s) = \sum_{n=0}^{\infty} \frac{e^{2in\pi\xi}}{(n+a)^s}, \quad (\Re(s) > 1;\ \xi \in \mathbb{R};\ 0 < a \leqslant 1). \tag{5}$$

and

$$l_s(\xi) = \sum_{n=0}^{\infty} \frac{e^{2in\pi\xi}}{(n+1)^s}, \quad (\Re(s) > 1;\ \xi \in \mathbb{R}). \tag{6}$$

It is well known that the Hurwitz-Lerch zeta function (1) possesses the following integral representation

$$\Phi(z, s, a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-at}}{1 - ze^{-t}} dt \tag{7}$$

$$(\Re(a) > 0,\ \Re(s) > 0, \text{ when } |z| < 1;\ \Re(s) > 1 \text{ when } |z| = 1).$$

Recently, a more general family of Hurwitz-Lerch zeta functions was investigated by Goyal and Laddha [4, p. 100, Eq. (1.5)]

$$\Phi_{\tau}^{*}(z, s, a) := \sum_{n=0}^{\infty} \frac{(\tau)_n}{n!} \frac{z^n}{(n+a)^s}, \tag{8}$$

$$\left(\tau \in \mathbb{C},\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \text{ when } |z| < 1;\ \Re(s - \tau) > 1 \text{ when } |z| = 1\right).$$

Here, and for the remainder of this paper, $(\tau)_\kappa$ denotes the Pochhammer symbol defined, in terms of the gamma function, that is

$$(\tau)_\kappa := \frac{\Gamma(\tau+\kappa)}{\Gamma(\tau)} = \begin{cases} 1 & (\kappa = 0, \tau \in \mathbb{C} \setminus \{0\}) \\ \tau(\tau+1)\ldots(\tau+n-1) & (\kappa = n \in \mathbb{N},\ \tau \in \mathbb{C}), \end{cases}$$

being understood conventionally that $(0)_0 := 1$ and assumed tacitly that the above Gamma quotient exists.

Garg et al. [5, p. 313, Eq. (1.7)], considered a further generalization of the Hurwitz-Lerch zeta functions $\Phi(z, s, a)$ and $\Phi_\tau^*(z, s, a)$ defined in the following form

$$\Phi_{\lambda,\mu,\nu}(z, s, a) := \sum_{n=0}^{\infty} \frac{(\lambda)_n (\mu)_n}{n! (\nu)_n} \frac{z^n}{(n+a)^s} \tag{9}$$

$$\left(\lambda, \mu \in \mathbb{C}; a, \nu \in \mathbb{C} \setminus \mathbb{Z}_0^-, s \in \mathbb{C} \text{ when } |z| < 1; \Re(s + \nu - \lambda - \mu) > 1 \text{ when } |z| = 1\right).$$

Various integral representations and two-sided bounding inequalities for $\Phi_{\lambda,\mu,\nu}(z, s, a)$ can be found in the works by Garg et al. [5] and Jankov et al. [6], respectively.

An extension of the above-defined function was investigated by Srivastava et al. [16, p. 491, Eq. (1. 20)] as

$$\Phi_{\lambda,\mu,\nu}^{\rho,\sigma,\kappa}(z, s, a) := \sum_{n=0}^{\infty} \frac{(\lambda)_{\rho n} (\mu)_{\sigma n}}{n! (\nu)_{\kappa n}} \frac{z^n}{(n+a)^s} \tag{10}$$

$$\Big(\lambda,\mu \in \mathbb{C}; a,\nu \in \mathbb{C}\setminus\mathbb{Z}_0^-; \rho,\sigma,\kappa \in \mathbb{R}^+, \kappa-\rho-\sigma > -1 \text{ when } s,z \in \mathbb{C};$$
$$\kappa-\rho-\sigma = -1 \text{ and } s \in \mathbb{C} \text{ when } |z| < \delta_0 := \kappa^\kappa \rho^{-\rho}\sigma^{-\sigma};$$
$$\kappa-\rho-\sigma = -1 \text{ and } \Re(s+\nu-\lambda-\mu) > 1 \text{ when } |z| = \delta_0\Big).$$

In 2011, Srivastava et al. [16, p. 503, Eq. (6.2)] investigate a new unification of the extended Hurwitz-Lerch zeta function $\Phi_{\lambda,\mu,\nu}^{\rho,\sigma,\kappa}(z,s,a)$, so-called *multi-parameter Hurwitz-Lerch zeta function*:

$$\begin{aligned}\Phi_{(\mu_j,\sigma_j;q)}^{(\lambda_j,\rho_j;p)}(z,s,a) &= \Phi_{\lambda_1,\dots,\lambda_p;\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p;\sigma_1,\dots,\sigma_q)}(z,s,a)\\ &= \left(\frac{\prod_{j=1}^q \Gamma(\mu_j)}{\prod_{j=1}^p \Gamma(\lambda_j)}\right)\sum_{k=0}^\infty \frac{\prod_{j=1}^p \Gamma(\lambda_j+k\rho_j)}{\prod_{j=1}^q \Gamma(\mu_j+k\sigma_j)}\frac{z^k}{k!(k+a)^s}\end{aligned} \tag{11}$$

$$\Big(p,q \in \mathbb{N}_0; \lambda_j \in \mathbb{C}\ (j=1,\dots,p); a,\mu_j \in \mathbb{C}\setminus\mathbb{Z}_0^-\ (j=1,\dots,q);$$
$$\rho_j,\sigma_k \in \mathbb{R}^+ (j=1,\dots,p; k=1,\dots,q);$$
$$\Delta_1 > -1 \ \text{ when } \ s,z \in \mathbb{C};$$
$$\Delta_1 = -1 \ \text{ and } \ s \in \mathbb{C} \text{ when } |z| < \nabla^*;$$
$$\Delta_1 = -1 \ \text{ and } \ \Re(\Xi) > \frac{1}{2} \text{ when } |z| < \nabla^*\Big),$$

where

$$\nabla^* = \left(\prod_{j=1}^p \rho_j^{-\rho_j}\right)\cdot\left(\prod_{j=1}^q \sigma_j^{\sigma_j}\right),$$

and

$$\Delta_1 = \sum_{j=1}^q \sigma_j - \sum_{j=1}^p \rho_j,$$

and

$$\Xi = s + \sum_{j=1}^q \mu_j - \sum_{j=1}^p \lambda_j + \frac{p-q}{2}.$$

In this sequel, definite integral expressions are derived for the Hurwitz-Lerch zeta function (or the renormalization constant of the generalized Hurwitz zeta distrubition (see, e. g., for [10])) and for a class of function related to the multi-parameter Hurwitz-Lerch zeta function. Its important corollaries, closed-form definite integral expression for the the polylogarithm function, the moment generating function of zeta distribution and the Fox-Wright functions are established.

In this note, the main tool we refer to is the Cahen formula for the Laplace integral form of Dirichlet series [1, 11]. Accordingly, the Dirichlet series

$$\mathscr{D}_{\mathbf{a}}(s) = \sum_{k=1}^\infty a_k e^{-sb_k},\ \Re(s) > 0,$$

having positive monotone increasing divergent to infinity sequence $(b_k)_{k\geqslant 1}$, possesses a Laplace representation [1, p. 97]

$$\mathscr{D}_{\mathbf{a}}(s) = s\int_0^\infty e^{-sx} \sum_{k:b_k\leqslant x} a_k\, dx = s\int_0^\infty e^{-sx} \sum_{k=1}^{[b^{-1}(x)]} a_k\, dx, \tag{12}$$

where the restriction to the set of positive integers of the function $b:\mathbb{R}_+ \longrightarrow \mathbb{R}_+$ forms the coefficient sequence $b|_{\mathbb{N}} = (b_k)$ associated with $\mathscr{D}_{\mathbf{a}}(s)$, b^{-1} denotes the (unique) inverse of the function b, and $[x]$ denotes the integer part of a real x, also see [14, 15]. Recently, Pogány [12, 13], by using the Cahen formula (12), has established a closed-form definite integral expressions for the COM-Poisson constant and of Le Roy-type hypergeometric function.

For the present study, we consider the following definition:

DEFINITION 1. A real valued function f, defined on an interval I, is called completely monotone on I, if f has derivatives of all orders and satisfies

$$(-1)^n f^{(n)}(x) \geqslant 0,\ n\in\mathbb{N}_0,\ \text{ and } x\in I. \tag{13}$$

The celebrated Bernstein Characterization Theorem gives a necessary and sufficient condition that the function f should be completely monotonic

$$f(x) = \int_0^\infty e^{-xt}\, d\mu(t),\ x>0, \tag{14}$$

where $\mu(t)$ is non-decreasing and the integral converges.

DEFINITION 2. [10, Definition 2.2] (Generalized Hurwitz zeta distribution) The generalized Hurwitz zeta random variable X_s is defined by

$$P(X_s = -s\log(k+a)) = \frac{z^k(k+a)^{-s}}{\Phi(z,s,a)},\ k\in\mathbb{N}_0,\ s>0,$$

where $\Phi(z,s,a)$ stands for the renormalization constant, and we call the distribution of X_s a generalized Hurwitz zeta distribution with parameter s.

DEFINITION 3. The moment generating function of a zeta distribution is defined by

$$M(t;s) = E(e^{tX}) = \frac{\mathrm{Li}_s(e^t)}{\zeta(s)}, t<0, \tag{15}$$

where the zeta distribution is defined for positive integers $k\geqslant 1$, and its probability mass function is given by

$$P(X_s = k) = \frac{k^{-s}}{\zeta(s)},$$

when $s>1$ is the parameter.

2. Integral form of the Hurwitz-Lerch zeta function

Here, using Cahen's formula (12), a single definite integral expression is established for the Hurwitz-Lerch zeta function $\Phi(z,s,a)$ defined in (1).

THEOREM 1. *The following integral representation*

$$\Phi(z,s,a) = \frac{1}{a^s} + \frac{z}{1-z} + \frac{sz}{z-1}\int_0^\infty e^{-sx} z^{[e^x-a]}dx, \tag{16}$$

holds true for all $0<z<1$ *and* $a>0$, *while* $[x]$ *denotes the integer part of a real* x. *Furthermore the function*

$$s \mapsto \Psi(z,s,a) := \frac{(z-1)\Phi(z,s,a)}{sz} + \frac{1-z}{sza^s} + \frac{1}{s},$$

is completely monotonic and log-convex on $(0,\infty)$ *for all* $0<z<1$ *and* $a>0$.

Proof. Rewriting the Hurwitz–Lerch zeta function into

$$\Phi(z,s,a) = \sum_{k=0}^{\infty} z^k e^{-s\log(k+a)}. \tag{17}$$

It turns out that it a classical Dirichlet series. Having in mind that $b(x) \equiv \log(x+a)$ is increasing and invertible on $(0,\infty)$ for all $a>0$. Keeping (12) and (17) in mind, we get

$$\begin{aligned}
\Phi(z,s,a) &= \frac{1}{a^s} + \sum_{k=1}^{\infty} z^k e^{-s\log(k+a)} \\
&= \frac{1}{a^s} + s\int_0^\infty e^{-sx} \sum_{k:\log(k+a)\leqslant x} z^k dx \\
&= \frac{1}{a^s} + s\int_0^\infty e^{-sx} \sum_{k=1}^{k(x)} z^k dx \\
&= \frac{1}{a^s} + \frac{sz}{1-z}\int_0^\infty e^{-sx}(1-z^{k(x)})dx \\
&= \frac{1}{a^s} + \frac{z}{1-z} + \frac{sz}{z-1}\int_0^\infty e^{-sx} z^{k(x)}dx,
\end{aligned} \tag{18}$$

where

$$k(x) = [e^x - a].$$

Moreover, by virtue of the integral representation (16) we conclude

$$\Psi(z,s,a) = \int_0^\infty e^{-sx} z^{[e^x-a]}dx. \tag{19}$$

Being simultaneously the spectral function $z^{[e^x-a]}$ positive, all prerequisites of the Bernstein Characterization Theorem for the complete monotone functions are fulfilled, that is, the function $s \mapsto \Psi(z,s,a)$ is completely monotone in the above mentioned range of the parameters involved. Moreover, since every completely monotonic function is log-convex, see [18, p. 167], we deduce that the function $s \mapsto \Psi(z,s,a)$ is log-convex. This completes the proof of Theorem 1.

REMARK 1. Substituting $t = e^x - a$ in (16) and (19) we have

$$\Phi(z,s,a) = \frac{1}{a^s} + \frac{z}{1-z} + \frac{sz}{z-1}\int_{1-a}^{\infty} \frac{z^{[t]}}{(t+a)^{s+1}}dt, \tag{20}$$

$$\Psi(z,s,a) = \int_{1-a}^{\infty} \frac{z^{[t]}}{(t+a)^{s+1}}dt. \tag{21}$$

COROLLARY 1. *The following inequalities hold true:*
a. *For $s > 0$ and $0 < z,a < 1$, we have*

$$\Phi(z,s,a) \leqslant \frac{1}{a^s} + \frac{z}{1-z}. \tag{22}$$

b. *For $s > 0$ and $0 < z,a < 1$, we have*

$$\Psi(z,s,a) \leqslant \frac{z^a - 1}{z^a \log(z)} + \frac{-\mathrm{Ei}(\log(z))}{z^{a+1}}, \tag{23}$$

where $\mathrm{Ei}(x)$ is the exponential integral function defined by

$$\mathrm{Ei}(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t}dt.$$

c. *For $s > 0$ and $0 < z,a < 1$, we have*

$$\Psi(z,s,a)\Psi(z,s+2,a) \geqslant \Psi^2(z,s+1,a). \tag{24}$$

d. *Let $s,t > 0$ $0 < z,a < 1$, we have*

$$\Psi(z,s,a)\Psi(z,t,a) \leqslant \left(\frac{z^a - 1}{z^a \log(z)} + \frac{-\mathrm{Ei}(\log(z)}{z^{a+1}}\right)\Psi(z,s+t,a). \tag{25}$$

Proof. For getting the inequality (22), just observe that the function $\Psi(z,s,a)$ is non-negative for all $s,a > 0$ and $0 < z < 1$ and consequently (22) holds true. As to the inequality (23), we apply the integral representation (21) we have

$$\begin{aligned}\Psi(z,s,a) &\leqslant \int_{1-a}^{\infty} \frac{z^t}{z(t+a)^{s+1}}dt \\ &= \frac{1}{z}\left[\int_{1-a}^{1} \frac{z^t}{(t+a)^{s+1}}dt + \int_{1}^{\infty} \frac{z^t}{(t+a)^{s+1}}dt\right] =: \frac{1}{z}\left(I_1 + I_2\right).\end{aligned} \tag{26}$$

Here

$$I_1 = \int_{1-a}^{1} \frac{z^t}{(t+a)^{s+1}} dt \leqslant \int_{1-a}^{1} z^t\, dt = \frac{z^a - 1}{z^{a-1}\log(z)}. \tag{27}$$

In addition, we have

$$I_2 = \int_{1}^{\infty} \frac{z^t}{(t+a)^{s+1}} dt \leqslant \int_{1}^{\infty} \frac{z^t}{t+a} dt. \tag{28}$$

In view of the following formula [3, Eq. (6), p. 134]

$$\int_{b}^{\infty} \frac{e^{-pt}}{t+a} dt = -e^{ap}\mathrm{Ei}(-(a+b)p), \; (\Re(p) > 0, |\arg(a+b)| < \pi), \tag{29}$$

we find that

$$I_2 \leqslant \frac{-\mathrm{Ei}(\log(z))}{z^a}. \tag{30}$$

Now applying (27) and (30) to (26) we get the inequality (23). Now, focus to the Turán type inequality (24). Since $s \mapsto \Psi(z,s,a)$ is log-convex on $(0,\infty)$ for $a > 0$ and $0 < z < 1$, it follows that for all $s_1, s_2 > 0, t \in [0,1]$ we have

$$\Psi(z, ts_1 + (1-t)s_2, a) \leqslant (\Psi(z,s_1,a))^t \, (\Psi(z,s_2,a))^{1-t}.$$

Choosing $s_1 = s, s_2 = s+2$ and $t = \frac{1}{2}$ the above inequality reduces to the Turán inequalty (24). Next, we derive the inequality (25). We set

$$f_a(z) = \frac{z^a - 1}{z^a \log(z)} + \frac{-\mathrm{Ei}(\log(z))}{z^{a+1}}.$$

By means of Theorem 1 and (23), it is clear that the function $s \mapsto \Psi(z,s,a)/f_a(z)$ maps $(0,\infty)$ into $(0,1)$ and it is completely monotonic on $(0,\infty)$. On the other hand, according to Kimberling [7] if a function f, defined on $(0,\infty)$, is continuous and completely monotonic and maps $(0,\infty)$ into $(0,1)$, then $\log f$ is super-additive, that is for all $x, y > 0$ we have

$$f(x)f(y) \leqslant f(x+y).$$

Therefore we conclude the asserted inequality (25).
On setting $a = 1$ in (16) (or in (20)), we get the following new integral representation for the polylogarithm function as follows:

COROLLARY 2. *The polylogarithm function* $\mathrm{Li}_s(z)$ *defined in (4) possesses the following integral representation*

$$\begin{aligned} \mathrm{Li}_s(z) &= z + \frac{z^2}{1-z} + \frac{sz}{z-1} \int_0^\infty e^{-sx} z^{[e^x]} dx \\ &= z + \frac{z^2}{1-z} + \frac{sz^2}{z-1} \int_0^\infty \frac{z^{[t]}}{(t+1)^{s+1}} dt, \end{aligned} \tag{31}$$

where $s>0$ *and* $0<z<1$. *Furthermore, the function*

$$s \mapsto \Psi_1(z,s) := \frac{(z-1)\mathrm{Li}_s(z)}{sz^2} + \frac{1-z}{sz} + \frac{1}{s},$$

is completely monotonic on $(0,\infty)$ *for all* $0<z<1$.

REMARK 2. Since the function $\Psi_1(z,s)$ is non-negative for all $s>0$ and $0<z<1$, we deduce that the following inequality holds true:

$$\mathrm{Li}_s(z) \leqslant z + \frac{z^2}{1-z}. \tag{32}$$

On letting $z=\mathrm{e}^{-t}, t>0$ in (31), using (15) we get the following closed integral form expression of the moment generating function of a zeta distribution, as follows:

COROLLARY 3. *The following integral formulas*

$$\begin{aligned} M(t;s) &= \frac{\mathrm{e}^{-t}}{\zeta(s)} + \frac{\mathrm{e}^{-2t}}{(1-\mathrm{e}^{-t})\zeta(s)} + \frac{s\mathrm{e}^{-t}}{(\mathrm{e}^{-t}-1)\zeta(s)} \int_0^\infty \mathrm{e}^{-sx}\mathrm{e}^{-t[\mathrm{e}^x]}dx \\ &= \frac{\mathrm{e}^{-t}}{\zeta(s)} + \frac{\mathrm{e}^{-2t}}{(1-\mathrm{e}^{-t})\zeta(s)} + \frac{s\mathrm{e}^{-2t}}{(\mathrm{e}^{-t}-1)\zeta(s)} \int_0^\infty \frac{\mathrm{e}^{-t[\xi]}}{(\xi+1)^{s+1}}d\xi, \end{aligned}$$

hold true for all $t>0$.

3. The integral expression of the multi-parameter Hurwitz-Lerch zeta function

Our main result in this section is asserted by the following theorem.

THEOREM 2. *Let*

$$(\lambda_i,\rho_i) = (\lambda,\rho), (\mu_i,\sigma_i) = (\mu,\sigma) \text{ for } 1 \leqslant i \leqslant p, \text{ and } (\lambda_{p+1},\rho_{p+1}) = (2,1).$$

If $\lambda<\mu$ *and* $\rho\leqslant\sigma$, *then the following integral representation holds true*

$$\begin{aligned} \tilde{\Phi}^{(\lambda_j,\rho_j;p+1)}_{(\mu_j,\sigma_j;p)}(z) &:= \Phi^{(\lambda_j,\rho_j;p+1)}_{(\mu_j,\sigma_j;p)}(z,1,1) \\ &= 1 + \frac{z\Gamma^p(\mu)}{(1-z)\Gamma^p(\lambda)} + \frac{pz\Gamma^p(\mu)}{(z-1)\Gamma^p(\lambda)} \int_0^\infty e^{-px} z^{\left[(\Delta^{(\rho,\sigma)}_{\lambda,\mu})^{-1}(e^x)\right]} dx, \end{aligned} \tag{33}$$

where $z\in(0,1)$ *and* $(\Delta^{(\rho,\sigma)}_{\lambda,\mu})^{-1}$ *stands for the inverse of the function*

$$\Delta^{(\rho,\sigma)}_{\lambda,\mu}(x) = \frac{\Gamma(\mu+\sigma x)}{\Gamma(\lambda+\rho x)}.$$

Proof. It turns out that

$$\Phi^{(\lambda_j,\rho_j;p+1)}_{(\mu_j,\sigma_j;p)}(z,1,1)=\Phi^{(1,\rho,\ldots,\rho;\sigma_1,\ldots,\sigma_q)}_{2,\lambda,\ldots,\lambda;\mu,\ldots,\mu}(z,1,1)=\frac{\Gamma^p(\mu)}{\Gamma^p(\lambda)}\sum_{k=0}^{\infty}e^{-p\log\left(\frac{\Gamma(\mu+\sigma x)}{\Gamma(\lambda+\rho x)}\right)}z^k, \tag{34}$$

is a classical Dirichlet series. Keeping in mind that $b(x)=\log\Gamma(\mu+\sigma x)-\log\Gamma(\lambda+\rho x)$ is strictly increasing on $(0,\infty)$ for all $\lambda<\mu$ and $\rho\leqslant\sigma$ and consequently $b(x)$ is invertible. Indeed, using the fact that the digamma function $x\mapsto\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$ is strictly increases on $(0,\infty)$, we get

$$\begin{aligned}b'(x)&=\sigma\psi(\mu+\sigma x)-\rho\psi(\lambda+\rho x)\\&>(\sigma-\rho)\psi(\lambda+\rho x)\\&\geqslant 0.\end{aligned}$$

This implies that the function $b(x)$ is strictly increasing on $(0,\infty)$ for each $\lambda<\mu$ and $\rho\leqslant\sigma$. Now, by combining (34) and (12) we thus get

$$\begin{aligned}\Phi^{(1,\rho,\ldots,\rho;\sigma_1,\ldots,\sigma_q)}_{2,\lambda,\ldots,\lambda;\mu,\ldots,\mu}(z,1,1)&=1+\frac{\Gamma^p(\mu)}{\Gamma^p(\lambda)}\sum_{k=1}^{\infty}e^{-p\log\left(\frac{\Gamma(\mu+\sigma x)}{\Gamma(\lambda+\rho x)}\right)}z^k\\&=1+\frac{p\Gamma^p(\mu)}{\Gamma^p(\lambda)}\int_0^\infty e^{-px}\sum_{k:\log(\Gamma(\mu+\sigma k)-\log\Gamma(\lambda+\rho k))\leqslant x}z^k dx\\&=1+\frac{p\Gamma^p(\mu)}{\Gamma^p(\lambda)}\int_0^\infty e^{-px}\sum_{k=1}^{j(x)}z^k dx\\&=1+\frac{pz\Gamma^p(\mu)}{(1-z)\Gamma^p(\lambda)}\int_0^\infty e^{-px}(1-z^{j(x)})dx\\&=1+\frac{z\Gamma^p(\mu)}{(1-z)\Gamma^p(\lambda)}+\frac{pz\Gamma^p(\mu)}{(z-1)\Gamma^p(\lambda)}\int_0^\infty e^{-px}z^{j(x)}dx,\end{aligned}$$

where

$$j(x)=\left[(\Delta^{(\rho,\sigma)}_{\lambda,\mu})^{-1}(e^x)\right].$$

The proof of Theorem 2 is complete.

On taking $p=1$ in (33), from (10) we compute the following result as follows:

COROLLARY 4. *If $\mu<\nu$ and $\sigma\leqslant\kappa$ then the following relation*

$$\Phi^{1,\sigma,\kappa}_{2,\mu,\nu}(z,1,1)=1+\frac{z\Gamma(\nu)}{(1-z)\Gamma(\mu)}+\frac{z\Gamma(\nu)}{(z-1)\Gamma(\mu)}\int_0^\infty e^{-x}z^{\left[(\Delta^{(\sigma,\kappa)}_{\mu,\nu})^{-1}(e^x)\right]}dx, \tag{35}$$

holds true for all $0<z<1$.

On setting $\sigma=\kappa=1$ in (35), we get the following result as follows:

COROLLARY 5. *If $\mu<\nu$, then the function $\Phi_{2,\mu,\nu}(z,1,1)$ defined in (9), admits* *the following integral formula:*

$$\Phi_{2,\mu,\nu}(z,1,1) = 1 + \frac{z\Gamma(\nu)}{(1-z)\Gamma(\mu)} + \frac{z\Gamma(\nu)}{(z-1)\Gamma(\mu)} \int_0^\infty e^{-x} z^{\left[(\Delta_{\mu,\nu}^{1,1})^{-1}(e^x)\right]} dx,$$

where $z \in (0,1)$.

The Fox-Wright function ${}_p\Psi_q[.]$ with p numerator parameters $a_1,\ldots,a_p$ and q denominator parameters $b_1,\ldots,b_q$, is defined by [19, p. 4, Eq. (2.4)]

$$\begin{aligned} {}_p\Psi_q\left[\begin{matrix}(a_1,A_1),\cdots,(a_p,A_p)\\(b_1,B_1),\cdots,(b_q,B_q)\end{matrix}\Big|z\right] &= {}_p\Psi_q\left[\begin{matrix}(\mathbf{a}_p,\mathbf{A}_p)\\(\mathbf{b}_q,\mathbf{B}_q)\end{matrix}\Big|z\right] \\ &= \sum_{k\geqslant 0} \frac{\prod_{l=1}^{p}\Gamma(a_l+kA_l)}{\prod_{l=1}^{q}\Gamma(b_l+kB_l)} \frac{z^k}{k!}, \end{aligned} \tag{36}$$

$$\left(a_i, b_j \in C, \text{ and } A_i, B_j \in R^+ \ (i=1,\ldots,p, j=1,\ldots,q)\right).$$

The convergence conditions and convergence radius of the series at the right-hand side of (36) we get from the known asymptotic of the Euler Gamma-function. The defining series in (36) converges in the whole complex z-plane when

$$\Delta = \sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i > -1.$$

If $\Delta = -1$, then the series in (36) converges for $|z| < \rho$, and $|z| = \rho$ under the condition $\Re(\mu) > \frac{1}{2}$, where

$$\rho = \left(\prod_{i=1}^{p} A_i^{-A_i}\right)\left(\prod_{j=1}^{q} B_j^{B_j}\right), \ \mu = \sum_{j=1}^{q} b_j - \sum_{k=1}^{p} a_k + \frac{p-q}{2}.$$

Setting in the definition (36)

$$A_1 = \ldots = A_p = 1 \quad \text{and} \quad B_1 = \ldots = B_q = 1,$$

we get the relatively more familiar generalized hypergeometric function ${}_pF_q[.]$ given by

$$\begin{aligned} {}_pF_q\left[\begin{matrix}\mathbf{a}_p\\ \mathbf{b}_q\end{matrix}\Big|z\right] &= \sum_{k\geqslant 0} \frac{\prod_{l=1}^{p}(a_l)_k}{\prod_{l=1}^{q}(b_l)_k} \frac{z^k}{k!} \\ &= \frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)} {}_p\Psi_q\left[\begin{matrix}(\mathbf{a}_p,\mathbf{1})\\(\mathbf{b}_q,\mathbf{1})\end{matrix}\Big|z\right]. \end{aligned} \tag{37}$$

Hence, by means of the definition (36) and (33) we deduce that the Fox-Wright function ${}_{p+1}\Psi_p[z]$ possesses the following integral representation.

COROLLARY 6. *Let $0<b<a$ and $0<A\leqslant B$, then following integral formula*

$$ {}_{p+1}\Psi_q\left[\begin{matrix}(1,1),(a,A)\\(b,B)\end{matrix}\Big|z\right]=\frac{\Gamma^p(a)}{\Gamma^p(b)}+\frac{z}{1-z}+\frac{pz}{(z-1)}\int_0^\infty \mathrm{e}^{-px}z^{\left[(\Delta_{a,b}^{(A,B)})^{-1}(\mathrm{e}^x)\right]}dx, \quad (38)$$

holds true for all $0<z<1$.

REMARK 3. For further some integral representation of the Fox-Wright function, we refer [8, 9].

If we set $A=B=1$ in (38), in view of (37), we get the following result.

COROLLARY 7. *If $0<a<b$, then the following integral formula holds true:*

$$\begin{aligned} {}_{p+1}F_p\left[\begin{matrix}1,a,\cdots,a\\b,\cdots,b\end{matrix}\Big|z\right]=1+\frac{z\Gamma^p(b)}{(1-z)\Gamma^p(a)}+\frac{pz\Gamma^p(b)}{(z-1)\Gamma^p(a)}\\ \times\int_0^\infty \mathrm{e}^{-px}z^{\left[(\Delta_{a,b}^{(1,1)})^{-1}(\mathrm{e}^x)\right]}dx, \end{aligned}\quad (39)$$

where $0<z<1$.

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