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A MULTILINEAR RELLICH INEQUALITY

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Abstract. We prove a multilinear variant of the Rellich inequality on the real line. In particular, we establish the weighted inequality

$$\left(\int_a^b w(\delta(x))\Big|\prod_{k=1}^m u_k(x)\Big|^p dx\right)^{1/p} \leqslant C\prod_{k=1}^m \|u_k''\|_{L^{p_k}(a,b)}, \;\; u_k \in C_0^2(a,b), \;\; k=1,\dots,m,$$

with a positive function w on $(0,b-a)$, where $-\infty \leqslant a < b \leqslant +\infty$, m is a positive integer, $\delta(x) = \min\{x-a, b-x\}$ is the distance function on (a,b), and $1/p = \sum_{j=1}^m 1/p_j$, $p_j > 1$, $j = 1,\dots,m$. As a corollary we derive the following estimate

$$\left(\int_a^b \Big|\prod_{j=1}^m u_j(x)\Big|^p \delta(x)^{-2mp} dx\right)^{1/p} \leqslant C\prod_{j=1}^m \|u_j''\|_{L^{p_j}(a,b)}.$$

1. Introduction

A considerable effort has been made in recent years to establish the (weighted) boundedness of integral operators in Lebesgue spaces. Such problems have been studied extensively in Harmonic Analysis, especially in the last two decades (see e.g. the monograph [7] and references cited therein). Our aim is to establish an m-linear weighted Rellich inequality

$$\Big\|\prod_{j=1}^m u_j\Big\|_{L^p_{w(\delta(\cdot))}(I)} \leqslant C\prod_{j=1}^m \Big\|u_j''\Big\|_{L^{p_j}(I)}, \;\; I := (a,b), \;\; -\infty \leqslant a < b \leqslant +\infty, \tag{1}$$

with a certain positive constant c independent of $u_k \in C_0^\infty(I)$, $k = 1,\dots,m$, where $\delta(x)$ is the distance function on I given by the formula

$$\delta(x) = \min\{x-a, b-x\} \tag{2}$$

and

$$\frac{1}{p} := \sum_{k=1}^m \frac{1}{p_k}, \quad 1 < p_k < \infty, \;\; k = 1,\dots,m. \tag{3}$$

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Throughout the paper we shall assume that m is a positive integer, and p is determined by (3). Observe that in this case $0 < p < \infty$.

We establish (1) by using appropriate multilinear weighted Hardy inequalities

$$\left\| \prod_{j=1}^{m} \int_{a}^{x} f_j(t)dt \right\|_{L^p_v(a,b)} \leqslant c \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(a,b)}, \tag{4}$$

$$\left\| \prod_{j=1}^{m} \int_{x}^{b} f_j(t)dt \right\|_{L^p_v(a,b)} \leqslant c \prod_{j=1}^{m} \|f_j\|_{L^{p}(a,b)}, \tag{5}$$

which are also proved in this paper. It should be noticed that necessary and sufficient conditions governing the two-weight bilinear Hardy inequality

$$\left(\int_{a}^{b} \left(\int_{a}^{x} f \right)^{q} \left(\int_{a}^{x} g \right)^{q} w(x)dx \right)^{1/q} \leqslant C \left(\int_{a}^{b} f^{p_1} w_1 \right)^{1/p_1} \left(\int_{a}^{b} g^{p_2} w_1 \right)^{1/p_2},$$

for non-negative f and g were found in [1] under different conditions on weights for various ranges of p_1, p_2 and q, with $q > 1$. The proofs used in this paper enable us to get appropriate two-weight bilinear Rellich inequalities based on the results of [1], but we do not consider this case here.

Historically, Rellich's [9] famous inequality states that if $u \in C_0^{\infty}(\mathbb{R}^n \backslash \{0\})$ and $n \neq 2$, then

$$\frac{n^2(n-4)^2}{16} \int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^4} dx \leqslant \int_{\mathbb{R}^n} |\Delta u(x)|^2 dx,$$

and the constant $n^2(n-4)^2/16$ is sharp. When $n = 2$ the inequality still holds but only for a restricted class of functions (see [2]). In these results the underlying set is $\mathbb{R}^n \backslash \{0\}$ and $|x|$ may be thought of as the distance of a point $x \in \mathbb{R}^n$ from the boundary $\{0\}$ of $\mathbb{R}^n \backslash \{0\}$. With this in mind it is natural to consider functions defined on a more general open subset Ω of $\mathbb{R}^n$ and to seek an inequality involving the distance function δ defined by $\delta(x) = \text{dist}\,(x, \partial\Omega)$. This approach led to the inequality

$$\frac{9}{16} \int_{\Omega} \frac{|u(x)|^2}{\delta(x)^4} dx \leqslant \int_{\Omega} |\Delta u(x)|^2 dx, \; u \in C_0^{\infty}(\Omega),$$

valid when Ω is convex and even under rather more general conditions: see [2], Corollary 6.2.7.

All this work is in the setting of L^2. The first step towards an L^p version appears to have been taken by Davies and Hinz [3], most notably when $\Omega = \mathbb{R}^n \backslash \{0\}$. In the recent paper [4] an L^p form of the Rellich inequality was obtained in general open subsets Ω of $\mathbb{R}^n$, together with a somewhat similar inequality involving the p-Laplacian. The main results of [4] can be summarized as follows:

THEOREM A. (the case $n=1$) *Suppose that* $-\infty < a < b \leqslant \infty$ *and let* $r \in (1,\infty)$; *put* $\delta(t) = \min\{t-a, b-t\}$. *Then for all* $u \in C_0^2(a,b)$,

$$\int_a^b \frac{|u(t)|^r}{\delta(t)^{2r}} dt \leqslant \left(\frac{r}{2r-1}\right)^r \left(\frac{r}{r-1}\right)^r \int_a^b \left|u''(t)\right|^r dt.$$

THEOREM B. (the higher-dimensional case) *Let* Ω *be a non-empty, proper open subset of* $\mathbb{R}^n$ *and let* $r \in (1,\infty)$; *suppose that* $u \in C_0^2(\Omega)$. *If* $r=2$, *then*

$$\int_\Omega \frac{|u(x)|^2}{\delta_{M,4}(x)^4} dx \leqslant \frac{16}{9} \int_\Omega |\Delta u(x)|^2 dx,$$

while if $r \in (1,\infty)\backslash\{2\}$, *then for some explicit constant* $K(r,n)$,

$$\int_\Omega \frac{|u(x)|^r}{\delta_{M,2r}(x)^{2r}} dx \leqslant K(r,n) \int_\Omega |\Delta u(x)|^r dx.$$

Here $\delta_{M,4}$ *and* $\delta_{M,2r}$ *are mean distance functions obtained by averaging, in a certain sense, the distance to the boundary of* Ω *in all possible directions.*

Finally we mention that two-weight estimates involving one-dimensional Rellich inequalities were studied in [5] in the linear setting.

2. Preliminaries

Let v be an a.e. positive function (i.e. a weight) on the interval $I := (a,b)$, $-\infty \leqslant a < b \leqslant \infty$. We denote by $L_v^r(I)$ (or by $L_v^r(a,b)$), $0 < r < \infty$, the Lebesgue space defined by the norm for $r \geqslant 1$ (quasi-norm if $0 < r < 1$):

$$\|g\|_{L_v^r(I)} = \left(\int_a^b |g(x)|^r v(x) dx\right)^{1/r}.$$

If $v \equiv const$, then $L_v^r(I)$ will be denoted by $L^r(I)$ (or by $L^r(a,b)$).

In the sequel we will denote by H_a and H_b' the Hardy-type operators of the form:

$$H_a f(x) = \frac{1}{x-a} \int_a^x f(t) dt, \quad x \in (a,b), \quad -\infty < a < b \leqslant \infty;$$

$$H_b' f(x) = \frac{1}{b-x} \int_x^b f(t) dt; \quad x \in (a,b), \quad -\infty \leqslant a < b < \infty.$$

It is known that (for example, it follows from the general two-weight theory for the Hardy operator, see, e.g., introduction of [8] and references therein) that operators H_a and H_b' are bounded in $L^r(a,b)$ for $1 < r < \infty$; moreover,

$$\|H_a\|_{L^r} := \|H_a\|_{L^r(a,b) \to L^r(a,b)} \leqslant r', \text{ and } \|H_b'\|_{L^r} := \|H_b'\|_{L^r(a,b) \to L^r(a,b)} \leqslant r', \tag{6}$$

where $-\infty < a < b \leqslant \infty$, and

$$r' = \frac{r}{r-1}.$$

To get the main results of this paper we will obtain weighted multilinear Hardy inequalities. In particular, we prove the following statements:

PROPOSITION 1. *Let $-\infty < a < b \leqslant \infty$, v be a weight function on (a,b). Then inequality (4) with a positive constant c independent of f_j, $f_j \in L^{p_j}(a,b)$, $j = 1,\dots,m$, holds if and only if*

$$A_{a,b} := \sup_{a<t<b} \left(\int_t^b v(x)dx \right)^{1/p} \Big(t-a\Big)^{m-1/p} < \infty.$$

Moreover, if c is the best possible constant in (4), then

$$A_{a,b} \leqslant c \leqslant CA_{a,b},$$

where

$$C = \begin{cases} \left(2 + 2^{mp-1} \prod_{i=1}^{m} \|H_a\|^p_{L^{p_i}(a,b)}\right)^{1/p}, & \text{if } b < \infty, \\ 2^{m-1/p} \prod_{j=1}^{m} \|H_a\|_{L^{p_i}(a,\infty)} & \text{if } \ b = \infty. \end{cases}$$

PROPOSITION 2. *Let $-\infty \leqslant a < b < \infty$, v be a weight function on (a,b). Then inequality (5) with a positive constant c independent of f_j, $f_j \in L^{p_j}(a,b)$, $j = 1,\dots,m$, holds if and only if*

$$B_{a,b} := \sup_{a<t<b} \left(\int_a^t v(x)dx \right)^{1/p} \Big(b-t\Big)^{m-1/p} < \infty.$$

Moreover, if c is the best possible constant in (5), then

$$B_{a,b} \leqslant c \leqslant CB_{a,b},$$

where

$$C = \begin{cases} \left(2 + 2^{mp-1} \prod_{i=1}^{m} \|H'_b\|^p_{L^{p_i}(a,b)}\right)^{1/p}, & \text{if } a > -\infty, \\ 2^{m-1/p} \prod_{j=1}^{m} \|H'_b\|_{L^{p_i}(-\infty,b)} & \text{if } \ a = -\infty. \end{cases}$$

3. The weighted Rellich inequality

In this section we formulate the main results of this paper. We discuss the case when a weight function is a function of the distance function (2) defined on (a,b).

THEOREM 1. *Let $-\infty < a < b < \infty$, $I := (a,b)$, and let w be a weight function on the interval $\big(0,(b-a)\big)$. If*

$$\widetilde{D}_{a,b} := \sup_{0<\tau<b-a} \left(\int_\tau^b w(x)x^{mp}dx \right)^{1/p} \tau^{m-1/p} < \infty,$$

then for all $u_j \in C_0^2(I)$, $j=1,\ldots,m$, *inequality* (1) *holds with the constant* C *given by the formula:*

$$C = 4^{1/p}\widetilde{D}_{a,b}\left[1+2^{mp-2}\prod_{i=1}^{m}(p_i')^p\right]^{1/p}. \tag{7}$$

The next statement deals with the cases $b=\infty$ and $a=-\infty$, respectively.

THEOREM 2. *Let* $-\infty < a < \infty$. *Suppose that* $I := (a,\infty)$. *Let* w *be a weight function on* $(0,\infty)$. *If*

$$\widetilde{D} := \sup_{t>0}\left(\int_t^{\infty} w(x)x^{mp}\,dx\right)^{1/p} t^{m-1/p} < \infty, \tag{8}$$

then for all $u_j \in C_0^2(I)$, $j=1,\ldots,m$, *inequality* (1) *holds, where*

$$C = 2^{m-1/p}\widetilde{D}\prod_{i=1}^{m} p_i'. \tag{9}$$

THEOREM 3. *Let* $-\infty < b < \infty$ *and let* $I := (-\infty,b)$. *Suppose that* w *be a positive function on* $(0,\infty)$. *If condition* (8) *is satisfied, then for all* $u_j \in C_0^2(I)$, $j=1,\ldots,m$, *inequality* (1) *holds, where* C *is defined by* (9).

By applying Theorems 1, 2 and 3 we can easily deduce the following statements:

COROLLARY 1. *Let* $-\infty < a < b < \infty$ *and let* $I := (a,b)$. *Then the following inequality holds for all* $u_j \in C_0^2(I)$, $j=1,\ldots,m$,

$$\left(\int_I \left|\prod_{j=1}^{m} u_j(x)\right|^p \delta(x)^{-2mp}\,dx\right)^{1/p} \leqslant C\prod_{j=1}^{m}\|u_j''\|_{L^{p_j}(I)}, \tag{10}$$

where

$$C = (2mp-1)^{-1/p}\left[1+2^{mp-2}\prod_{i=1}^{m}(p_i')^p\right]^{1/p}.$$

COROLLARY 2. *Let* $-\infty < a < \infty$ *and let* $I := (a,\infty)$. *Then inequality* (10) *holds for all* $u_j \in C_0^2(I)$, $j=1,\ldots,m$, *where*

$$C = 2^{m-1/p}(2mp-1)^{-1/p}\prod_{j=1}^{m} p_j'.$$

COROLLARY 3. *Let* $-\infty < b < \infty$. *Suppose that* $I := (-\infty,b)$. *Then inequality* (10) *holds for all* $u_j \in C_0^2(I)$, $j=1,\ldots,m$, *where* C *is defined by*

$$C = 2^{m-1/p}(2mp-1)^{-1/p}\prod_{j=1}^{m} p_j'.$$

4. Proofs of the results

First we prove Proposition 1.

Proof of Proposition 1. *Necessity*, and consequently, the lower estimate $c \geqslant A_{a,b}$ follows by the standard way taking the test functions $f_j = \chi_{(a,t)}$ in inequality (4). Now we show sufficiency, and hence, the upper estimate for the best constant c in (4).

(i) $b < \infty$.

Observe that

$$\Big(\int_t^b v(x)dx\Big)^{1/p}(t-a)^{m-1/p} = \Big(\int_t^b v(x)dx\Big)^{1/p}\prod_{j=1}^m (t-a)^{1/p'_j}$$

for all $a < t < b$. That is why the condition $A_{a,b} < \infty$, Hölder's inequality and the representation $(a,b) = \cup_{k\geqslant 0}[a+\frac{b-a}{2^{k+1}}, a+\frac{b-a}{2^k})$ yield (we assume that f_j are non-negative and $f_j \in L^{p_j}(I)$, $j = 1,\dots,m$):

$$\begin{aligned}
\Big\|\prod_{j=1}^m \int_a^x f_j dt\Big\|^p_{L^p_v(a,b)} &= \sum_{k\geqslant 0}\int_{a+\frac{b-a}{2^{k+1}}}^{a+\frac{b-a}{2^k}}\prod_{j=1}^m\Big(\int_a^x f_j(t)dt\Big)^p v(x)dx\\
&\leqslant \sum_{k\geqslant 0}\Big(\int_{a+\frac{b-a}{2^{k+1}}}^{a+\frac{b-a}{2^k}} v(x)dx\Big)\prod_{j=1}^m\Big(\int_a^{a+\frac{b-a}{2^k}} f_j(t)dt\Big)^p\\
&\leqslant A^p_{a,b}\sum_{k\geqslant 0}\prod_{j=1}^m\Big(\frac{b-a}{2^{k+1}}\Big)^{-p/p'_j}\Big(\int_a^{a+\frac{b-a}{2^k}} f_j(t)dt\Big)^p\\
&= A^p_{a,b}\prod_{j=1}^m\Big(\frac{b-a}{2}\Big)^{-p/p'_j}\Big(\int_a^b f_j(t)dt\Big)^p\\
&\quad + A^p_{a,b}\sum_{k\geqslant 1}\prod_{j=1}^m\Big(\frac{b-a}{2^{k+1}}\Big)^{-p/p'_j}\Big(\int_a^{a+\frac{b-a}{2^k}} f_j(t)dt\Big)^p\\
&\leqslant A^p_{a,b}\prod_{j=1}^m\Big(\frac{b-a}{2}\Big)^{-p/p'_j}\Big(\int_a^b f_j^{p_j}(t)dt\Big)^{p/p_j}(b-a)^{p/p'_j}\\
&\quad + A^p_{a,b}\prod_{j=1}^m\Big[\sum_{k\geqslant 1}\Big(\frac{b-a}{2^{k+1}}\Big)^{-p_j/p'_j}\Big(\int_a^{a+\frac{b-a}{2^k}} f_j(t)dt\Big)^{p_j}\Big]^{p/p_j}
\end{aligned}$$

$$\leqslant 2A_{a,b}^p \prod_{i=1}^m \|f_j\|_{L^{p_j}}^p + 2^{mp-1} A_{a,b}^p \prod_{i=1}^m \Bigg[\sum_{k\geqslant 1} \Big(\frac{b-a}{2^k}\Big)^{-p_j/p_j'} \times \Big(\frac{(b-a)}{2^k}\Big)^{-1} \frac{(b-a)}{2^k} \Big(\int_a^{a+(b-a)/2^{k-1}} f_j(t)dt \Big)^{p_j} dx \Bigg]^{p/p_j}$$

$$\leqslant 2A_{a,b}^p \prod_{i=1}^m \|f_j\|_{L^{p_j}}^p + 2^{mp-1} A_{a,b}^p \prod_{i=1}^m \Bigg[\int_a^b \Big(\int_a^x f_j(t)dt \Big)^{p_j} (x-a)^{-p_i} dx \Bigg]^{p/p_j}$$

$$\leqslant 2A_{a,b}^p \prod_{i=1}^m \|f_j\|_{L^{p_j}}^p + 2^{mp-1} A_{a,b}^p \prod_{i=1}^m \|H_a\|_{L^{p_i}}^p \Big(\int_a^b (f_j(t))^{p_j} dt \Big)^{p/p_j}$$

$$= \Big(2A_{a,b}^p + 2^{mp-1} A_{a,b}^p \prod_{i=1}^m \|H_a\|_{L^{p_i}}^p \Big) \|f_j\|_{L^{p_j}}^p .$$

Hence,

$$\Big\| \prod_{i=1}^m \Big(\int_a^x f_i(t)dt \Big) \Big\|_{L_w^p} \leqslant A_{a,b} \Big(2 + 2^{mp-1} \prod_{i=1}^m \|H_a\|_{L^{p_i}}^p \Big)^{1/p} \prod_{i=1}^m \|f_i\|_{L^{p_i}}^p .$$

(ii) $b=\infty$. In this case we represent the interval (a,∞) as the union of intervals $(a+2^k, a+2^{k+1}]$, where $k\in\mathbb{Z}$. Consequently, arguing as above we find that

$$\Big\| \prod_{j=1}^m \int_a^x f_j dt \Big\|_{L_v^p(a,b)}^p = \sum_{k\in\mathbb{Z}} \Big(\int_{a+2^k}^{a+2^{k+1}} v(x)dx \Big) \prod_{j=1}^m \Big(\int_a^{a+2^{k+1}} f_j(t)dt \Big)^p$$

$$\leqslant A_{a,\infty}^p \sum_{k\in\mathbb{Z}} \prod_{j=1}^m 2^{-kp/p_j'} \Big(\int_a^{a+2^{k+1}} f_j(t)dt \Big)^p$$

$$\leqslant A_{a,\infty}^p \prod_{j=1}^m \Bigg[\sum_{k\in\mathbb{Z}} 2^{-kp_j/p_j'} \Big(\int_a^{a+2^{k+1}} f_j(t)dt \Big)^{p_j} \Bigg]^{p/p_j}$$

$$\leqslant 2^{pm-1} A_{a,\infty}^p \prod_{j=1}^m \Bigg[\int_a^\infty (x-a)^{-p_j} \Big(\int_a^x f_j(t)dt \Big)^{p_j} \Bigg]^{p/p_j}$$

$$\leqslant 2^{pm-1} A_{a,\infty}^p \prod_{j=1}^m \|H_a\|_{L^{p_j}}^p \prod_{j=1}^m \|f_j\|_{L^{p_j}}^p .$$

The proposition has been proved. □

Proof of Proposition 2. The proof of this statement is similar to that of Proposition 1. We only point out that in this case we use the representations:

$$(a,b) = \cup_{j=0}^\infty \Big(b-(b-a)/2^k, b-(b-a)/2^{k+1} \Big],$$

$$(-\infty,b)=\cup_{j\in\mathbb{Z}}\Big(b-2^{k+1},b-2^{k}\Big]. \quad \square$$

Proof of Theorem 1. Let d be the midpoint of the interval (a,b). Then by using Propositions 1 and 2 we have that

$$\begin{aligned}
\Big\|\prod_{j=1}^{m}u_j\Big\|^p_{L^p_{w(\delta(\cdot))}(a,b)} &\leqslant \Big\|\prod_{j=1}^{m}u_j\Big\|^p_{L^p_{w(\delta(\cdot))}(a,d)}+\Big\|\prod_{j=1}^{m}u_j\Big\|^p_{L^p_{w(\delta(\cdot))}(d,b)}\\
&\leqslant \int_a^d w(x-a)\Big|\prod_{j=1}^{m}u_j(x)\Big|^p dx+\int_d^b w(b-x)\Big|\prod_{j=1}^{m}u_j(x)\Big|^p dx\\
&\leqslant \int_a^b w(x-a)\Big|\prod_{j=1}^{m}\int_a^x (x-t)|u_j''(t)|dt\Big|^p dx\\
&\quad+\int_a^b w(b-x)\Big|\prod_{j=1}^{m}\int_x^b (t-x)u_j''(t)dt\Big|^p dx\\
&\leqslant \int_a^b w(x-a)(x-a)^{mp}\Big(\prod_{j=1}^{m}\int_a^x |u_j''(t)dt|\Big)^p dx\\
&\quad+\int_a^b w(b-x)(b-x)^{mp}\Big(\prod_{j=1}^{m}\int_x^b |u_j''(t)|dt\Big)^p dx\\
&\leqslant \widetilde{C}\prod_{j=1}^{m}\|u_j''\|^p_{L^{p_j}(a,b)},
\end{aligned}$$

where $\widetilde{C}$ is defined by

$$\widetilde{C}=\Big(2+2^{mp-1}\prod_{i=1}^{m}\|H_a\|^p_{L^{p_i}(I)}\Big)\widetilde{A}^p_{a,b}+\Big(2+2^{pm-1}\prod_{i=1}^{m}\|H_b'\|^p_{L^{p_i}(I)}\Big)\widetilde{B}^p_{a,b} \tag{11}$$

with

$$\widetilde{A}_{a,b}:=\sup_{a<t<b}\Big(\int_t^b w(x-a)(x-a)^{mp}dx\Big)^{1/p}\Big(t-a\Big)^{m-1/p},$$

and

$$\widetilde{B}_{a,b}:=\sup_{a<t<b}\Big(\int_a^t w(b-x)(b-x)^{mp}dx\Big)^{1/p}\Big(b-t\Big)^{m-1/p}.$$

Consequently,

$$\Big\|\prod_{j=1}^{m}u_j\Big\|_{L^p_{w(\delta(\cdot))}(a,b)}\leqslant \widetilde{C}^{1/p}\prod_{j=1}^{m}\|u_j''\|_{L^{p_j}(a,b)},$$

with the constant $\widetilde{C}$ defined by (11).

Observe now that an appropriate change of variables in integrals yields that

$$\widetilde{A}_{a,b} = \widetilde{B}_{a,b} = \widetilde{D}_{a,b} < \infty.$$

Consequently, taking also (6) into account we find that

$$\widetilde{C}^{1/p} = 4^{1/p}\widetilde{D}_{a,b}\left[1 + 2^{mp-3}\left(\prod_{i=1}^{m}\|H_a\|^p_{L^{p_i}(I)} + \prod_{i=1}^{m}\|H'_b\|^p_{L^{p_i}(I)}\right)\right]^{1/p} \leqslant C,$$

where C is defined by (7). □

Proof of Theorem 2. Suppose that $I = (a,\infty)$. Observe that if $x \in (a,\infty)$ we have that $\delta(x) = x - a$. Arguing as in the proof of Theorem 1 and applying Proposition 1 and the representations $u_j(x) = \int_a^x (x-t)u''_j(t)dt$ for $u_j \in C_0^2(I)$, $j = 1,\dots,m$, we get

$$\left\|\prod_{j=1}^{m} u_j\right\|_{L^p_{w(\delta(\cdot))}(I)} \leqslant 2^{m-1/p}\widetilde{A}_{a,\infty}\left(\prod_{j=1}^{m}\|H_a\|_{L^{p_i}(I)}\right)\prod_{j=1}^{m}\|u''_j\|_{L^{p_j}(I)},$$

where

$$\widetilde{A}_{a,\infty} := \sup_{t>a}\left(\int_t^b w(x-a)(x-a)^{mp}dx\right)^{1/p}\left(t-a\right)^{m-1/p}.$$

Since

$$\widetilde{A}_{a,\infty} = \widetilde{D} < \infty,$$

and (6) holds, we are done.

Proof of Theorem 3. Let $I = (-\infty,b)$. The proof of this case is similar to that of the previous one. In this case we use representations $u_j(x) = \int_x^b (t-x)u''_j(t)dt$ which hold for $u_j \in C_0^2(I)$, $j = 1,\dots,m$, and Proposition 2. Hence, we find that

$$\left\|\prod_{j=1}^{m} u_j\right\|_{L^p_{w(\delta(\cdot))}(I)} \leqslant 2^{m-1/p}\widetilde{B}_{-\infty,b}\left(\prod_{j=1}^{m}\|H'_b\|_{L^{p_i}(I)}\right)\prod_{j=1}^{m}\|u''_j\|_{L^{p_j}(I)},$$

where

$$\widetilde{B}_{-\infty,b} := \sup_{t<b}\left(\int_{-\infty}^t w(b-x)(b-x)^{mp}dx\right)^{1/p}\left(b-t\right)^{m-1/p}.$$

Again, it can be checked that

$$\widetilde{B}_{-\infty,b} = \widetilde{D} < \infty,$$

where $\widetilde{D}$ is defined by (8).

Since (6) holds for the norms $\|H'_b\|_{L^{p_i}(I)}$, $i = 1\dots,m$, the result follows. □

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